SYMMETRIZED CURVE-STRAIGHTENING

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ABSTRACT. The 'traditional' curve-straightening flow is based on one of the standard Sobolev inner products and it is known to break certain symmetries of reflection. The purpose of this paper is to show that there are alternative Riemannian structures on the space of curves that yield flows that preserve symmetries. This feature comes at a price. In one symmetrizing metric the gradient vector fields are considerably more demanding to compute. In another symmetrizing metric smoothness is lost. This investigation will also explain the phenomena of 'spinning' as observed in several examples in the traditional flow. Three classes of alternative Riemannian structures are examined. The first class includes the traditional metric as a special case and is shown to never preserve both rotation symmetries and symmetries of reflection. The second class consists of a single metric corresponding to one of the standard Sobolev metrics, and is shown to preserve both types of symmetries. The third class also includes the traditional metric but it is shown that there is a unique different metric in this class, which preserves both types of symmetries. This particular metric generally yields smooth vector fields, which when evaluated at a smooth function do not give a smooth element of the corresponding tangent space. The third class is nevertheless 'preferred' since it has the distinction that it 'respects' the projection induced by the derivative operator onto the tangent bundle of the space of derivatives. The paper concludes with a number of graphical illustrations that show preserved symmetry and removal of spinning.

1. INTRODUCTION

1.1 Background. Consider a Riemannian manifold X with metric $\langle , \rangle$, and a functional $F : X \to \mathbb{R}$. The directional derivative $DF(x) : T_x X \to \mathbb{R}$ is related to the gradient $\nabla F(x) \in T_x X$ by $\langle \nabla F(x), v \rangle_x = DF(x)v$. The curve-straightening flow with $\nu \geq 0$ is the negative gradient flow associated with the modified total squared curvature functional

$$\gamma \mapsto \int_\gamma \tfrac{1}{2} k^2 + \nu ds \, .$$

Although there is a certain amount of freedom in the choice of X and its Riemannian structure, papers dealing with the curve-straightening flow proper have so far consistently used the same space X and metric. Specifically, X is the largest possible Sobolev space on which the modified

total squared curvature functional is defined with the simplest standard metric. This choice of facilitates the computation of the gradient and its projection onto tangent spaces in the case of constraints. It also simplifies several of the analytical arguments required when verifying the Palais-Smale condition.

1.2 Purpose. The purpose of this paper is to answer questions concerning the failure of the traditional curve-straightening flow to preserve certain symmetries of reflection. There are many examples that illustrate this behavior; see [3]. Robert L. Bryant and Joel Langer have on separate occasions asked the author why the symmetry is broken. The short, but not particularly satisfying answer, is that the metric

$$\langle v, w \rangle = v(0)w(0) + \int_0^1 \dot{v}(s)\dot{w}(s)ds$$

is not symmetric on the parameter interval $[0,1]$. In the case of closed curves most of the effects of this asymmetric choice are masked by the periodicity.

1.3 Problem. Recall that solutions of the equation $\nabla F(x) = 0$ are called critical points. The choice of metric affects the form of the equation $\nabla F(x) = 0$. A critical point satisifes $DF(x)v = 0$ for all v. It follows that the choice of metric is not important when the main concern is to find all solutions of $\nabla F(x) = 0$. The lack of symmetry in the metric $\langle , \rangle$ becomes an issue when the concern is the symmetry of x_τ in the flow

$$\frac{dx_\tau}{d\tau} = -\nabla F(x_\tau).$$

A related question, posed by David A. Singer, concerns an example given in [2]. The example shows the curve-straightening flow of an initial curve, which is 'almost' a double covered Euler figure eight. The flow has as limit the single covered Euler figure eight. The purpose of the example is to show a 'complete' negative gradient trajectory, which starts close to the unstable double covered Euler figure eight and ends at the stable single covered Euler figure eight. The initial curve has an axis of symmetry and subsequent curves along the trajectory also have an axis of symmetry. The peculiar behavior is that the axis of symmetry rotates during the flow and David's question is why this is the case. The results in [1] concerning preserved symmetry do not apply to this case since the rotation number in the example is 0 as opposed to 1. The root cause of the behavior is the extra degree of freedom due to the possibility of 'spinning' closed curves. In Section 2.4 the technical manifold theoretic reason behind spinning is exposed.

1.4 Organization. In Section 2 a slight extension of the concepts of odd and even functions is introduced. This leads to the notion of parametric symmetries. In the context of curves in the plane this extension efficiently identifies non-closed curves with an axis of symmetry or 180 degree rotational symmetry. In the case of closed curves the parametric symmetries are only present in some of the parameterizations so the choice of initial point is in this case important. The spaces

of parametrically symmetric curves are analyzed as subsets in the space of all curves. In [1] 'externally' symmetric closed curves of rotation number 1 are shown to be a closed submanifold using a technique by Palais. The externally symmetric curves are not to be confused with the parametrically symmetric curves considered here. In Section 3 the general formulas derived in [4] are recalled and specialized to the context of the curve-straightening flow. Unconstrained as well as constrained curves are considered. The main result is the proof of the existence of curve-straightening flows that preserve symmetries. To keep the present paper self-contained an appendix is included with direct proofs of the general formulas derived in [4].

