EXAM 2

1. *(15 points)*

(a) Describe the largest possible domain for the function $f(x, y) = \sqrt{x\,y}$.

It is necessary to have $x\,y \geq 0$, so the axes $y = 0$ or $x = 0$ are allowed.

It is also fine with x, y both positive as well as x, y both negative.

(b) Calculate the first partial derivative $g_x(x, y) = \frac{\partial g}{\partial x}(x, y)$ when $g(x, y) = x^2\,y \sin(3\,x + 5\,y)$.

```
3 x^2 y Cos[3 x + 5 y] + 2 x y Sin[3 x + 5 y]
```

(c) Calculate $h_{xy}(x, y) = \frac{\partial^2 h}{\partial x\,\partial y}(x, y)$ when $h(x, y) = y^2 + x^2\,y + x\,y^3$.

```
2 x + 3 y^2
```

2. *(15 points)*

Find the local maximum and minimum of the function $f(x, y) = x^4 + y^4 - 4\,x\,y + 1$.

The gradient is $< 4\,x^3 - 4\,y,\ -4\,x + 4\,y^3 >$.

$x^3 = y$
$y^3 = x$ implies $x^9 = x$ so $x = 0$ or $x^8 = 1$ and hence $x = \pm 1$.

It follows that $y = x$ and there are three critical points: $(0, 0)$, $\pm(1, 1)$.

$f_{x\,x} = 12\,x^2$, $f_{y\,y} = 12\,y^2$, $f_{x\,y} = -4$

so $D_{(0,0)} = -16$ saddle, $D_{\pm(1,1)} > 0$ so both are local min.

Argue carefully why your claim is correct.

3. *(15 points)*

Use Lagrange multipliers to find the maximum value of $f(x, y) = x + 4\,y$ subject to the constraint $g(x, y) = x^2 + 2\,y^2 = 36$.

Continuous function and the ellipse is closed and bounded so there is a maximum and a minimum.

```
Grad[x + 4 y, {x, y}]
{1, 4}
Grad[x^2 + 2 y^2, {x, y}]
{2 x, 4 y}
Solve[{1, 4, x^2 + 2 y^2} == {λ 2 x, λ 4 y, 36}, {x, y, λ}]
{{x → -2, y → -4, λ → -1/4}, {x → 2, y → 4, λ → 1/4}}
```

4. *(20 points)*

A particle is moving along the space curve $\vec{r}(t) = < 2\cos(t),\ 2\sin(t),\ t >$.

(a) How far is it from $\vec{r}(0)$ to $\vec{r}(\pi)$ using the straight-line shortcut?

$\| \vec{r}(\pi) - \vec{r}(0) \| = \sqrt{16 + \pi^2}$

(b) Calculate the speed $v(t)$ of the particle.

$v(t) = \| \vec{r}\,'(t) \| = \sqrt{5}$

(c) Determine the length along the path from $\vec{r}(0)$ to $\vec{r}(\pi)$.

$\int_0^{\pi} \| \vec{r}\,'(t) \| \, dt = \pi \sqrt{5}$

(d) Determine the acceleration of the particle and use

$\vec{r}\,''(t) = v'(t)\, T(t) + \kappa(t)\, v^2(t)\, N(t)$

to determine the curvature $\kappa(t)$.

Clearly $v'(t) = 0$ since the speed is constant. $\vec{r}\,''(t) = -2 < \cos(t), \sin(t), 0 >$ so $\| \vec{r}\,''(t) \| = 2$.

It follows that $\kappa(t) = \frac{\| \vec{r}\,''(t) \|}{v^2(t)} = \frac{2}{5}$ since $\| N(t) \| = 1$.

5. *(15 points)* Determine the limits $\lim_{(x,y)\to(0,0)} f(x, y)$.

(a) $f(x, y) = \frac{\sqrt{x^2+y^2+4}}{x\,y\,+1}$

The functions is continuous and defined at (0, 0) so plug in and get 2.

(b) $f(x, y) = \frac{x^2\,y^2}{x^4+y^4}$

Observe that $f(x, 0) = 0$ and $f(x, x) = \frac{1}{2}$ so there can be no limit at (0, 0).

