11 Game Theory Example.

11.1 Background.

Imagine the time of World War II. As a lieutenant in the allied forces your mission is to defend against river crossings. Each night a patrol of twenty soldiers from the axis forces will attempt to cross the river. They use any one of three different locations to cross. Sometimes the group is split in half, and the two halves use two different locations to cross. There are six possibilities: $20_I, 20_{II}, 20_{III}, 10_I10_{II}, 10_I10_{III}, 10_{II}10_{III}$. Location I is far upstream at a very narrow and fast flowing portion of the river. Once across, the terrain consists of the bottom of a steep hill T. Once the steep hill is crossed a tiny hill H is visible downstream. Location II is straight ahead where the river is the widest and flows very slowly. Upstream the high hill H looms in the distance, and slightly downstream the tiny hill T guards the river. Location III is quite a distance downstream. The river is of average width, and once across there is a marsh with a smattering of landmines. Once the marsh is navigated successfully the tiny hill T is visible upstream the river. There are two ways for you to defend: from the top of the high hill H or from the top of the tiny hill T.

11.2 Estimated Outcomes.

Based on past experience the following is known. If plan 20_I is used, then trouble awaits at the crossing due to wild water, and afterwards due to the steep hill. A defense from H is devastating because of the proximity and the difficult terrain. Only four soldiers are expected to survive. A defense from T is far less effective and fourteen soldiers are expected to survive. If plan 20_{II} is used, then the crossing itself is the problem. A long time is spent in the water and the soldiers are defenseless all that time. A defense from H is fairly effective but the distances to the targets present a problem. Eight soldiers are expected to survive. Meanwhile, a defense from T is more robust. The targets are within closer range and there is plenty of time to act. Ten soldiers are expected to survive. If plan 20_{III} is used, then the crossing is comparatively easy. The trouble this time is the landmines and the marsh. A defense from H has no effect. The mines do their job but eighteen soldiers survive. A defense from T is overwhelming because the remaining soldiers are physically exhausted passing through the marsh. Only five soldiers survive. When the twenty soldiers split into two groups they have a harder time to give each other cover. This added risk is sometimes balanced by the possibility of a two-sided attack. If plan 10_I10_{II} is used, then a defense from H results in six surviving soldiers. The distance to the upstream crossing point hampers a defense from T, and the result is that twelve soldiers survive. If plan 10_I10_{III} is used, then a defense at H obliterates the upstream party and the downstream party loose one soldier due to the mines. A total of nine soldiers survive. A defense at T is slightly more difficult and ten

soldiers survive. If plan $10_{II}10_{III}$ is used, then a defense at H allows thirteen soldiers to survive. There is no two-sided attack so a defense at T is potent and only six soldiers survive.

11.3 Payoff matrix.

A payoff matrix gives a convenient representation of the different outcomes. In the present example the payoff matrix is given by:

	20_I	20_{II}	20_{III}	10_I10_{II}	10_I10_{III}	$10_{II}10_{III}$
H	4	8	18	6	9	13
T	14	10	5	12	10	6

The magnitudes of the numbers have opposite meanings to the two sides. Large numbers are good for the axis but bad for the allies. Small numbers are good for the allies but bad for the axis. Note that each night both sides have to choose one of the plans available to them. It is assumed that neither side has prior knowledge of the choice of plan of the other side. Once the choice of plans is completed, the payoff matrix gives the outcome in each case.

11.4 Strategies.

The term strategy comes into play when several nights are considered. A *pure* strategy is to choose the same plan each night. A *mixed* strategy is to allow a different choice of plan for different nights. It may be argued that a pure strategy is too predictable. The average number of soldiers that survive gives a statistical measure of the performance of a particular strategy. Using this kind of measure it is conceivable that a pure strategy yields the best performance. In other words, sometimes it is okay to be predictable. One way to incorporate unpredictability into a strategy is to let the choice of plan be dictated by a probability distribution. Incidentally, this includes pure strategies because the probability distribution is allowed to assign the probability one. The mixed strategy considered here assumes that a fixed probability distribution is used for all nights.

