1. It is possible to view the complex numbers as an extension of the real numbers by a solution of the equation $x^2 = -1$.
2. This fits the pattern seen as the equations $2x = 1$ and $x^2 = 2$ extend integers to rational numbers and rational numbers to real numbers, respectively.
3. The symbol i is typically used to indicate the selected solution so that $i^2 = -1$.
4. As a set one writes $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ where the addition and the multiplication only has notational significance.
5. When $y = 0$ the complex number is said to be *real* and is written x.
6. When $x = 0$ the complex number is said to be *imaginary* and is written iy.
7. When both $x = 0$ and $y = 0$ the complex number is simply *zero* and it is written 0.
8. By preserving the usual algebraic laws, one *defines* $(a + ib) + (c + id) = (a + c) + i(b + d)$.
9. Observe that $+$ here appears in three different ways.
10. Similarly, $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$.
11. As seen, $i^2 = -1$ is used without mentioning.
12. One writes $\mathrm{Re}(a + ib) = a$ for the real part, and $\mathrm{Im}(a + ib) = b$ for the imaginary part.
13. These are functions $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \to \mathbb{R}$.
14. Complex conjugation is given by $\overline{a + ib} = a - ib$, which is a function from $\mathbb{C}$ to $\mathbb{C}$.
15. Conjugation splits over both addition and multiplication so that $\overline{z + w} = \overline{z} + \overline{w}$, and $\overline{zw} = \overline{z}\,\overline{w}$.
16. Observe that with $z = a + ib$ one has $z\overline{z} = a^2 + b^2 \geq 0$.
17. The magnitude is given by $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$.
18. The magnitude splits over multiplication so that $|zw| = |z||w|$.
19. The magnitude does not split over addition but $|z + w| \leq |z| + |w|$ holds.
20. Observe that $a(b + ic) = ab + iac$ so multiplication by a real number corresponds to scalar multiplication.
21. To complete the algebraic picture it is necessary to deal with $z^{-1} = 1 / z$.
22. The requirement is of course that $z^{-1}z = 1$, which leads to $(x + iy)(a + ib) = 1$.
23. This complex equation corresponds to two linear real equations: $ax - by = 1, bx + ay = 0$.
24. If $z = 0$, there is no solution.
25. $abx - b^2y = b, abx + a^2y = 0$ implies $y = -\dfrac{b}{a^2 + b^2}$ and $x = \dfrac{a}{a^2 + b^2}$.
26. Just as for the real numbers, zero is the only number with no multiplicative inverse.
27. Complex notation yields $z^{-1} = \dfrac{\overline{z}}{z\overline{z}} = \dfrac{1}{z}$ where the real fractional notation has been promoted.

28. The scheme $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}}$ is worth remembering for more general computations.

29. $\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{1}{c^2 + d^2}((ac + bd) + i(bc - ad))$.

30. The ordering of real numbers has no counterpart in the complex numbers, although the magnitudes of different complex numbers may be compared.

31. Square roots are characterized by $z^2 = w$ or $(x + iy)^2 = a + ib$.

32. The corresponding real equations are $x^2 - y^2 = a, 2xy = b$.

33. When $w = 0$ the only possibility is $z = 0$, which mimics the case of real numbers.

34. When w is real so $b = 0$, then either $y = 0$ when $a > 0$, or $x = 0$ when $a < 0$.

35. In the first case $x = \pm\sqrt{a}$, and in the second case $y = \pm\sqrt{-a}$.

36. When w is purely imaginary so that $a = 0$, then either $x = y$ when $b > 0$, or $x = -y$ when $b < 0$.

37. In the first case $x = y = \pm\sqrt{b / 2}$, and in the second case $x = -y = \pm\sqrt{-b / 2}$.

38. Suppose $w = i$, then $z = \pm\frac{1}{\sqrt{2}}(1 + i)$.

39. Observe that $(x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2 = \left|z\right|^4 = \left|w\right|^2$ so $\left|z\right|^2 = \left|w\right|$.

40. Write $w = \left|w\right|(\frac{a}{\left|w\right|} + i\frac{b}{\left|w\right|})$ and recognize that $\frac{a}{\left|w\right|} = \cos\varphi, \frac{b}{\left|w\right|} = \sin\varphi$ for some $\varphi \in \mathbb{R}$ referred to as $\arg w$.