2. PARAMETRIC SYMMETRIES

2.1 Generalized even and odd. The concepts of even and odd are used here in a slightly more general setting. Given an interval $[a,b]$, let $m=(a+b)/2$ be the midpoint. A function $f:[a,b]\to\mathbb{R}$ is *generalized even* with respect to the midpoint if $f(x)=f(a+b-x)$ for all $x\in[a,b]$. The function is *generalized odd* with respect to the midpoint if $f(x)+f(a+b-x)=2f(m)$. Note that only the constant functions have both symmetries.

2.2 Parametric symmetry about a point. Suppose a regular (i.e., differentiable and nonzero speed) curve $\gamma:[0,1]\to\mathbb{C}$ of length L in the complex plane is related to a continuous function $\theta:[0,1]\to\mathbb{R}$ by

$$\gamma'(s)=Le^{i\theta(s)}.$$

The curve γ is said to have *parametric* rotational symmetry with respect to the point $\gamma(1/2)$ if the tangent angle θ is generalized even with respect to $1/2$. The curve γ is said to have *parametric* reflection symmetry with respect to the normal line through $\gamma(1/2)$ if θ is generalized odd with respect to $1/2$.

Examples: Consider parameterized curves, which start at $(-1,0)$ and end at $(1,0)$. The first curve follows the real axis to the origin, continues along a circle in the upper half-plane, and finishes by following the real axis to $(1,0)$. The tangent angle function is given by

$$\theta(s)=\begin{cases}0 & 0\le s<1/3\\ 6\pi(s-1/3) & 1/3\le s<2/3\\ 2\pi & 2/3\le s\le 1\end{cases}.$$

The curve has parametric reflection symmetry with respect to the imaginary axis. The curve does not have parametric rotational symmetry. The second curve consists of two straight segments and two circles and

$$\theta(s) = \begin{cases} 0 & 0 \le s < 1/4 \\ 8\pi(s-1/4) & 1/4 \le s < 1/2 \\ 8\pi(3/4-s) & 1/2 \le s < 3/4 \\ 0 & 3/4 \le s \le 1 \end{cases}.$$

This curve has parametric rotational symmetry. This curve does not have parametric reflection symmetry. Note that when the curve is thought of as a set of points it has 'external' reflection symmetry across the imaginary axis.

2.3 Useful maps. To be in a position to employ manifold theoretical tools the following facts are established. Suppose the endpoints $\gamma(0)$ and $\gamma(1)$ are fixed, then the tangent angle θ is subject to a nonlinear constraint of the form

$$L\int_0^1 e^{i\theta(s)}ds = \gamma(1) - \gamma(0)\,.$$

The directional derivative of

$$\int_0^1 e^{i\theta(s)}ds\,,$$

in the direction v, has the form

$$i\int_0^1 v(s)e^{i\theta(s)}ds\,.$$

The space of tangent angles appropriate in the context of total squared curvature is given by the vector space $H = \left\{\theta : [0,1] \to \mathbb{R} \;\middle|\; \theta \text{ is absolutely continuous, } \dot\theta \in L^2[0,1]\right\}$. In the next two results there is no need to assume this much smoothness in θ, but no harm is done.

Proposition 1. *Let* $W = \left\{w \in H \mid w(s) = w(1-s)\right\}$ *and suppose* $\theta \in W$ *is given such that* θ *is not a constant function. The linear map* $\Gamma_\theta : W \to \mathbb{C}$ *defined by*

$$\Gamma_\theta(w) = \int_0^1 w(s)e^{i\theta(s)}ds\,,$$

is onto.

Proof. Integrate by parts and get

$$\Gamma_\theta(w) = w(1)\int_0^1 e^{i\theta(s)}ds - \int_0^1 \left\{\dot w(s)\int_0^s e^{i\theta(t)}dt\right\}ds\,.$$

Given any $a \in [0,1/2]$ define $w_a \in W$ by $w_a(s) = \frac{1}{2}(|s-a| + |s-(1-a)| - 2)$ with

$$\dot w_a = \begin{cases} -1 & 0 \le s < a \\ 0 & a \le s \le 1-a \\ 1 & 1-a < s \le 1 \end{cases},$$

and $w_a(1) = 0$. Define a curve $\sigma : [0,1/2] \to \mathbb{C}$ by $\sigma(a) = \Gamma_\theta(w_a)$. The curve is twice differentiable with

$$\sigma'(a) = -\int_a^{1-a} e^{i\theta(s)} ds \,,$$

and since $\theta \in W$

$$\sigma''(a) = 2e^{i\theta(a)} \,.$$

It follows that σ is not constant. Assume $\sigma(a_0) \neq 0$ for $a_0 \in [0,1/2]$. If there is a function $f : [0,1/2] \to \mathbb{R}$ such that $\sigma(a) = f(a)\sigma(a_0)$ for all $a \in [0,1/2]$, then f is as smooth as σ and $2e^{i\theta(a)} = \sigma''(a) = f''(a)\sigma(a_0)$. Since θ is continuous this implies that θ is a constant. This is ruled out by assumption and hence there are two linearly independent vectors of the form $\sigma(a_0) = \Gamma_\theta(w_{a_0})$ and $\sigma(a_1) = \Gamma_\theta(w_{a_1})$. □

