Quiz #10, 11/1/16

Problem #1

The sustitution u=cosθ with du=-sinθ dθ is useful here.

$$\int_0^5 \left(\int_0^{\pi} r^4 \sin(\theta) \cos^2(\theta)\, d\theta \right) dr = \int_0^5 r^4\, dr \int_0^{\pi} \sin(\theta) \cos^2(\theta)\, d\theta$$

$$\int_0^5 r^4 \, dr$$

625

$$\int_0^{\pi} \sin(\theta) \cos^2(\theta)\, d\theta$$

$\frac{2}{3}$

$$625 \times \frac{2}{3}$$

$\frac{1250}{3}$

Problem #2

Since both the density distribution and the object are symmetric about the y-axis, the x-coordinate of the center of mass must be 0.

Split the triangle along the y-axis. By symmetry, the two pieces must have the same y-coordinate for their center of mass.

The whole triangle must have that y-coordinate for its center of mass.

Half the triangle has mass 3/2.

$$\frac{1}{\frac{3}{2}} \int_0^1 \left(\int_0^{3-3x} y\, dy \right) dx$$

1

$$\int_0^{3-3x} y\,dy$$

$$\frac{9}{2} - 9x + \frac{9x^2}{2}$$

$$\int_0^1 \left(\frac{9}{2} - 9x + \frac{9x^2}{2}\right) dx$$

$$\frac{3}{2}$$

The center of mass is (0,1).

Quiz #9, 10/25/16

Problem #1

First in x then in y!

$$\text{TableForm}\left[\left\{\int\left(\frac{x}{y^2}+y\right)dx,\ \int_0^2\left(\frac{x}{y^2}+y\right)dx,\ \int\left(\int_0^2\left(\frac{x}{y^2}+y\right)dx\right)dy,\ \int_1^2\left(\int_0^2\left(\frac{x}{y^2}+y\right)dx\right)dy\right\}\right]$$

$$\frac{x^2}{2y^2} + xy$$

$$\frac{2}{y^2} + 2y$$

$$-\frac{2}{y} + y^2$$

4

First in y then in x!

$$\text{TableForm}\left[\left\{\int\left(\frac{x}{y^2}+y\right)dy,\ \int_1^2\left(\frac{x}{y^2}+y\right)dy,\ \int\left(\int_1^2\left(\frac{x}{y^2}+y\right)dy\right)dx,\ \int_0^2\left(\int_1^2\left(\frac{x}{y^2}+y\right)dy\right)dx\right\}\right]$$

$$-\frac{x}{y} + \frac{y^2}{2}$$

$$\frac{3+x}{2}$$

$$\frac{1}{2}\left(3x + \frac{x^2}{2}\right)$$

4

The two ways are about the same.

Problem #2

First in x then in y!

```
TableForm[{∫(x^2 + 2 y) dx, ∫_y^∛y (x^2 + 2 y) dx, ∫(∫_y^∛y (x^2 + 2 y) dx) dy, ∫_0^1 (∫_y^∛y (x^2 + 2 y) dx) dy}]
```

$\frac{x^3}{3} + 2\,x\,y$

$\frac{y}{3} + 2\,y^{4/3} - 2\,y^2 - \frac{y^3}{3}$

$-\frac{1}{84}\,y^2\,\left(-14 - 72\,y^{1/3} + 56\,y + 7\,y^2\right)$

$\frac{23}{84}$

First in y then in x!

```
TableForm[{∫(x^2 + 2 y) dy, ∫_x^x^3 (x^2 + 2 y) dy, ∫(∫_x^x^3 (x^2 + 2 y) dy) dx, ∫_0^1 (∫_x^x^3 (x^2 + 2 y) dy) dx}]
```

$x^2\,y + y^2$

$x^2 + x^3 - x^5 - x^6$

$\frac{x^3}{3} + \frac{x^4}{4} - \frac{x^6}{6} - \frac{x^7}{7}$

$\frac{23}{84}$

The second way is cleaner.

Quiz #8, 10/18/16

Problem #1

Objective: A polynomial so continuous!

```
f[x_, y_] := y^2 - x^2
```

Constraint: An ellipse, so closed an bounded.

```
g[x_, y_] := 1/4 x^2 + y^2
```

There must exist both a maximum and a minimum.