41. The *polar form* is given by $w = \rho(\cos\varphi + i\sin\varphi)$ where $\rho = \left|w\right|$.

42. With $r = \left|z\right|$ and $\theta = \arg z$ it is seen that $r^2 = \rho$ and $(\cos\theta + i\sin\theta)^2 = \cos\varphi + i\sin\varphi$.

43. $(\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta) = \cos 2\theta + i\sin 2\theta$, which suggests $\theta = \varphi / 2$.

44. The key is to realize that $2\theta = \varphi + 2\pi$ also works so there is also the alternative $\theta = \varphi / 2 + \pi$.

45. To get explicit formulas, replace y by $b / (2x)$ so that $x^2 - \frac{b^2}{4x^2} = a$ or $4x^4 - 4ax^2 - b^2 = 0$.

46. The quadratic formula yields $x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$, and consequently $y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}$.

47. Finally, $z = \pm(\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + i\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}})$ if $b > 0$ and $z = \pm(\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} - i\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}})$ if $b < 0$.

48. By regarding each complex number z as a vector in the Euclidean plane it is seen that $|z|$ corresponds to the length of the vector, and $\arg z = \theta + n \cdot 2\pi$ for some $n \in \mathbb{Z}$ where θ is the angle positive angle between the 'horizontal' axis and the vector.
49. In this so-called complex plane, the horizontal axis is known as the real axis and the vertical axis as the imaginary axis.
50. For nonzero numbers, and unlike the real numbers, there are always two numbers that 'act like' the square root, and both belong to the same circle at diametrically opposite points.
51. For zero the circle has degenerated down to a point.
52. For general n'th roots the picture is analogous in that they all are on the same circle and the are evenly distributed.
53. Consider the equation $z^3 = i$. This time $\theta = \pi/6, 5\pi/6, 3\pi/2$. The last one is in fact $-i$ and $(-i)^3 = (-i)(-i)(-i) = i$ as required.
54. In the complex plane the operation Re corresponds to the projection onto the real axis, and similarly Im is the projection onto the imaginary axis.
55. Complex conjugation is a reflection in the real axis.
56. The solutions of $|z - c| = r > 0$ belong to the circle with center c and radius r.
57. Solutions of $\mathrm{Im}\, z = a \in \mathbb{R}$ belong the horizontal line at 'height' a.
58. Solutions of $\mathrm{Re}\, z = a \in \mathbb{R}$ belong to the vertical line at distance a from the imaginary axis.
59. With $z = x + iy$ observe that $x = \dfrac{z + \overline{z}}{2}$ and $y = \dfrac{z - \overline{z}}{2i}$.
60. The general equation of a straight line $ax + by = c$ may be written as $a\dfrac{z + \overline{z}}{2} + b\dfrac{z - \overline{z}}{2i} = c$, or $ez + if\overline{z} = g$ for some real numbers e, f, g.
61. For $r > 0$ and $c \in \mathbb{C}$, the solutions of $|z - c| < r$ correspond to the open disk with center c and radius r.
62. $\mathrm{Re}\, z > 0$ is the open right-hand half-plane, and $\mathrm{Im}(z) > 0$ is the open upper half-plane.
63. A set is *bounded* if $|z| < r$ for some $r > 0$ and all z in the set.
64. A set is *open* if at each z there is an open disk centered at z that is contained in the set.
65. This terminology is consistent because if $|z - c| < r$, then $|w - c| \leq |w - z| + |z - c| < r$ as long as one requires $|w - z| < r - |z - c|$.
66. Complements of open sets are said to be *closed*.
67. Many sets are neither open nor closed, $\mathbb{C}$ as well as the empty set are both open and closed.
68. A set that is both closed and bounded is said to be *compact*.