Proposition 2. *Let* $V = \left\{ v \in H \mid v(s) + v(1-s) = 2v(1/2) \right\}$ *and suppose* $\theta \in V$ *is given such that* θ *is not a constant function. The linear map* $\Lambda_\theta : V \to \mathbb{C}$ *defined by*

$$\Lambda_\theta(v) = \int_0^1 v(s) e^{i\theta(s)} ds \,,$$

is onto if

$$\int_0^1 e^{i\theta(s)} ds \neq 0 \,,$$

and has a 1-dimensional image spanned by $ie^{i\theta(1/2)}$ *otherwise.*

Proof. Since $\Lambda_{\theta - c} = e^{-ic}\Lambda_\theta$ for any real number c, it is acceptable to assume that $\theta(s) + \theta(1-s) = 0$. By splitting the interval of integration in half and using a substitution it is seen that

$$\int_0^1 e^{i\theta(s)} ds = 2\int_0^{1/2} \cos\theta(s) ds \,,$$

so

$$\operatorname{Im}\left(\int_0^1 e^{i\theta(s)} ds \right) = 0 \,.$$

First consider the case

$$\int_0^1 e^{i\theta(s)} ds \neq 0 \,.$$

Let $\overline{v} \in V$ be given by $\overline{v}(s) = 1$. Define $v_a \in V$ by

$$\dot{v}_a = \begin{cases} -1 & 0 \leq s < a \\ 0 & a \leq s \leq 1-a \\ -1 & 1-a < s \leq 1 \end{cases} \,,$$

and $v_a(1) = 0$. Define a curve $\sigma : [0,1/2] \to \mathbb{C}$ by $\sigma(a) = \Lambda_\theta(v_a)$. The curve is twice differentiable with

$$\sigma'(a) = \int_0^a e^{i\theta(s)} ds + \int_0^{1-a} e^{i\theta(s)} ds \text{ , and}$$

$$\sigma''(a) = e^{i\theta(a)} - e^{i\theta(1-a)} = e^{i\theta(a)} - e^{-i\theta(a)} = 2i\sin\theta(a)\,.$$

Suppose there is a function $f : [0,1/2] \to \mathbb{R}$ such that $\sigma(a) = f(a)\Lambda_\theta(\overline{v})$ for all $a \in [0,1/2]$, then f is as smooth as σ and

$$2i\sin\theta(a) = \sigma''(a) = f''(a)\Lambda_\theta(\overline{v}) = f''(a)\int_0^1 e^{i\theta(s)}ds\,.$$

This implies

$$\mathrm{Re}\left(\int_0^1 e^{i\theta(s)}ds\right) = 0\,.$$

It follows that

$$\int_0^1 e^{i\theta(s)}ds = 0\,,$$

a contradiction. Hence there are two linearly independent vectors of the form $\Lambda_\theta(\overline{v})$ and $\sigma(a) = \Lambda_\theta(v_a)$. Next consider the case

$$\int_0^1 e^{i\theta(s)}ds = 0\,.$$

For all $v \in V$ it is true that

$$\int_{1/2}^1 v(s)e^{i\theta(s)}ds = -\int_{1/2}^0 v(1-u)e^{i\theta(1-u)}du = \int_0^{1/2} (\,2v(1/2) - v(s)\,)\,e^{-i\theta(s)}ds\,,\ \text{and}$$

$$\int_0^1 v(s)e^{i\theta(s)}ds = \int_0^{1/2} v(s)\left(e^{i\theta(s)} - e^{-i\theta(s)}\right)ds + 2v(1/2)\int_0^{1/2} e^{-i\theta(s)}ds\,.$$

Since

$$0 = \int_0^1 e^{i\theta(s)}ds = 2\int_0^{1/2}\cos\theta(s)ds$$

it follows that

$$\mathrm{Re}\left(\int_0^1 v(s)e^{i\theta(s)}ds\right) = 0\,.$$

To finish the proof it suffices to find at least one $v \in V$ such that

$$\int_0^1 v(s)e^{i\theta(s)}ds \neq 0\,.$$

To this end, note that the curve σ, which is defined in the first part of the proof, moves away from the origin unless $\theta(a) = n\cdot\pi$ for some integer n. In this case the continuity of θ implies that θ is a constant function. This contradicts the assumption on θ. It follows that any a such that $\sigma(a) \neq 0$ yields a function $v_a \in V$ with the property

$$\int_0^1 v_a(s)e^{i\theta(s)}ds \neq 0\,. \qquad \square$$

2.4 Submanifolds of parametrically symmetric curves. The space of 'all' 'geometric curves' with initial point at the origin is represented by $H \times \mathbb{R}^+$ where $\mathbb{R}^+$ denotes the positive real numbers. Given an element $(\theta, L) \in H \times \mathbb{R}^+$, the corresponding curve is given by