Use the Lagrange multipler technique.

```
Grad[f[x, y], {x, y}]
```

$\{-2\,x,\ 2\,y\}$

```
Grad[g[x, y], {x, y}]
```

$$\left\{\frac{x}{2}, 2\,y\right\}$$

```
Solve[{-2 x == λ x/2, 2 y == λ 2 y, 1/4 x^2 + y^2 == 1}, {x, y, λ}]
```

```
{{x → 0, y → -1, λ → 1}, {x → 0, y → 1, λ → 1},
 {x → -2, y → 0, λ → -4}, {x → 2, y → 0, λ → -4}}
```

There are two points that yield max: (0,-1) and (0,1).

```
y^2 - x^2 /. {x → 0, y → -1, λ → 1}
```

```
1
```

```
y^2 - x^2 /. {x → 0, y → 1, λ → 1}
```

```
1
```

There are two points that yield min: (-2,0) and (2,0).

```
y^2 - x^2 /. {x → -2, y → 0, λ → -4}
```

```
-4
```

```
y^2 - x^2 /. {x → 2, y → 0, λ → -4}
```

```
-4
```

Problem #2

Objective: Polynomial so continuous!

```
f[x_, y_] := x^2 + (y - 1)^2
```

Constraint: Closed but not bounded so the existence theorem is not applicable.

The distance from (x,y) on the parabola to (0,1) goes to infinity as x→±∞ so if for a bounded x we at least have a minimum.

```
g[x_, y_] := y - x^2
```

```
Grad[f[x, y], {x, y}]
```

$$\{2\,x,\ 2\,(-1+y)\}$$

```
Grad[g[x, y], {x, y}]
```

$$\{-2\,x,\ 1\}$$

The Lagrange technique discovers three candidates.

```
Solve[{2 x, 2 (-1 + y), y - x^2} == {-2 λ x, λ, 0}, {x, y, λ}]
```

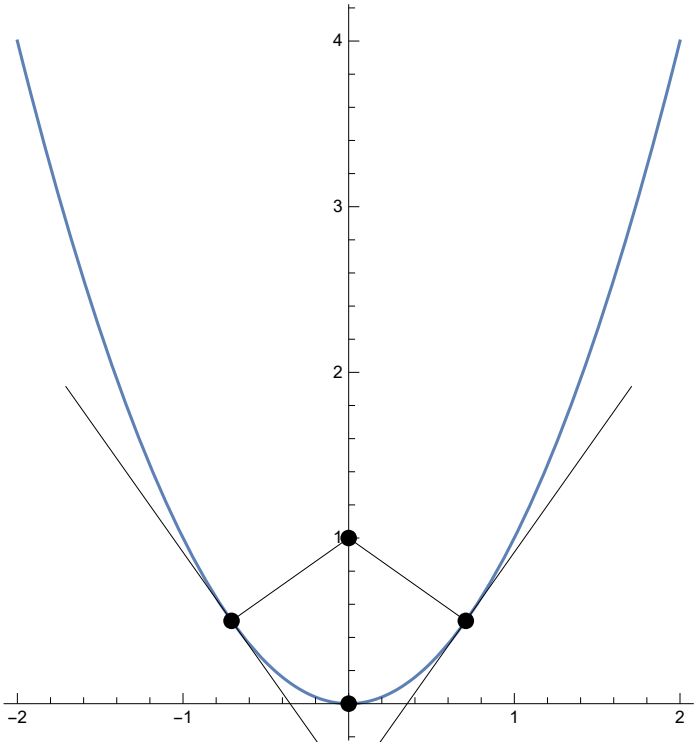
$$\left\{\{x\to 0,\ y\to 0,\ \lambda\to -2\},\ \left\{x\to -\frac{1}{\sqrt{2}},\ y\to \frac{1}{2},\ \lambda\to -1\right\},\ \left\{x\to \frac{1}{\sqrt{2}},\ y\to \frac{1}{2},\ \lambda\to -1\right\}\right\}$$

Here are the lengths.

```
Sqrt[f[x, y]] /. # & /@
  {{x → 0, y → 0, λ → -2}, {x → -1/Sqrt[2], y → 1/2, λ → -1}, {x → 1/Sqrt[2], y → 1/2, λ → -1}}
```

$$\left\{1,\ \frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right\}$$

The points $\left(\pm 1/\sqrt{2},\ 1/2\right)$ have smaller distance to (0,1)!