(c) $f(x, y) = \frac{x^4-y^4}{x^2+y^2}$

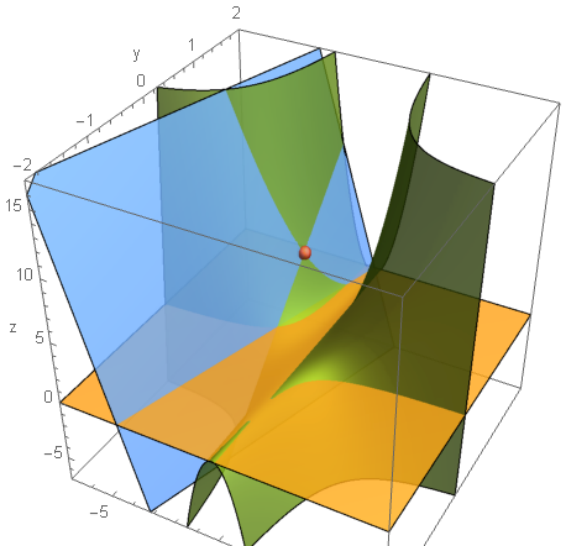
Away from the origin, $\frac{x^4-y^4}{x^2+y^2} = \frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2} = x^2 - y^2$, which is continuous and defined, so plug in and get 0.

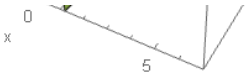
6. *(20 points)* Determine the equation of the tangent plane.

(a) To the surface given by $f(x, y) = x + x\,y + 2\,x^2\,y$ at the point corresponding to $(x, y) = (-2, 1)$.

Let $F(x, y, z) = z - x - x\,y - 2\,x^2\,y = 0$ with gradient $< -1 - y - 4\,x\,y, -x - 2\,x^2, 1 >$, which is $< 6, -6, 1 >$ at the point.

$6\,x - 6\,y + z = D$ is the equation, plug in $(-2, 1, 4)$ to get $D = -14$.



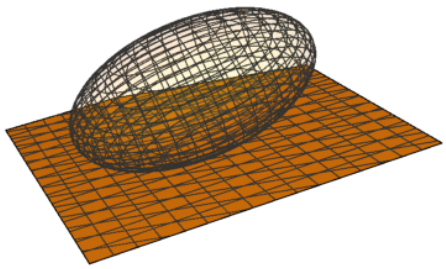


(b) To the level set given by $x^2 + 3\,y^2 + 5\,z^2 = 9$ at the point $(x,\ y,\ z) = (1,\ -1,\ 1)$.

The gradient is $< 2\,x,\ 6\,y,\ 10\,z >$ which becomes $< 2,\ -6,\ 10 >$ at the point $(1,\ -1,\ 1)$.

The equation is $2\,x - 6\,y + 10\,z = D$, plug in $(1,\ -1,\ 1)$ to get $D = 18$.

Answer: $x - 3\,y + 5\,z = 9$



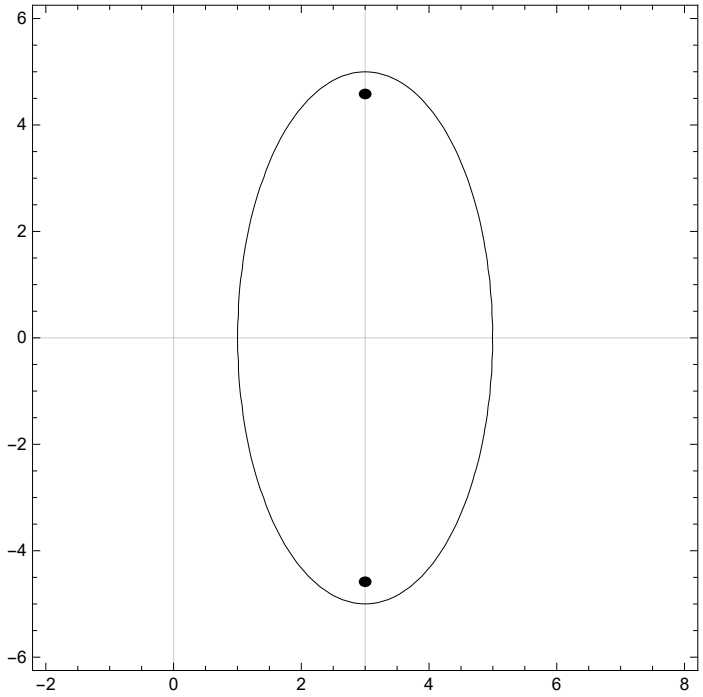
EXAM 1

1. (*15 points*) Determine the vertices and the foci of $\frac{(x-3)^2}{4} + \frac{y^2}{25} = 1$.