$$\gamma(s) = L\int_0^s e^{i\theta(t)}dt\,.$$

Let $\Phi : H \to H$ be given by $\Phi(\theta)(s) = \theta(s) - \theta(1-s)$. Put $m = 1/2$ and let $\Psi : H \to H$ be given by $\psi(\theta)(s) = \theta(s) + \theta(1-s) - 2\theta(m)$. Suppose g is a fixed complex number and $\overline{\theta}$, $\overline{L} > 0$ are fixed real numbers. Consider the following subsets:

$$\Omega_g^{rot} = \left\{ (\theta, L) \in H \times \mathbb{R}^+ \;\middle|\; L\int_0^1 e^{i\theta(s)}ds = g, \Phi(\theta) = 0 \right\},$$

$$\Omega_{g,\overline{L}}^{rot} = \left\{ \theta \in H \;\middle|\; \overline{L}\int_0^1 e^{i\theta(s)}ds = g, \Phi(\theta) = 0 \right\},$$

$$\Omega_{g,\overline{\theta}}^{refl} = \left\{ (\theta, L) \in H \times \mathbb{R}^+ \;\middle|\; L\int_0^1 e^{i\theta(s)}ds = g, \theta(m) = \overline{\theta}, \Psi(\theta) = 0 \right\}, \text{ and}$$

$$\Omega_{g,\overline{\theta},\overline{L}}^{refl} = \left\{ \theta \in H \;\middle|\; \overline{L}\int_0^1 e^{i\theta(s)}ds = g, \theta(m) = \overline{\theta}, \Psi(\theta) = 0 \right\}.$$

All the inner products used in this paper have the property that convergence in norm implies point-wise convergence. Moreover, both Φ and Ψ are continuous maps.

Theorem 3: *Suppose the vector space H is equipped with a norm such that convergence in norm implies point-wise convergence, and H is a complete space. Furthermore, assume that both Φ and Ψ are continuous, then the following holds:*

(1) Ω_g^{rot} and $\Omega_{g,\overline{\theta}}^{refl}$ are closed submanifolds of $H \times \mathbb{R}^+$, and

(2) $\Omega_{g,\overline{L}}^{rot}$ and $\Omega_{g,\overline{\theta},\overline{L}}^{refl}$ are closed submanifolds of H.

Proof: Since Φ and Ψ are continuous linear maps, the space $H_\Phi = \left\{ \theta \in H \;\middle|\; \Phi(\theta) = 0 \right\}$ is a closed linear subspace of H. It also follows that $H_{\Psi,0} = \left\{ \theta \in H \;\middle|\; \theta(m) = 0, \Psi(\theta) = 0 \right\}$ is a closed linear subspace in H and $H_{\Psi,\overline{\theta}} = H_{\Psi,0} + \overline{\theta}$ is a closed linear submanifold of H. To deal with the case of unconstrained length case (1) define

$$\phi_\Phi : H_\Phi \times \mathbb{R}^+ \to \mathbb{C}\,, \text{ and}$$
$$\phi_{\Psi,\overline{\theta}} : H_{\Psi,\overline{\theta}} \times \mathbb{R}^+ \to \mathbb{C}$$

by

$$\phi_\Phi(\theta, L) = \phi_{\Psi,\overline{\theta}}(\theta, L) = L\int_0^1 e^{i\theta(s)}ds\,.$$

Note that $\Omega_g^{rot} = \phi_\Phi^{-1}(g)$ and $\Omega_{g,\overline{\theta}}^{refl} = \phi_{\Psi,\overline{\theta}}^{-1}(g)$. Recall that

$$\gamma(s) = L\int_0^s e^{i\theta(t)}dt$$

so that $\gamma(1) = g$. The directional derivatives in the direction $v = \left(v_\theta, v_L\right)$ are given by

$$D\phi_{\Phi}(\theta,L)v = D\phi_{\Psi}(\theta,L)v = v_L\int_0^1 e^{i\theta(s)}ds + iL\int_0^1 v_\theta e^{i\theta(s)}ds = \frac{v_L}{L}\,g + i\left(v_\theta(1)g - \int_0^1 \dot{v}_\theta(s)\gamma(s)ds\right).$$

The tangent spaces are given by $TH_{\Phi} = H_{\Phi}$ and $TH_{\Psi,\bar{\theta}} = H_{\Psi,0}$. According to the standard manifold theory ([5] p. 197, the Submersion Theorem) it is enough to show that the directional derivative is onto to prove that $\Omega_g^{rot} = \phi_{\Phi}^{-1}(g)$ and $\Omega_{g,\bar{\theta}}^{refl} = \phi_{\Psi,\bar{\theta}}^{-1}(g)$ are closed submanifolds. To this end, assume first that $g \neq 0$. If v_θ is a constant function so that $v_\theta(s) = a \in R$, then $v_\theta \in TH_{\Phi}$ and $v_\theta \in TH_{\Psi}$. For any $v_L \in TR^+ = R$ the directional derivative is given by

$$D\phi_{\Phi}(\theta,L)\big(a,v_L\big) = D\phi_{\Psi,\bar{\theta}}(\theta,L)\big(a,v_L\big) = \Big(\frac{v_L}{L} + ia\Big)g\,.$$