Obeserve how the tangent lines are perpendicular to the line-segment between the parabola and (0,1) at all three points.

The vertex (0,0) is not a maximum, only a local maximum on the parabola.

The point (0,1) is not the focus of this parabola, the point (0,1/4) is the focus.

The vertex is the point closest to the focus on the parabola.

As the point varies like (0,α), there is a 'bifurcation' where the minimum splits into to two minima when α=1/2 !

Quiz #7, 10/11/16

Problem #1

```
f[x_, y_] := x^2 e^-y
```

```
Grad[f[x, y], {x, y}]
```

$$\left\{2\, e^{-y}\, x,\ -e^{-y}\, x^2\right\}$$

```
{2 e^-y x, -e^-y x^2} /. {x → 3, y → 0} +
```

```
{6, -9}
```

```
v := 3 {1, 0} + 4 {0, 1}
```

```
Norm@v
```

```
5
```

```
u := 1/Norm@v v
```

```
Dot[{6, -9}, u]
```

$$-\frac{18}{5}$$

Problem #2

```
f[x_, y_] := (x - y) (1 - x y)
```

```
Grad[f[x, y], {x, y}]
```

```
{1 - x y - (x - y) y, -1 - x (x - y) + x y}
```

```
Grad[f[x, y], {x, y}] // Simplify
```

```
{1 - 2 x y + y^2, -1 - x^2 + 2 x y}
```

```
Solve[{1 - 2 x y + y^2, -1 - x^2 + 2 x y} == {0, 0}, {x, y}, Reals]
```

```
{{x → -1, y → -1}, {x → 1, y → 1}}
```

```
fxx = D[1 - 2 x y + y^2, x]
```

```
-2 y
```

```
fxy = D[1 - 2 x y + y^2, y]
```

```
-2 x + 2 y
```

```
fyy = D[-1 - x^2 + 2 x y, y]
```

```
2 x
```

```
discriminant = fxx fyy - fxy^2 // Simplify
```

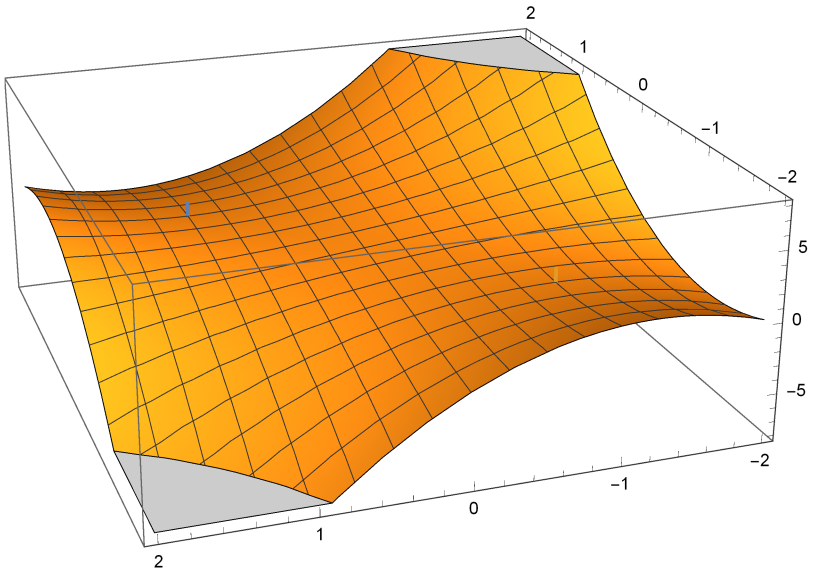
```
-4 (x^2 - x y + y^2)
```

```
{discriminant /. {x -> -1, y -> -1}, discriminant /. {x -> 1, y -> 1}}
```

```
{-4, -4}
```

(-1,-1) saddle point and (1,1) saddle point.

```
Show[p2, p1]
```



Quiz #6, 10/4/16

Problem #1

```
f[x_, y_] := x^4 y - 2 x^3 y^2
```

First partial derivatives.

```
{D[f[x, y], x], D[f[x, y], y]}
```

$\{4\,x^3\,y - 6\,x^2\,y^2,\ x^4 - 4\,x^3\,y\}$

Second partial derivatives.

```
{D[f[x, y], {x, 2}], D[f[x, y], {x, 1}, {y, 1}], D[f[x, y], {y, 2}]}
```

$$\left\{12\,x^2\,y - 12\,x\,y^2,\ 4\,x^3 - 12\,x^2\,y,\ -4\,x^3\right\}$$

Quick check:

```
{D[D[f[x, y], x], y], D[D[f[x, y], y], x]}
```

$$\left\{4\,x^3 - 12\,x^2\,y,\ 4\,x^3 - 12\,x^2\,y\right\}$$

Same ☺!

