1. Suppose $a, b, c \in \mathbb{R}$, $a \neq 0$, and $b^2 < 4ac$. Show that the two roots of $az^2 + bz + c = 0$ are complex conjugates of each other. Also show that the conclusion may fail if not all a, b, c are real. What can be said in the general case $a_n z^n + \cdots + a_1 z + a_0 = 0$?
2. Determine all solutions to the equation $z^3 + 8i = 0$ and give them in the form $a + ib$.
3. Which of these functions are analytic where defined: $f(z) = \overline{z}$, $g(z) = z\overline{z}$, $h(z) = \dfrac{\overline{z}}{|z|^2}$?
4. Consider the family of functions $f_{a,b}^c : \mathbb{C} \to \mathbb{C}$ given by $f_{a,b}^c(z) = e^{az} + b \sin z + c \cos z$ where a, b, c are complex numbers. Is the zero function a member of this family, i.e., are there complex numbers a, b, c so that $f_{a,b}^c(z) = 0$ for all $z \in \mathbb{C}$?
5. Calculate $(-i)^{-i}$.
6. Solve $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$.
7. Suppose A is a real number and B is a complex number. Show that $|z|^2 + A^2 = |z + A|^2 - 2\,\mathrm{Re}(Az)$ and $|z|^2 + 2\,\mathrm{Re}(Bz) = |z + \overline{B}|^2 - |B|^2$.
8. Show that $|z| = 1$ if and only if $1 / z = \overline{z}$.
9. Show that $|z + w|^2 - |z - w|^2 = 4\,\mathrm{Re}(z\overline{w})$ for any complex numbers z, w.
10. Show that if $(z + i)^n = (z - i)^n$, then z is real.
11. Prove that $|z| \geq \dfrac{1}{\sqrt{2}}(|\mathrm{Re}\, z| + |\mathrm{Im}\, z|)$.
12. If $z \in \mathbb{C}$ is real, then clearly $z + \dfrac{1}{z}$ is also real. Characterize all other $z \in \mathbb{C}$ so that $z + \dfrac{1}{z}$ is real.
13. Show that $|a_0 z^n + a_1 z^{n-1} + \cdots + a_n| = |\overline{a}_n z^n + \overline{a}_{n-1} z^{n-1} + \cdots + \overline{a}_0|$ whenever $|z| = 1$.
14. Suppose θ_1, θ_2 are real. Show that $|e^{i\theta_1} - e^{i\theta_2}| \leq |\theta_1 - \theta_2|$.
15. Show that $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin[(n + \frac{1}{2})\theta]}{2\sin(\theta / 2)}$ for $0 < \theta < 2\pi$.
16. Suppose $\mathrm{Re}\, f(x + iy) = y^2 - x^2 + 2y^3 - 6x^2 y + 7$. What is $f(z)$?
17. Let $f(z) = (\overline{z})^2 / z$ when $z \neq 0$ and $f(0) = 0$. Show that the Cauchy-Riemann equations are satisfied at $z = 0$ but the derivative fails to exist there.

18. Consider matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$. Discuss how this collection of matrices and the matrix product relate to $\mathbb{C}$. In particular, determine which matrix corresponds to the imaginary unit and describe the effect on vectors in $\mathbb{R}^2$ when this linear map is applied.
19. Solve $\cos z = 2i$.
20. Assume $f(z)$ is analytic and $\operatorname{Re} f(z) = \operatorname{Re} f(x + iy) = x^2 - y^2 + xy$. Determine all possible f.
21. Determine all analytic functions $f(z) = u(x,y) + iv(x,y)$, $z \neq 0$ such that $u(x,y) = \dfrac{y}{x^2 + y^2}$.
22. Determine all analytic functions $f(z) = u(x,y) + iv(x,y)$ such that $u(x,y) = e^{x^2 - y^2} \cos(2xy)$.
23. Determine all analytic functions $f(z) = u(x,y) + iv(x,y)$ such that $u(x,y) = \cos(x^2 - y^2)\cosh(2xy) + y$.
24. Suppose $v(x,y) = y + e^x \sin y$.
    (a) Is v harmonic?
    (b) Is there a function $f(z)$ such that $f(0) = 1$ and $\operatorname{Im}(f(z)) = \operatorname{Im}(f(x + iy)) = v(x,y)$?
    (c) If so, what is $f(z)$?
25. Let $u, v \in \mathbb{C}$ be two distinct nonzero complex numbers. When regarded as arrows extending from the origin they generate a triangle with one corner at the origin. Show that the area of this triangle is given by $\left|\operatorname{Im} u\bar{v}\right| / 2$.
26. Show that $\mathbb{C}$ has no subset P of 'positive complex numbers' such that P is closed under addition and multiplication and if $z \in \mathbb{C}$ then exactly one of $z \in P$, $z = 0$ or $-z \in P$ is true. In other words, $\mathbb{C}$ is cannot be ordered.