Given any $c \in \mathbb{C}$, choose $v_L = L\,\mathrm{Re}(c/g)$ and $a = \mathrm{Im}(c/g)$. To finish suppose that $g = 0$. The derivatives simplify as

$$D\phi_{\Phi}(\theta,L)v = D\phi_{\Psi,\bar{\theta}}(\theta,L)v = -i\int_0^1 \dot{v}_\theta(s)\gamma(s)ds\,.$$

In the case of $D\phi_{\Phi}$ it suffices to mimic the steps in the proof of Proposition 1 to show that the map is onto, and in the case of $D\phi_{\Psi,\bar{\theta}}$ use Proposition 2. The difference in the latter case is that the map is onto a one-dimensional subspace of $\mathbb{C}$, which only depends on $\bar{\theta}$. To finish the proof the same reasoning is applied in the case (2). Divide by $\bar{L}$ before applying the two Propositions.

□

3. SYMMETRIZING RIEMANNIAN STRUCTURES

3.1 Riemannian structures. The Riemannian structure defined on $T\big(H\times\mathbb{R}^+\big) = H\times\mathbb{R}$ has the following form $\big\langle\big(v_\theta,v_L\big),\big(w_\theta,w_L\big)\big\rangle = \big\langle v_\theta,w_\theta\big\rangle_H + v_L w_L$. Following [4], let $\alpha \geq 0$, $\beta \geq 0$ (not both zero) and $p \in [0,1]$ be given and consider the following three different possibilities for $\langle,\rangle_H$

$$\langle v,w\rangle_I = \alpha v(0)w(0) + \beta v(1)w(1) + \int_0^1 \dot{v}(s)\dot{w}(s)ds\,,$$

$$\langle v,w\rangle_{II} = \int_0^1 v(s)w(s)ds + \int_0^1 \dot{v}(s)\dot{w}(s)ds\,,\text{ and}$$

$$\langle v,w\rangle_{III} = v(p)w(p) + \int_0^1 \dot{v}(s)\dot{w}(s)ds\,.$$

3.2 The modified total squared curvature. Given a real number $\nu \geq 0$, the general curve-straightening functional $J^\nu : H\times\mathbb{R}^+ \to \mathbb{R}$ is given by

$$J^\nu(\theta,L) = \frac{1}{2L}\int_0^1 \dot{\theta}^2(s)ds + \nu L\,.$$

The gradient of J^ν is a pair of the form $\big(\nabla F, v_L\big) \in TH\times T\mathbb{R}^+ = H\times\mathbb{R}$, where

$$v_{_L} = -\frac{1}{2L^2}\int_0^1 \dot\theta^2(s)ds + \nu\,, \text{ and}$$

$$F(\theta) = \int_0^1 \frac{\dot\theta^2(s)}{2L}\,ds\,.$$

In [4] the general case with

$$F(x) = \int_a^b f(t,x(t),\dot x(t))\,dt$$

is developed. The quantities

$$E_x^f(t) = f_{\dot x}(t,x(t),\dot x(t)) - \int_a^t f_x(s,x(s),\dot x(s))\,ds\,, \text{ and}$$

$$W_x^f = \int_a^b f_x(t,x(t),\dot x(t))\,dt$$

are introduced. This general case specializes to the present context with

$$f\big(t,\theta(t),\dot\theta(t)\big) = \frac{\dot\theta^2(t)}{2L}\,.$$

In terms of the signed curvature k this yields $E_\theta^f = f_{\dot\theta} = \dot\theta\,/\,L = k$, and $W_\theta^f = 0$.

3.3 The unconstrained case and the metric I. It is shown in [4] (and in the appendix) that

$$\nabla_{_I}F(x)(t) = \int_a^t E_x^f(s)ds + \left(W_x^f - \beta\int_a^b E_x^f(s)ds\right)\frac{\alpha(t-a)+1}{\alpha\beta(b-a)+\alpha+\beta}\,.$$

This specializes to

$$\nabla_{_I}F(\theta)(t) = \frac{1}{L}(\theta(t)-\theta(0)) - \frac{\beta}{L}(\theta(1)-\theta(0))\frac{\alpha t+1}{\alpha\beta+\alpha+\beta}\,.$$

Theorem 4. *In metric I, $\nabla_{_I}F$ preserves rotation symmetries if $\alpha = 0$, and reflection symmetries if $\alpha = \beta$. It follows that both symmetries cannot be preserved simultaneously.*

Proof. Suppose $\theta(1-t) = \theta(t)$, then

$$\nabla_{_I}F(\theta)(1-t) = \frac{1}{L}(\theta(t)-\theta(0)) - \frac{\beta}{L}(\theta(1)-\theta(0))\frac{\alpha(1-t)+1}{\alpha\beta+\alpha+\beta}\,.$$

If $\nabla F(\theta)(t) = \nabla F(\theta)(1-t)$, then $\alpha(1-t) = \alpha t$ and $\alpha = 0$. Suppose $\theta(t) = -\theta(1-t)$, then

$$\nabla_{_I}F(\theta)(1-t) = \frac{1}{L}(-\theta(t)-\theta(0)) + \frac{2\theta(0)\beta}{L}\cdot\frac{\alpha(1-t)+1}{\alpha\beta+\alpha+\beta}\,.$$

If $\nabla F(\theta)(t) = -\nabla F(\theta)(1-t)$, then

$$1 = \frac{\alpha\beta+2\beta}{\alpha\beta+\alpha+\beta}\,.$$

It follows that $\alpha = \beta$, and hence both symmetries cannot be preserved since this implies $\alpha = \beta = 0$. □

Remark. Because the metric I fails to preserve both types of symmetries it is excluded from further considerations in this paper.