The ellipse has been shifted 3 units in the positive x direction.

The major axis is vertical. Vertices: $(3,\pm 5)$; Foci: $\left(3,\ \pm\sqrt{25-4}\right) = \left(3,\ \pm\sqrt{21}\right)$.

```
ContourPlot[(x - 3)^2/4 + y^2/25, {x, -2, 8}, {y, -6, 6},
 Contours → {1}, ContourShading → None, GridLines → {{0, 3}, {0}},
 Epilog → {Disk[{3, √21}, 1/10], Disk[{3, -√21}, 1/10]}]
```



2. (*15 points*) Consider $x(t) = 4 + 3\,t^2$, $y(t) = 5 + 2\,t^3$ with $0 \le t \le 1$.

(a) What are the endpoints of the curve?

$x(0) = 4$, $y(0) = 5$; $x(1) = 7$, $y(1) = 7$ so (4,5) and (7,7).

(b) What is the length of the path between the two endpoints?

$\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ where $x'(t) = 6\,t$ and $y'(t) = 6\,t^2$. The integral becomes $6\int_0^1 t\sqrt{1 + t^2}\, dt$ so substitute $u = 1 + t^2$.

```
Integrate[√(D[4 + 3 t^2, t]^2 + D[5 + 2 t^3, t]^2), {t, 0, 1}]
```

$-2 + 4\sqrt{2}$

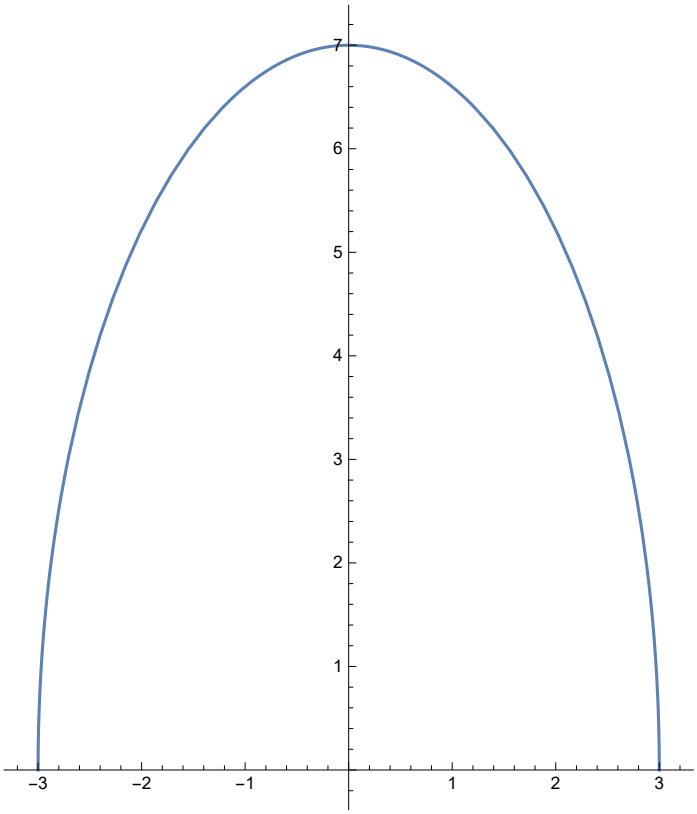
3. (*15 points*) Calculate the area of the region enclosed by $x(t) = 3\cos(t)$, $y(t) = 7\sin(t)$.

This is a standard ellipse so work on the upper half.

Use $\int y\,dx$ with $y \geq 0$. Replace dx with $x'(t)\,dt = -3\sin(t)\,dt$.

Integrate from π to 0 and multiply by 2 to get the whole area 21π.

```
ParametricPlot[{3 Cos[t], 7 Sin[t]}, {t, 0, π}]
```




```
Integrate[7 Sin[t] D[3 Cos[t], t], {t, π, 0}]
```

$$\frac{21\,\pi}{2}$$

4. *(15 points)*

(a) Consider the vectors $< x, 4, 1 >$ and $< 1, 1, 1 >$. Is it possible for these vectors to be perpendicular? If so, for which x?

The dot product to the rescue: $x + 4 + 1 = 0$ so $x = -5$.

(b) Consider the vectors $< x, 1, 0 >$ and $< 1, 1, 1 >$. Is it possible for these vectors to have angle $\frac{\pi}{4}$ between them? If so, for which x?