--- No later than Jan 31 ---

27. Show that if $D \subset \mathbb{C}$ and $f : D \to \mathbb{C}$ is analytic with $\left|f\right|$ constant on D, then f is constant on D.
28. Suppose $\operatorname{Re} f \geq 0$ of an analytic $f : \mathbb{C} \to \mathbb{C}$. Show that f is constant.
29. Determine the form of all entire functions f, g that satisfy $f^2 + g^2 = 1$.
30. Suppose $\sum_{k=0}^{\infty} \left|a_k\right|$ is finite and $\sum_{k=0}^{\infty} k\left|a_k\right|$ is infinite for some complex numbers $a_0, a_1, \cdots$. Determine the only possible value for the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k z^k$.
31. Using the Taylor series or otherwise, expand $f(z) = 1 / z$ about the point $z = 1$ and indicate the region of convergence.
32. Expand the function $\dfrac{1}{z^2 + 1}$ in a Laurent series valid in $|z| > 1$.

33. Use algebra to determine the Laurent series in powers of z when $f(z) = \dfrac{1+2z}{z^2+z^3}$ and indicate the region of convergence.

34. Let $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$. Determine the power series or Laurent series in powers of z as appropriate in the three domains (a) $|z| < 1$, (b) $1 < |z| < 2$, and (c) $2 < |z|$.

35. Give two Laurent series expansions in powers of z when $f(z) = \dfrac{1}{z^2(1-z)}$ and specify the regions in which those expansions are valid.

36. Calculate the residues at the singularities of $f(z) = \dfrac{5z-2}{z(z-1)}$.

37. Classify the singular point as removable, pole, or singular when $f(z)$ is:

(a) $\dfrac{1}{(2-z)^3}$, (b) $\dfrac{\cos z}{z}$, (c) $\dfrac{\sin z}{z}$, (d) $ze^{1/z}$.

38. Determine the order of the pole and the residue:

(a) $\dfrac{1-\cosh z}{z^3}$, (b) $\dfrac{1-e^{2z}}{z^4}$, (c) $\dfrac{e^{2z}}{(z-1)^2}$.

39. Let $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ and determine a Laurent series that represents f throughout $1 < |z| < 2$.

40. Expand the function $\dfrac{7z-3}{z(z-1)}$ in a Laurent series in $0 < |z-1| < 1$.

41. Expand the function $(z^2-2z+1)e^{\frac{z+1}{z-1}}$ in a Laurent series about $z_0 = 1$.

42. Evaluate $\displaystyle\int_C 1$ when C is the curve given by $y = x^2$ and the endpoints are $(0,0)$ and $(1,1)$.

Express your answer by giving a, b in $\dfrac{1}{4}(2\sqrt{a} + \sinh^{-1}(b))$.

43. Evaluate $\displaystyle\int_C 1\,dz$ when C is the curve given by $y = x^2$ and the endpoints are $(0,0)$ and $(1,1)$.

Express your answer by giving a, b in $a + ib$.

44. Evaluate $\displaystyle\int_C x$ when C is the curve given by $y = x^2$ and the endpoints are $(0,0)$ and $(1,1)$.

Express your answer by giving a, b in $\dfrac{1}{12}(5\sqrt{a} - b)$.