Problem #2

(a)

```
f[x_, y_] := (x^4 - y^4)/(x^2 + y^2)
```

```
Factor[x^4 - y^4]
```

$$(x - y)\,(x + y)\,\left(x^2 + y^2\right)$$

```
Simplify[f[x, y]]
```

$$x^2 - y^2$$

This is continuous, so plug in and get that the limit is 0!

(b)

```
f[x_, y_] := (x y)/(x^2 + y^2)
```

```
f[x, y] /. {x → r Cos[θ], y → r Sin[θ]}
```

$$\frac{r^2\,\text{Cos}[\theta]\,\text{Sin}[\theta]}{r^2\,\text{Cos}[\theta]^2 + r^2\,\text{Sin}[\theta]^2}$$

```
(r^2 Cos[θ] Sin[θ])/(r^2 Cos[θ]^2 + r^2 Sin[θ]^2) // Simplify
```

$$\text{Cos}[\theta]\,\text{Sin}[\theta]$$

```
Cos[θ] Sin[θ] // TrigReduce
```

$$\frac{1}{2}\,\text{Sin}[2\,\theta]$$

When $\theta = 0$ this is 0. When $\theta = \pi / 4$ this is 1/2.

Approaching the origin along the x-axis versus along the line y = x gives different values, so the limit does not exist.

Quiz #5, 9/27/16

Problem #1

To get started, determine the value of t at the given point.

Since the given point is (2,4,1) and $y = 4\sqrt{t}$ the only possibility is t=1, which is consistent with $x = t^2 + 1$ and $z = e^{t^2 - t}$.

Now, in $\vec{p} + t\,\vec{v}$ use $\vec{p} = \langle 2, 4, 1 \rangle$.

To get $\vec{v}$, take the derivative and plug in t=1.

The derivative, using the power rule and chain rule, is $\left\langle 2\,t,\ \frac{2}{\sqrt{t}},\ e^{t^2 - t} \cdot (2\,t - 1) \right\rangle$.

At t=1 this is <2,2,1>.

Problem #2

The curve.

```
r@t_ := {t^2 + t, t^2 - t, t^3}
```

The velocity.

```
dr@t_ = D[r@t, t]
```

$$\{1 + 2\,t,\ -1 + 2\,t,\ 3\,t^2\}$$

The acceleration.

```
ddr@t_ = D[r@t, {t, 2}]
```

```
{2, 2, 6 t}
```

The speed.

```
v@t_ = Simplify[Norm[dr@t], t > 0]
```

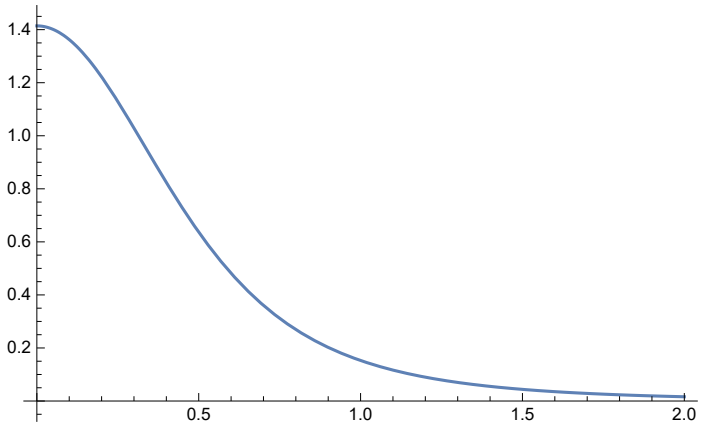
$$\sqrt{2 + 8\,t^2 + 9\,t^4}$$

The curvature is given by $\| r'(t) \times r''(t) \| \,/\, \| r'(t) \|^3 = \| r'(t) \times r''(t) \| \,/\, v(t)^3$.

```
κ@t_ = Simplify[Norm[Cross[dr@t, ddr@t]] / (v@t)^3, t > 0]
```

$$2\sqrt{2}\,\sqrt{\frac{2 + 9\,t^2 + 9\,t^4}{\left(2 + 8\,t^2 + 9\,t^4\right)^3}}$$

```
Plot[κ@t, {t, 0, 2}]
```