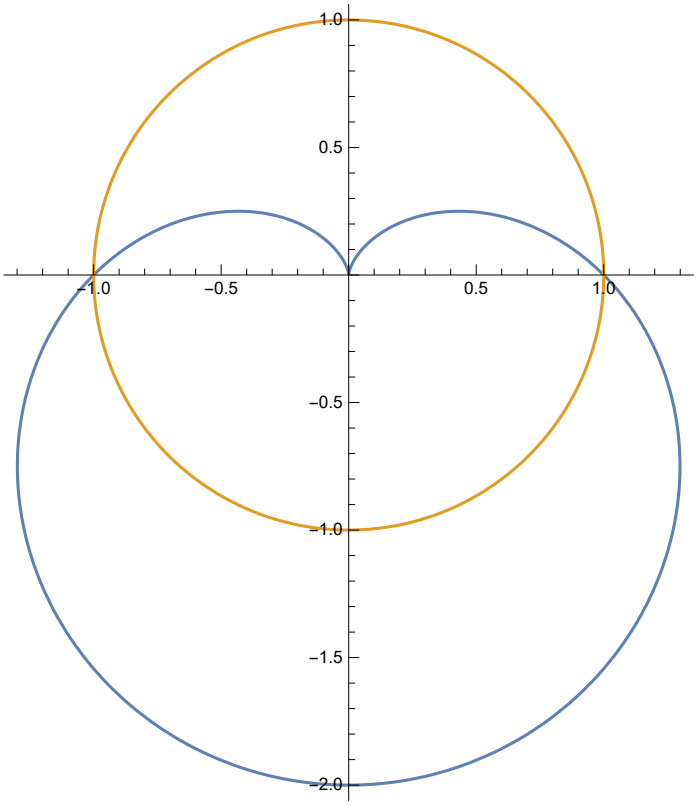
This time $x+1 = \sqrt{x^2+1}\ \sqrt{3}\ \cos(\pi/4)$ so $2\,(x+1)^2 = 3\left(x^2+1\right)$ which rewrites as $x^2 - 4\,x + 1 = 0$, a basic quadratic equation, with solutions $x = 2 \pm \sqrt{3}$

```
Solve[x^2 - 4 x + 1 == 0, x]
```

$\{\{x \to 2 - \sqrt{3}\}, \{x \to 2 + \sqrt{3}\}\}$

5. *(15 points)* Determine the area of the region outside $r = 1 - \sin(\theta)$ but inside $r = 1$ as given in polar coordinates. Include a plot!

Integrate $\frac{r^2}{2} = \frac{(1-\sin(\theta))^2}{2}$ from 0 to π and subtract from π. Use double-angle formula. Should be familiar from a quiz!




```
π/2 - Integrate[(1 - Sin[θ])^2/2, {θ, 0, π}]
```

$2 - \frac{\pi}{4}$

```
1/2 Integrate[1^2 - (1 - Sin[θ])^2, {θ, 0, π}] // Simplify
```

$2 - \frac{\pi}{4}$

6. (*15 points*) What is the distance from the plane $x + 2\,y + 3\,z = 42$ to the point (1, 2, 3)?

(42,0,0) is in the plane! Now form a vector by taking the difference bewtween it and the given point. Remember to turn $< 1, 2, 3 >$ into a unit vector and also possibly take the absolute value.

```
Dot[{42 - 1, 0 - 2, 0 - 3}, {1, 2, 3}] / Norm[{1, 2, 3}]
```

$2\sqrt{14}$

7. (*10 points*) Match the following six images to the equations:

(a) $y^2 = x^2 + 2\,z^2$; (b) $x^2 - y^2 + z^2 = 1$; (c) $y = x^2 - z^2$;

(d) $x^2 + 4\,y^2 + 9\,z^2 = 1$; (e) $-x^2 + y^2 - z^2 = 1$; (f) $y = 2\,x^2 + z^2$

Here is the match!

(d) $x^2 + 4\,y^2 + 9\,z^2 = 1$ (Ellipsoid); (b) $x^2 - y^2 + z^2 = 1$ (Hyperboloid of One Sheet); (e) $-x^2 + y^2 - z^2 = 1$ (Hyperboloid of Two Sheets)

(f) $y = 2\,x^2 + z^2$ (Elliptic Paraboloid); (a) $y^2 = x^2 + 2\,z^2$ (Cone); (c) $y = x^2 - z^2$ (Hyperbolic Paraboloid)