45. Suppose $f(z) = e^{i\arg(z)}$. Evaluate $\displaystyle\int_C |f(z)||dz|$ when C is the curve given by $y = x^2$ and the endpoints are $(0,0)$ and $(1,1)$.

46. Let $f(z) = 1 / z$. With $f(z) = f(x + iy) = u(x,y) + iv(x,y)$, what is u and v? Express $\int_C f(z)\,dz$ in the form $\int_C (P\,dx + Q\,dy) + i\int_C (R\,dx + S\,dy)$. The first integral corresponds to the work by $\overline{f}$ along C. Determine the potential in this case. What does the existence of this potential tell you about the value of $\int_C f(z)\,dz$ when C is a closed curve? The second integral corresponds to the flux of $\overline{f}$ through C. Is there a 'potential' in this case as well? Determine a function Φ such that $\Phi_x = v$ and $\Phi_y = u$. What is the problem? Why is it not the case that $\int_C f(z)\,dz = 0$ although the real and the imaginary integral parts both seem to evaluate to zero?

47. Let C_r be the semicircle from $-ir$ to ir with positive real parts. Estimate the magnitude of $\int_{C_r} \frac{z^2 - 5}{z^4 + 7z^3 + 6}\,dz$ for large r.

48. Let C be the unit circle and consider $\int_C f(z)\,dz$. For which f is Cauchy's formula applicable so that the integral's value must be zero?

(a) $f(z) = \frac{z^2 + 1}{e^{iz} - 1}$, (b) $f(z) = \frac{z^3}{z^2 + iz + 3}$, (c) $f(z) = \sin(\frac{1}{z - 1})$, (d) $f(z) = \tan z$

49. Calculate $\int_C \overline{z}\,dz$. (a) C is the straight line segment from 1 to i, (b) C is the quarter of the unit circle from 1 to i.

50. Calculate $\int_C \frac{1}{z^2 + 4}\,dz$ when C is the straight line segment from i to 2.

--- No later than Feb 28 ---

51. Let $C = \{z \in \mathbb{C} : |\, z \,| = 2\}$ in the positive sense and calculate $\oint_C \frac{z}{(9 - z^2)(z + i)}\,dz$.

52. Let $C = \{z \in \mathbb{C} : |\, z - i \,| = 2\}$ in the positive sense and calculate $\oint_C \frac{1}{z^2 + 4}\,dz$.