The unit tangent vector T(t).

```
theT@t_ = 1/v@t dr@t
```

$$\left\{\frac{1 + 2\,t}{\sqrt{2 + 8\,t^2 + 9\,t^4}}, \frac{-1 + 2\,t}{\sqrt{2 + 8\,t^2 + 9\,t^4}}, \frac{3\,t^2}{\sqrt{2 + 8\,t^2 + 9\,t^4}}\right\}$$

The normal vector N(t) as $\frac{1}{\kappa\, v^2}\left(\vec{r}\,'' - v'\,T\right)$

```
theN@t_ = FullSimplify[1/(κ@t (v@t)^2) (ddr@t - D[v@t, t] theT@t), t > 0]
```

$$\left\{\frac{2-4\,t-9\,t^3-9\,t^4}{\sqrt{8+68\,t^2+216\,t^4+306\,t^6+162\,t^8}},\ \frac{2+4\,t+9\,t^3-9\,t^4}{\sqrt{8+68\,t^2+216\,t^4+306\,t^6+162\,t^8}},\ \frac{3\sqrt{2}\ t\left(1+2\,t^2\right)}{\sqrt{\left(2+8\,t^2+9\,t^4\right)\left(2+9\,t^2+9\,t^4\right)}}\right\}$$

Interesting observations:

```
Limit[theT@t, t → ∞]
```

```
{0, 0, 1}
```

```
Limit[theN@t, t → ∞]
```

$$\left\{-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right\}$$

```
Limit[κ@t, t → ∞]
```

```
0
```

Alternative way to calculate N(t) as $\frac{1}{||T'(t)||}\,T'(t)$

```
theN@t_ = FullSimplify[1/Simplify[Norm[D[theT@t, t]], t > 0] D[theT@t, t], t > 0]
```

$$\left\{\frac{2-4\,t-9\,t^3-9\,t^4}{\sqrt{8+68\,t^2+216\,t^4+306\,t^6+162\,t^8}},\ \frac{2+4\,t+9\,t^3-9\,t^4}{\sqrt{8+68\,t^2+216\,t^4+306\,t^6+162\,t^8}},\ \frac{3\sqrt{2}\ t\left(1+2\,t^2\right)}{\sqrt{\left(2+8\,t^2+9\,t^4\right)\left(2+9\,t^2+9\,t^4\right)}}\right\}$$

Building blocks:

$\vec{r}\,' \times \vec{r}\,''(t)$

```
Cross[{2 t + 1, 2 t - 1, 3 t^2}, {2, 2, 6 t}]
```

$$\left\{-6\,t+6\,t^2,\ -6\,t-6\,t^2,\ 4\right\}$$

$||\vec{r}' \times \vec{r}''(t)||$

```
Simplify[Norm[Cross[{2 t + 1, 2 t - 1, 3 t^2}, {2, 2, 6 t}]], t ∈ Reals]
```

$$2\sqrt{2}\ \sqrt{2 + 9\,t^2 + 9\,t^4}$$

v'(t)

```
Simplify[D[v@t, t], t ∈ Reals]
```

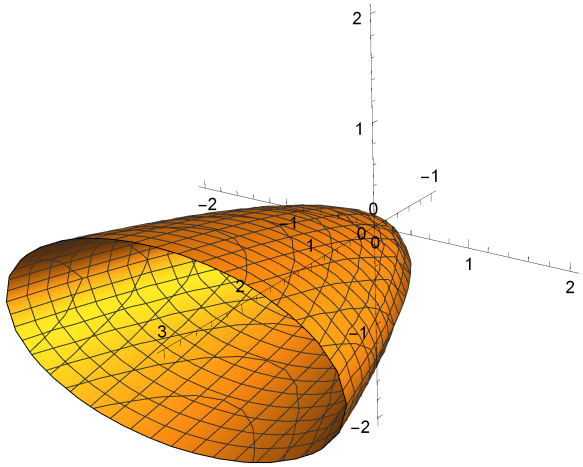
$$\frac{2\,t\,\left(4 + 9\,t^2\right)}{\sqrt{2 + 8\,t^2 + 9\,t^4}}$$