69. A set is said to be *connected* if each pair of points in the set can be joined by a polygonal that never leaves the set.
70. The set $|z| < 1$ is connected and so is its complement.
71. The complement of $|\mathrm{Re}\, z| < 1$ is not connected.
72. A nonempty connected open set is referred to as a *region*.
73. Domains under consideration in complex analysis are frequently regions so if no assumptions are given assume the domain is a region.
74. Observe that a region need not be bounded.
75. A sequence of complex numbers z_k converges to z if for each $r > 0$, $|z_k - z| < r$ holds for all but a finite number of z_k.
76. If z_k belongs to a closed set and z_k converges to z, then z belongs to the set as well.
77. It is always possible to extract a convergent so-called subsequence from a bounded sequence.
78. Observe that a sequence can only converge to a one point, but a bounded sequence that does not converge may have several subsequences that converge to different values.
79. Consider a complex-valued function f and assume $w_k = f(z_k)$ converges to $w = f(z)$ anytime z_k converges to z, then f is said to be *continuous*.
80. The magnitude, the complex conjugate, Re, and Im are all continuous.
81. If it is assumed that $\arg z \in [0, 2\pi)$, then $\arg z$ is not continuous as seen by the sequence $z_k = 1 - i \cdot \frac{1}{k}$.
82. The difference quotient $\dfrac{f(z+h) - f(z)}{h}$ is defined for complex-valued functions f as long as h is not zero.
83. If f only takes on real-values then the numerator is always real and it follows that for real h the difference quotient is real. On the other hand, if h is purely imaginary, then the difference quotient is also purely imaginary. The only way a limit of difference quotients exists in this case is if it is equal to zero.
84. For the difference quotient to be of interest it must be assumed that f is genuinely complex-valued.
85. Just as in the real case $\dfrac{(z+h)^2 - z^2}{h} = 2z + h$, so it follows that for any sequence h_k approaching zero, the difference quotient approaches $2z$.
86. Write $\lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h} = f'(z)$ in this case. f' is said to be the derivative or complex derivative in this case.

87. Observe that $\dfrac{\overline{z+h}-\overline{z}}{h}=\dfrac{\overline{h}}{h}$, and if h is real this quotient is 1 but if h is purely imaginary the quotient is -1. It follows that the complex conjugate is not differentiable.

88. As indicated the magnitude, Re, and Im are not differentiable either.

89. A function that is differentiable on the region on which it is define is said to be *analytic* or *holomorphic*.

90. Since $(\sum_{k=0}^{n} a_k z^k)' = \sum_{k=1}^{n} k a_k z^{k-1}$ is seen that polynomials are analytic in the entire complex plane.

91. Functions that are analytic everywhere in the complex plane are said to be *entire*.

92. Assume $f(z)=u(x,y)+iv(x,y)$ analytic with $z=x+iy$. It must be that the difference quotients are

$$\frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h},\ \frac{u(x,y+h)+iv(x,y+h)-u(x,y)-iv(x,y)}{ih}$$

when approaching z along a horizontal line versus a vertical line. Take limits and conclude that the partial derivatives satisfy $u_x=v_y$ and $u_y=-v_x$, the so-called Cauchy-Riemann equations.

93. Observe that if u,v are sufficiently smooth, then $u_{xx}=v_{xy}=v_{yx}=-u_{yy}$ and $v_{xx}=-u_{xy}=-u_{yx}=-v_{yy}$ so both satisfy the Laplace equation $\Delta w=w_{xx}+w_{yy}=0$. Such functions are said to be harmonic.

94. Consider the entire function $f(z)=z^2=(x+iy)^2=x^2-y^2+i\cdot 2xy$. Suppose only the real part or the imaginary part of f is known. For instance, let $v(x,y)=2xy$. Clearly v is harmonic and one of the Cauchy-Riemann equations forces $u_x=2x$. The most general possibility for u is given by $u(x,y)=x^2+g(y)$. According to the other of the Cauchy-Riemann equations it must be that $u_y=g'(y)=-2y$ and hence $g(y)=-y^2+C$ for some real constant C. This shows that it is possible to recover, up to a constant, an analytic f from its real or imaginary part, which is a remarkable fact.

95. When $f(z)=\overline{z}=\overline{x+iy}=x-iy$ is seen that $u_x=1$ and $v_y=-1$, which violates Cauchy-Riemann and hence f is not analytic.

96. Integrating a complex-valued function on an interval of the real line is straight forward as seen in $\int_{[a,b]} f=\int_{[a,b]} u+iv=\int_a^b u(x)dx+i\int_a^b v(x)dx$. The complex-valued result can be interpreted in terms of scaled averages of the real and the imaginary parts of f.