53. Let $C = \{z : |z| = 1\}$. Calculate $\oint_C e^{2/z} dz$.

54. Let $C = \{z : |z| = 3\}$. Calculate $\oint_C \frac{1}{z^4 + z^3 - 2z^2}\,dz$.

55. Let $C = \{z : |z - 1| = 2\}$. Calculate $\oint_C \frac{1}{z^4 + 4}\,dz$.

56. Let $C = \{z : |z| = 1\}$. Calculate $\oint_C \frac{1}{4z^2 + 1}\,dz$.

57. Let $C = \{z : |z - i| = 1\}$. Calculate $\oint_C \frac{1}{4z^2 + 1} dz$.

58. Let $C = \{z : |z| = 1\}$. Calculate $\oint_C \cot z \, dz$.

59. Let $C = \{z : |z| = 1\}$. Calculate $\oint_C \frac{1}{(e^z - 1)^2} dz$.

60. Let $C = \{z : |z| = 2\}$. Calculate $\oint_C \frac{1}{z^2 + 2z - 2i + 1} dz$.

61. Let $C = \{z : |z| = 1\}$. Calculate $\oint_C \frac{1}{\sin(1 / z)} dz$.

62. Let $C = \{z : |z| = \frac{1}{2}\}$. Calculate $\oint_C \frac{1}{\sin(1 / z) - 1} dz$.

63. Show that $\int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx = \frac{\pi}{4}$.

64. Show that $\int_0^\infty \frac{1}{(x^2 + 1)^2} dx = \frac{\pi}{4}$.

65. Show that $\int_0^\infty \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}$.

66. Show that $\int_{-\infty}^\infty \frac{\cos x}{(x^2 + 1)^2} dx = \frac{\pi}{e}$.

67. Show that $\int_0^\infty \frac{\cos ax}{x^2 + 1} dx = \frac{\pi}{2} e^{-a}$.

68. Show that $\int_0^{2\pi} \frac{1}{5 + 4 \sin \theta} d\theta = \frac{2\pi}{3}$.

69. Show that $\int_{-\pi}^{\pi} \frac{1}{1 + \sin^2 \theta} d\theta = \pi\sqrt{2}$.

70. Consider the function $f(z) = (z - 2)(z - 4)$ and the integral $\oint_C \frac{dz}{f(z)}$. What are the values of this integral in the three cases $C = \{z : |z| = 1\}$, $C = \{z : |z| = 3\}$, $C = \{z : |z| = 5\}$?

71. Use complex techniques to determine the value of $\int_{-\infty}^\infty \frac{6}{4 + 5x^2 + x^4} dx$.

(Hint: Let $Z = z^2$ when searching for singularities.)

72. Consider the integral $\int_C \frac{dz}{z}$ where C is three-quarter of the circle $|z| = 2$ going in the positive direction from $2i$ to 2.

(a) Calculate the integral using a parameterization of C.

(b) Calculate the integral using an anti-derivative and the complex counterpart to the fundamental theorem of calculus.

(c) What is the proper way to deal with the presence of branch cuts in this context?

73. Consider e^{it} when $0 \le t \le 2\pi$. Is there some x such that $\int_0^{2\pi} e^{it} dt = 2\pi e^{ix}$ and $0 < x < 2\pi$? What does this say about the mean-value theorem?

74. Suppose C is closed and does not pass through z_0. Let n be a nonzero integer. Show that $\int_C (z - z_0)^{n-1} dz = 0$.

75. Suppose f is analytic within and on a simple closed C and assume that z_0 is not on C. Show that $\int_C \frac{f'(z)}{z - z_0} dz = \int_C \frac{f(z)}{(z - z_0)^2} dz$.

76. Let C be the unit circle. Show that $\int_C \frac{e^{az}}{z} dz = 2\pi i$ for any real a. What is the value of $\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta)\, d\theta$?

--- No later than Mar 31 ---

77. Calculate the integral $\int_{|z|=1} \frac{e^{2z}}{z^2} dz$.

78. Calculate the integral $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx$.

79. Calculate the integral $\int_0^{\infty} \frac{x \sin x}{(x^2 + 1)(x^2 + 4)}$.

80. Calculate the integral $\int_{|z-2|=2} \frac{\cos z}{(z^2 - 4)^2} dz$.

81. Suppose f continuous in a closed and bounded domain and analytic in the interior. Assuming $f(z) \neq 0$, show that the minimum of $|f(z)|$ cannot be in the interior.

82. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and $|f(z)| \le A|z|$ for some positive real number A. Show that $f(z) = az$ for some complex a. More generally, if $|f(z)| \le A|z|^n$ for some integer $n \ge 1$ and all $|z| > R$, then f is a polynomial of degree n or less.

83. Determine the number of solutions of $8z^3 + 12z^2 + 1 = 0$ inside $|z| = 1/2$.

84. Determine the number of solutions of $z^{100} + 3z^{99} + 1 = 0$ outside $|z| = 1$.

The latest version of Mathematica suggests that when its Solve command is combined with its conversion-to-numerical-values command N that -3. is a solution. This is clearly nonsense. Plotting does not help much either, but if you have the opportunity give it a try. It is reassuring in cases such as this one that there are mathematical results that will answer certain questions with certainty where it seems like no other methods are available.

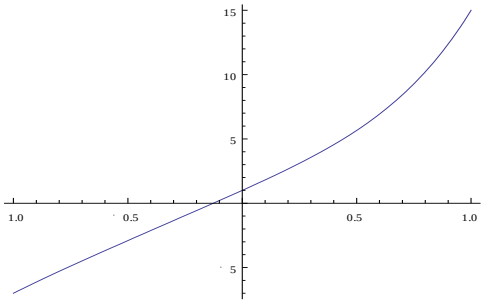
85. The function $f(z) = (z-1)(z+1)(z+2) = z^3 + 2z^2 - z - 2$ has a unique zero in the right half-plane and two distinct zeros in the left half-plane. Use the principle of the argument and show all the details that determine the number of zeros in both half planes. Work through both cases from scratch and do not appeal to the fundamental theorem of algebra to get one of the counts from the other. Make sure you understand the geometric difference in the two cases.