Quiz #4, 9/20/16

Problem #1

With the x-axis coming out of the screen to the left.

```
ContourPlot3D[x == y^2 + 4 z^2, {x, -1, 3}, {y, -2, 2},
 {z, -2, 2}, AxesOrigin → {0, 0, 0}, Boxed → False]
```



Observe how the formula forces x to be non-negative. With x=k one gets ellipses, so elliptic paraboloid.

Problem #2

The key is:

21: IV
22: VII
23: II
24: III
25: VI
26: I
27: VIII
28: V

Quiz #3, 9/13/16

Problem #1

The j - k corresponds to <0,1,-1> and i + j to <1,1,0>.

The cross product will produce a vector orthogonal to both.

```
Cross[{0, 1, -1}, {1, 1, 0}]
```

```
{1, -1, -1}
```

Now create a unit vector!

```
Norm[{1, -1, -1}]
```

$\sqrt{3}$

To get a second vector just flip it so these two unit vector do the job $\pm \frac{1}{\sqrt{3}}$<1,-1,-1> as you can check using the dot product.

Problem #2

Let t = 0 in x = 4 - t, y = 2 t -1, z = -3 t, so x(0) = 4, y(0) = -1, z(0) = 0 .

The difference between the given point (3,5,-1) and point (4,-1,0) on the line produces the vector <-1,6,-1>, which is in the plane.

Since the given line is in the plane, its v = <-1,2,-3> produces a second vector in the plane.

The cross product <-1,2,-3> × <-1,6,-1> will be orthogonal to the plane.

```
Cross[{-1, 2, -3}, {-1, 6, -1}]
```

```
{16, 2, -4}
```

Scale this orthogonal vector by a factor of 1/2 to get <A, B, C> = < 8, 1, -2 > and the reduced equation 8 x + y - 2 z = D.

To determine D plug in any point known to be in the plane, like (3,5,-1) or (4,-1,0). Any such a point yields D = 31.

Quiz #2, 9/6/16

Problem #1

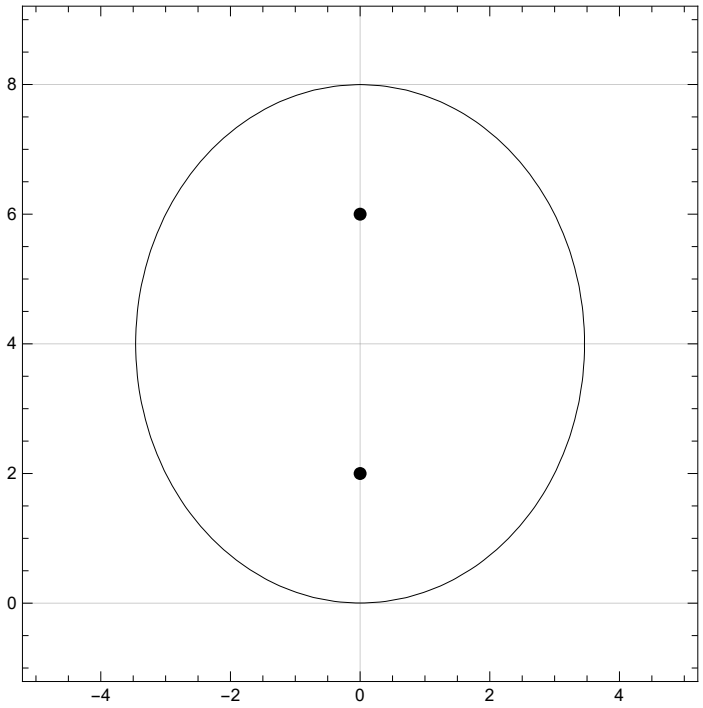
Sincce it says ellipse use $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

The vertices are on the y-axis and shifted 4 units in the positive direction so
$\frac{x^2}{a^2} + \frac{(y-4)^2}{b^2} = 1.$

Plug in the top vertex (0,8) to see that $b^2 = 16$.