3.4 The unconstrained case and the metrics II and III. To simplify the notation write $sh(s) = \sinh(s)$ and $ch(s) = \cosh(s)$. The general cases of [4] are given by

$$\nabla_{II}F(x)(t) = \int_a^t E_x^f(s)\,\mathrm{ch}(t-s)ds + \left(W_x^f - \int_a^b E_x^f(s)\,\mathrm{sh}(b-s)ds\right)\frac{\mathrm{ch}(t-a)}{\mathrm{sh}(b-a)}, \text{ and}$$

$$\nabla_{III}F(x)(t) = \begin{cases} \int_p^t E_x^f(s)ds + W_x^f, & a \le t \le p \\ \int_p^t E_x^f(s)ds + W_x^f\,(t-p+1) & p \le t \le b \end{cases}.$$

They specialize to

$$\nabla_{II}F(\theta)(t) = \int_0^t k(s)\,\mathrm{ch}(t-s)ds - \int_0^1 k(s)\,\mathrm{sh}(1-s)ds\,\frac{\mathrm{ch}(t)}{\mathrm{sh}(1)}, \text{ and}$$

$$\nabla_{III}F(\theta)(t) = \frac{1}{L}(\theta(t) - \theta(p)).$$

Theorem 5. *In metric II, $\nabla_{II}F$ preserves both rotation and reflection symmetries.*

Proof. Assume that $\theta(t) = \theta(1-t)$. It follows that $\dot\theta(t) = -\dot\theta(1-t)$ so that $k(t) = -k(1-t)$. Now

$$\nabla_{II}F(\theta)(t) = \int_0^t k(s)ch(t-s)ds - \int_0^1 k(s)sh(1-s)ds\,\frac{ch(t)}{sh(1)}, \text{ and}$$

$$\begin{aligned}\nabla_{II}F(\theta)(1-t) &= \int_0^{1-t} k(s)ch(1-t-s)ds - \int_0^1 k(s)sh(1-s)ds\,\frac{ch(1-t)}{sh(1)} \\ &= -\int_t^1 k(s)ch(t-s)ds - \int_0^1 k(s)sh(1-s)ds\,\frac{ch(1-t)}{sh(1)}.\end{aligned}$$

It follows that if the following is an identity

$$\int_0^1 k(s)ch(t-s)ds = \int_0^1 k(s)sh(1-s)ds\,\frac{ch(t) - ch(1-t)}{sh(1)},$$

then $\nabla_{II}F(\theta)(t) = \nabla_{II}F(\theta)(1-t)$ holds. With the help of hyperbolic identities this leads to

$$ch(t)\int_0^1 k(s)ch(s)ds - sh(t)\int_0^1 k(s)sh(s)ds =$$

$$\left\{sh(1)\int_0^1 k(s)ch(s)ds - ch(1)\int_0^1 k(s)sh(s)ds\right\}\frac{ch(t) - ch(1-t)}{sh(1)}.$$

Collect terms and get

$$sh(1)ch(1-t)\int_0^1 k(s)ch(s)ds =$$

$$(\,sh(1)sh(t) - ch(1)ch(t) + ch(1)ch(1-t)\,)\int_0^1 k(s)sh(s)ds.$$

One more application of a hyperbolic identity produces

$$sh(1)\int_0^1 k(s)ch(s)ds = (\,ch(1)-1\,)\int_0^1 k(s)sh(s)ds\,,$$

which converts to

$$\int_0^1 k(s)sh(s)ds = ch(1)\int_0^1 k(s)sh(s)ds - sh(1)\int_0^1 k(s)ch(s)ds = -\int_0^1 k(s)sh(1-s)ds\,.$$

A substitution shows that this is indeed an identity when $k(s) = -k(1-s)$ holds. The case $\theta(t) = -\theta(1-t)$ is similar and the proof is omitted here for the sake of brevity. □

Theorem 6. *In metric III, $\nabla_{III}F$ preserves both rotation symmetries and symmetries of reflection if and only if $p = 1/2$. In metric III, $\nabla_{III}F$ always preserves rotation symmetries.*