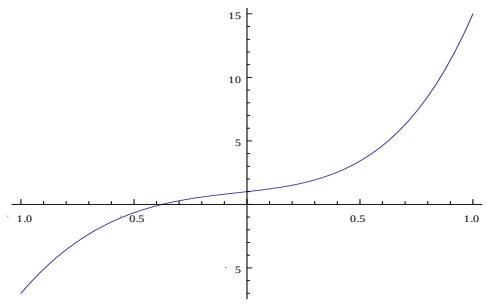
86. The level curves of $x^2 - y^2$ and xy intersect at right angles. Use regular calculus to establish this fact. Now consider $f(z) = z^3$. What is the analogous claim in this case and how do you establish this without using regular calculus but instead using complex techniques?

87. Use the principle of the argument to decide if $f(z) = z^3 - iz^2 - (3+2i)z - 2 - i$ has any zeros in the half-plane $\mathrm{Re}(z) > 0$.

88. The equation $z^5 + 2z^4 + 2z^3 + z^2 + 8z + 1 = 0$ appears in an application where the concern is the existence of complex solutions z such that $|z| \le 1$ but with positive real part.

The first plot shows one solution with negative real part.





Since the coefficients in the equation are real there is still the possibility that there is one pair, or maybe even two pairs, of conjugate roots inside the unit circle.

(a) Use Rouché's theorem and decide if the concern is legitimate.

(b) As it turns out the equation had a typo and should be $z^5 + 2z^4 + 8z^3 + z^2 + 2z + 1 = 0$. Does this change the conclusion? The second plot is the corrected plot.

89. Determine the number of solutions of $z^3 - z^2 + 3z + 5 = 0$ in the right half-plane.

90. Show that $z^4 + 6z + 3 = 0$ has no solutions of magnitude 2 or more and exactly one solution of magnitude 1 or less.

91. Show $3ze^z = 1$ has exactly one solution of magnitude less than 1.

92. Show that the function $\sin^2 z + 4z + 1$ has a zero inside $|z| = 1$.

93. Show that at least one zero of $z^3 + az^2 + bz + b$ has magnitude 3 or less.

94. Determine the number of zeros of $f(z) = 4z^4 + 2(1 - i)z + 1$ in $1/2 < |z| < 1$.

95. Determine the maximum of $|\cos 2z|$ in $\{z = x + iy : 0 \le x \le \pi/2, 0 \le y \le 2\}$.

96. Find a Möbius transformation $w = f(z)$ that sends $z_1 = -2, z_2 = 2i$ and $z_3 = 0$ to $w_1 = 0, w_2 = 1$ and $w_3 = -i$. Also show that $\{z : |z + 1 - i| < \sqrt{2}\}$ is mapped to $\{w : |w - (1 - i)/2| < 1/\sqrt{2}\}$.

97. Find a Möbius transformation $w = f(z)$ that maps $|z| \le 1$ to $\mathrm{Re}(z) \ge 0$ such that $f(0) = 1$ and $f(-1) = 0$.

98. Let $C_1 = \{z \in \mathbb{C} : |z| = 1\}$ and $C_2 = \{z \in \mathbb{C} : |z - 1| = 2\}$. Evaluate the two integrals $\oint_{C_1} \frac{24}{z^2(z^3 - 8)} dz$ and $\oint_{C_2} \frac{24}{z^2(z^3 + 8)} dz$ where both C_1 and C_2 are traversed in the positive direction.

99. Calculate the value of $\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 4n + 5}$.

100. What about sums like $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 5}$, $\sum_{n=0}^{\infty} \frac{1}{(n + 2)^2 + 1}$, $\sum_{n=0}^{\infty} \frac{1}{n^2 + a^2}$, or what to do when a tends to zero?