Finally, c=2 since that is the distance from the center (0,4), so
$a^2 = b^2 - c^2 = 16 - 2^2 = 12$, **and** $\frac{x^2}{12} + \frac{(y-4)^2}{16} = 1.$

```
ContourPlot[x^2/12 + (y - 4)^2/16, {x, -5, 5}, {y, -1, 9},
 Contours → {1}, ContourShading → None, GridLines → {{0}, {0, 4, 8}},
 Epilog → {Disk[{0, 2}, 1/10], Disk[{0, 6}, 1/10]}]
```



Problem #2

The projection is onto $\vec{v} = \langle 3, 4, 0 \rangle$, which has magnitude

$$\sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5.$$

The aligned unit vector is $\vec{u} = \frac{1}{5}\langle 3, 4, 0 \rangle$.

The projection of $\vec{W} = \langle 4, 0, 3 \rangle$ is given by $\left(\vec{w} \cdot \vec{u}\right)\vec{u} = \frac{12}{25}\langle 3, 4, 0 \rangle$.

```
v = {3, 4, 0};
```

```
Norm[v]
```

5

```
u = 1 / Norm@v v
```

$$\left\{\frac{3}{5}, \frac{4}{5}, 0\right\}$$

```
w = {4, 0, 3};
```

```
Dot[w, u]
```

$$\frac{12}{5}$$

```
Dot[w, u] u
```

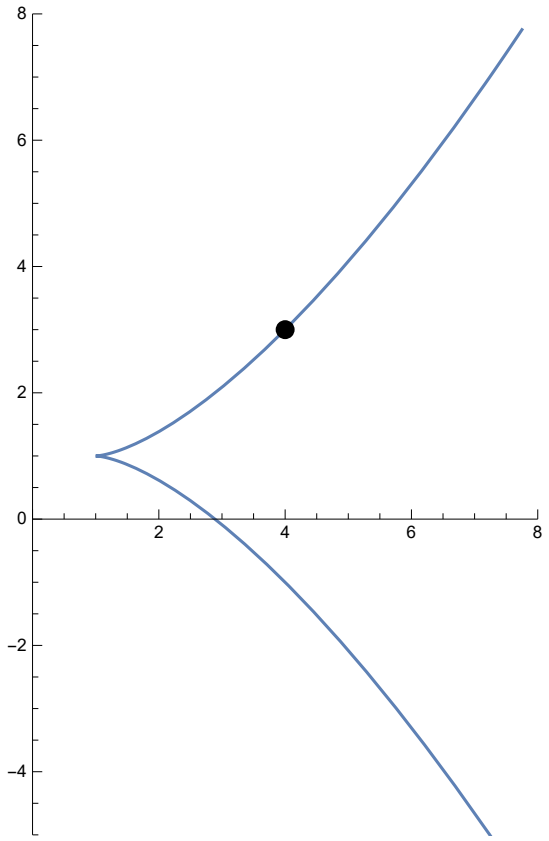
$$\left\{\frac{36}{25}, \frac{48}{25}, 0\right\}$$

Quiz #1, 8/30/16

Problem #1

```
x@t_ := 3 t^2 + 1; y@t_ := 2 t^3 + 1; p := {4, 3}
```

```
plot1 = ParametricPlot[{x@t, y@t}, {t, -3/2, 3/2},
  Epilog → Disk[p, .15], PlotRange → {{0, 8}, {-5, 8}}, AspectRatio → Automatic]
```



There is a unique t where the curve is at (4,3).

```
Solve[{x@t, y@t} == p, t]
```

```
{{t → 1}}
```

Here is the slope.

```
{D[y@t, t], D[x@t, t], D[y@t, t] / D[x@t, t]}
```

```
{6 t^2, 6 t, t}
```