Proof. Since

$$\nabla_{III}F(\theta)(t) = \frac{1}{L}(\theta(t) - \theta(p))\,,$$

it follows that $\theta(t) = \theta(1-t)$ implies $\nabla_{III}F(\theta)(t) = \nabla_{III}F(\theta)(1-t)$. If $\theta(t) = -\theta(1-t)$, then $\theta(1/2) = 0$, and $\nabla_{III}F(\theta)(t) + \nabla_{III}F(\theta)(1-t) = -2\theta(p)/L$. Hence

$$\nabla_{III}F(\theta)(t) = -\nabla_{III}F(\theta)(1-t)$$

for all θ such that $\theta(t) = -\theta(1-t)$ if and only if $p = 1/2$. □

3.5 Functionals in the constrained case. In the unconstrained case nothing is assumed about the endpoints. Suppose instead $(A,B) \in \mathbb{C}$ is given and

$$(A,B) = \gamma(1) = \gamma(1) - \gamma(0) = L\int_0^1 e^{i\theta(s)}ds\,.$$

It is useful to introduce two real-valued functionals $G_A, G_B : H \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$G_A(\theta,L) = L\int_0^1 \cos\theta(s)ds - A\,,\ G_B(\theta,L) = L\int_0^1 \sin\theta(s)ds - B\,.$$

This time

$$f_A\big(t,\theta(t),\dot\theta(t)\big) = L\cos\theta(t) - A\,,\ f_B\big(t,\theta(t),\dot\theta(t)\big) = L\sin\theta(t) - B\,,\text{ and}$$
$$\big(f_A\big)_\theta = -L\sin\theta\,,\ \big(f_B\big)_\theta = L\cos\theta\,,\ \big(f_A\big)_{\dot\theta} = \big(f_B\big)_{\dot\theta} = 0\,.$$

It follows that

$$E_\theta^{f_A}(t) = L\int_0^t \sin\theta(s)ds = y(t)\,,\ W_\theta^{f_A} = -B\,,\text{ and}$$
$$E_\theta^{f_B}(t) = -L\int_0^t \cos\theta(s)ds = -x(t)\,,\ W_\theta^{f_B} = A\,.$$

3.6 The metric II and the functionals in the constrained case.

Theorem 7. *In metric II, $\nabla_{II}G_A$ preserves rotational symmetries as well as symmetries of reflection, and $\nabla_{II}G_B$ preserves rotational symmetries.*

Proof. The two cases are similar so $\nabla_{II}G_B$ is left to the reader. The H component is given by

$$\nabla G_A^H(\theta)(t) = \int_0^t y(s)ch(t-s)ds - \left(B + \int_0^1 y(s)sh(1-s)ds \right) \frac{ch(t)}{sh(1)}.$$

Suppose $\theta(t) = \theta(1-t)$ so that

$$y(1-t) = L\int_0^{1-t} \sin\theta(s)ds = L\int_t^1 \sin\theta(s)ds = B - y(t).$$

It follows that

$$\nabla G_A^H(\theta)(t) - \nabla G_A^H(\theta)(1-t) =$$
$$\int_0^1 y(s)ch(t-s)ds - B\int_t^1 ch(t-s)ds - \left(B + \int_0^1 y(s)sh(1-s)ds \right) \frac{ch(t) - ch(1-t)}{sh(1)}\cdot$$

A substitution and some algebra show that

$$B + \int_0^1 y(s)sh(1-s)ds = Bch(1) - \int_0^1 y(s)sh(s)ds\,.$$

First collect the terms containing B and evaluate the remaining integral

$$-B\left\{ sh(1-t) + \frac{ch(1)}{sh(1)}(\,ch(t) - ch(1-t)) \right\} = -B\left\{ \frac{ch(1)-1}{sh(1)}\,ch(t) \right\}.$$

The rest of the terms simplify as follows

$$\int_0^1 y(s)ch(t-s)ds + \int_0^1 y(s)sh(s)ds\,\frac{ch(t) - ch(1-t)}{sh(1)} =$$
$$ch(t)\int_0^1 y(s)ch(s)ds - sh(t)\int_0^1 y(s)sh(s)ds + \int_0^1 y(s)sh(s)ds\,\frac{ch(t) - ch(1)ch(t) + sh(1)sh(t)}{sh(1)} =$$
$$\frac{1}{sh(1)}\left(ch(t)\int_0^1 y(s)\{\,sh(1)ch(s) - ch(1)sh(s)\}\,ds + ch(t)\int_0^1 y(s)sh(s)ds \right) =$$
$$\frac{ch(t)}{sh(1)}\int_0^1 y(s)\{\,sh(s) + sh(1-s)\}\,ds = -B\frac{(1-ch(1))}{sh(1)}\,ch(t)$$

A final consolidation of the terms proves that $\nabla G_A^H(\theta)(t) = \nabla G_A^H(\theta)(1-t)$.