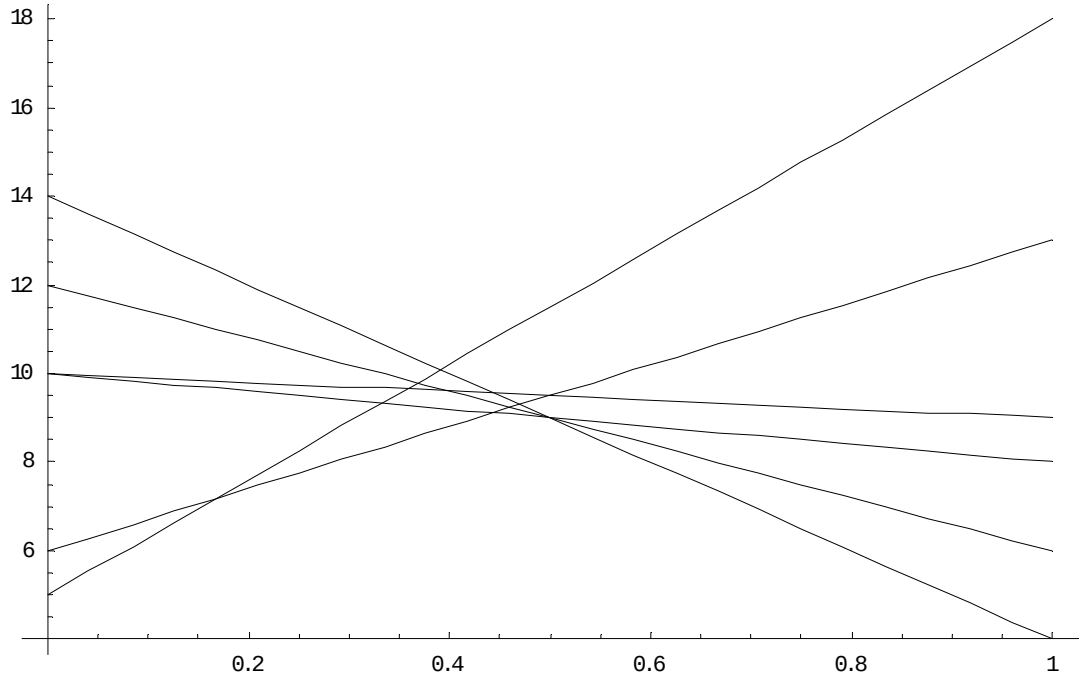
11.5 The allied view.

Suppose the plans H and T are executed with probability x and $1-x$ respectively. Given the x, the allies may analyze the situation from the opponent's point of view. The six available strategies have the following payoff

20_I	20_{II}	20_{III}	10_I10_{II}	10_I10_{III}	$10_{II}10_{III}$
$14-10x$	$10-2x$	$5+13x$	$12-6x$	$10-x$	$6+7x$

The opponent may in the same diagram draw all these six lines on the interval $x \in [0,1]$. Since in this case the opponent is the axis, large numbers are good. From the diagram it easy to see which

plan is most beneficial at any given $x \in [0,1]$. It is also possible to identify the worst-case scenario from the opponent's point of view.



Observe how only the lines $14-10x$ and $5+13x$ are involved as maximum. The switch takes place at $x = 9/23$ when the two are equal. The switch is at the lowest maximum, which corresponds to the worst-case scenario from the point of view of the axis. It must be assumed that the axis is capable of this kind of analysis and hence the allied strategy is to defend from hill H with probability $9/23$, and from hill T with probability $14/23$. The expected number of surviving soldiers is given by

$$5+13\cdot\frac{9}{23}=\frac{115+117}{23}=\frac{232}{23}=10.\overline{0869565217391304347826}\,.$$

If the opponent operates irrationally, the average may be lower than 232/23 as illustrated in

20_I	20_{II}	20_{III}	10_I10_{II}	10_I10_{III}	$10_{II}10_{III}$
$\frac{232}{23}$	$\frac{212}{23}$	$\frac{232}{23}$	$\frac{222}{23}$	$\frac{221}{23}$	$\frac{201}{23}$

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Observe that any mix of plans 20_I and 20_{III} gives the expected value 232/23, and it is impossible for the axis to do better than that.

11.6 The axis view.

Since there are six plans to choose from, five probabilities must be determined in general. In this particular example it is seen that some of the plans *dominate* others. A plan x is said to dominate a plan y, if for each opposing strategy the outcome of x is at least as good as the outcome of y, and there is at least one opposing strategy where x is strictly better. In this case the plan y is

dominated by the plan x. In the payoff matrix

$$\begin{array}{cccccc} 20_I & 20_{II} & 20_{III} & 10_I10_{II} & 10_I10_{III} & 10_{II}10_{III} \\ 14-10x & 10-2x & 5+13x & 12-6x & 10-x & 6+7x \end{array},$$

the plan 10_I10_{III} dominates 20_{II}. This is also readily seen when the graphs are examined. The graphs, in fact, reveal that only 20_I and 20_{III} should be considered. Assume that each night 20_I is picked with probability y and 20_{III} is picked with probability $1-y$. From the opponent's point of view the expected value of H is given by $18-14y$, and the expected value of T by $5+9y$. The opponent this time is the allied side so low numbers are good. When $0 \le y \le \frac{13}{23}$ the lower line is given by $5+9y$. When $\frac{13}{23} \le y \le 1$ the lower line is given by $18-14y$. The worst-case scenario this time is when the largest value is attained. From a diagram it is clear that $y = \frac{13}{23}$ corresponds to the worst case. The value in this case is 232/23. Note that if the side of the axis follows this strategy, then any mix of the H and T plans yields the value 232/23. This is worth stating separately. By using the computed probability distributions each night, both sides are guaranteed to do no worse than 232/23, in average. In this particular example only the allies benefit from irrational behavior of the opponent. In general both sides are expected to benefit from irrational behavior of the opponent. Finally, if the value 232/23 is computed using the diagram of six lines, then it suffices to balance the two strategies involved in the intersection so that this value is attained. In the example this leads to $14y+5(1-y) = 232/23$ when $x = 0$ and $4y+18(1-y) = 232/23$ when $x = 1$. In either case the solution is given by $y = 13/23$.