Plug in the relevant t!

```
D[y@t, t] / D[x@t, t] /. t → 1
```

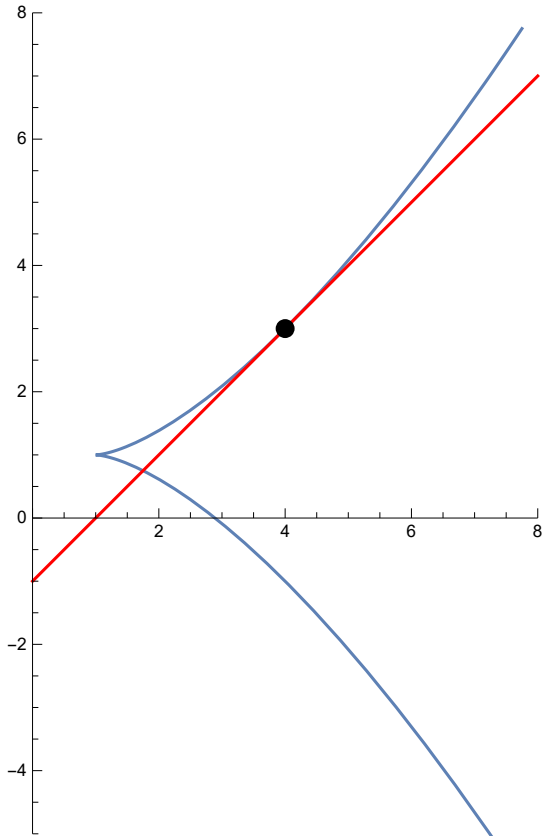
```
1
```

Equation must have the form y=x+b since m=1.

```
Solve[3 == 4 + b, b]
```

```
{{b → -1}}
```

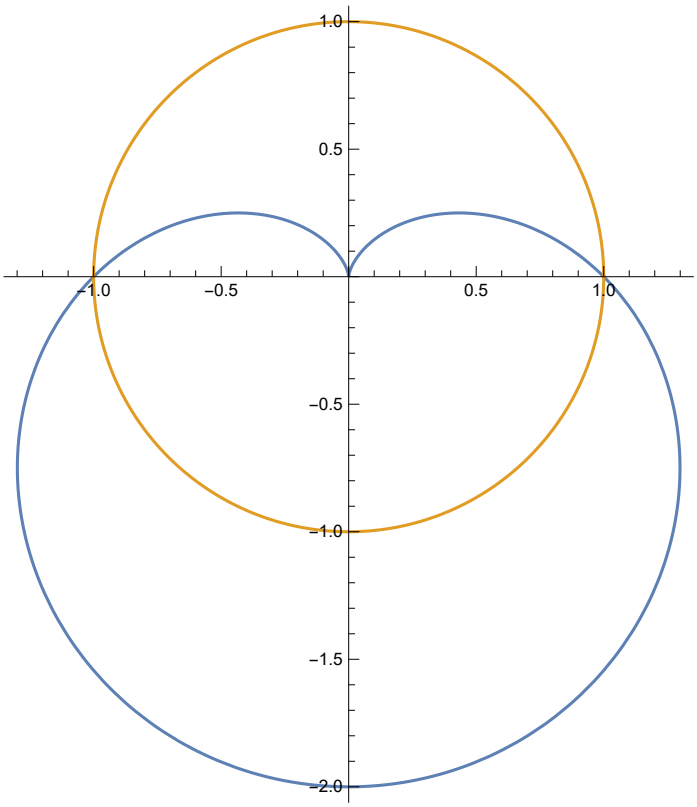
```
Show[{plot1, Plot[x - 1, {x, 0, 8}, PlotStyle → Red]}]
```



Problem #2

```
r1@θ_ := 1 - Sin@θ; r2@θ_ := 1
```

```
PolarPlot[{r1@θ, r2@θ}, {θ, 0, 2 π}]
```




```
(r1@θ)^2 / 2
```

$$\frac{1}{2}\,(1-\text{Sin}[\theta])^2$$

Use the double angle formula as we did in class.

```
TrigReduce[(r1@θ)^2 / 2]
```

$$\frac{1}{4}\,(3-\text{Cos}[2\,\theta]-4\,\text{Sin}[\theta])$$

```
Integrate[(r1@θ)^2 / 2, {θ, π, 2 π}]
```

$$2+\frac{3\,\pi}{4}$$

The second one is immediate since it is half of the unit disk.

```
Integrate[(r2@θ)^2/2, {θ, π, 2 π}]
```

$$\frac{\pi}{2}$$

```
2 + 3 π/4 - π/2 // Simplify
```

$$\frac{8+\pi}{4}$$

It is of course possible to combine the two integrals.

```
Integrate[(r1@θ)^2/2 - (r2@θ)^2/2, {θ, π, 2 π}] // Simplify
```

$$\frac{8+\pi}{4}$$