Next, suppose that $\theta(t) = -\theta(1-t)$ so that

$$\int_{1/2}^1 \sin\theta(s)ds = -\int_{1/2}^0 \sin\theta(1-u)du = -\int_0^{1/2} \sin\theta(s)ds\,,$$

and hence $\mathrm{B} = 0$. Conclude that

$$y(1-t) = L\int_0^{1-t} \sin\theta(s)ds = -L\int_t^1 \sin\theta(s)ds = y(t) - B = y(t).$$

It follows that

$$\nabla G_A^H(\theta)(t) + \nabla G_A^H(\theta)(1-t) =$$
$$\int_0^t y(s)ch(t-s)ds + \int_0^{1-t} y(s)ch(1-t-s)ds - \int_0^1 y(s)sh(1-s)ds\,\frac{ch(t) + ch(1-t)}{sh(1)}$$

A pair of substitutions yields

$$\nabla G_A^H(\theta)(t) + \nabla G_A^H(\theta)(1-t) = \int_0^1 y(s)ch(t-s)ds - \int_0^1 y(s)sh(s)ds\,\frac{ch(t)+ch(1-t)}{sh(1)}\,.$$

Standard hyperbolic identities and a substitution show that

$$\nabla G_A^H(\theta)(t) + \nabla G_A^H(\theta)(1-t) =$$

$$ch(t)\int_0^1 y(s)ch(s)ds - sh(t)\int_0^1 y(s)sh(s)ds - \int_0^1 y(s)sh(s)ds\,\frac{ch(t)+ch(1)ch(t)-sh(1)sh(t)}{sh(1)} =$$

$$\frac{ch(t)}{sh(1)}\int_0^1 y(s)\{sh(1)ch(s)-ch(1)sh(s)\}ds - \int_0^1 y(s)sh(1-s)ds\,\frac{ch(t)}{sh(1)} = 0$$

It follows that $\nabla G_A^H(\theta)(t) = -\nabla G_A^H(\theta)(1-t)$. □

3.7 The metric II and symmetrized curve-straightening. The functional J^ν is defined on 'all' curves with initial point at the origin. The gradient vector field ∇J^ν is a 'section of the tangent bundle' of the space of all curves. There is a projection of this gradient vector field onto the various tangent bundles associated with endpoint constraints and a possible length constraint. The flow induced by the projected vector field is referred to as a curve-straightening flow. As is shown in [4], the projected vector field in the tangent bundle of constrained curves is not in general tangent to the spaces of symmetric curves. Moreover, the spaces of symmetric curves have normal bundles with infinite dimensional fibers and a corresponding non-trivial projection onto the tangent bundle of symmetric curves. These difficulties are circumvented by the following result.

Theorem 8. *The curve-straightening flow in metric II preserves both rotation symmetries and symmetries of reflection.*

Proof. Write $\nabla J^\nu = (\nabla F, v_L) \in TH \times T\mathbb{R}^+ = H \times \mathbb{R}$. Denote the projection of ∇F onto the space of curves with constrained endpoints by $\nabla^\pi F$. There are two real scalar fields λ_A and λ_B such that $\nabla F = \nabla^\pi F + \lambda_A \nabla G_A^H + \lambda_B \nabla G_B^H$. The tangency conditions are given by $DG_A^H(\nabla^\pi F) = 0$ and $DG_B^H(\nabla^\pi F) = 0$. Since all the vector fields before the projection preserve rotational symmetries, rotational symmetries are also preserved after the projection. In the case of symmetries of reflection things are more complicated. The trouble is created by the failure of ∇G_B^H to preserve symmetries of reflection. Consider the following linear system

$$\begin{bmatrix} DG_A^H(\theta)\nabla G_A^H(\theta) & DG_A^H(\theta)\nabla G_B^H(\theta) \\ DG_B^H(\theta)\nabla G_A^H(\theta) & DG_B^H(\theta)\nabla G_B^H(\theta) \end{bmatrix} \begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} = \begin{bmatrix} DG_A^H(\theta)\nabla F(\theta) \\ DG_B^H(\theta)\nabla F(\theta) \end{bmatrix}.$$

It follows that

$$\lambda_B = \big(DG_A^H(\theta)\nabla G_A^H(\theta)DG_B^H(\theta)\nabla F(\theta) - DG_B^H(\theta)\nabla G_A^H(\theta)DG_A^H(\theta)\nabla F(\theta)\big)/\det.$$

Since $\theta(t) = -\theta(1-t)$ it follows that

$$\nabla F(\theta)(t) = \int_0^t k(s)ch(t-s)ds - \int_0^1 k(s)sh(1-s)ds\,\frac{ch(t)}{sh(1)}\,.$$

Recall that $B = 0$ so that

$$\nabla G_A^H(\theta)(t) = \int_0^t y(s)ch(t-s)ds - \int_0^1 y(s)sh(1-s)ds\frac{ch(t)}{sh(1)}.$$

Also note that having $B = 0$ eliminates the term in DG_B^H corresponding to a variation of the length. Now

$$DG_B^H(\theta)\nabla F(\theta) = \int_0^1 \cos\theta(t)f(t)dt$$

where

$$f(t) = \int_0^t k(s)ch(t-s)ds - \int_0^1 k(s)sh(1-s)ds\frac{ch(t)}{sh(1)}.$$

Substitutions and hyperbolic identities yield

$$\begin{aligned} f(t)+f(1-t) &= \int_0^1 k(s)ch(t-s)ds - \int_0^1 k(s)sh(1-s)ds\frac{ch(t)+ch(1-t)}{sh(1)} \\ &= ch(t)\int_0^1 k(s)ch(s)ds - sh(t)\int_0^1 k(s)sh(s)ds - \int_0^1 k(s)sh(1-s)ds\frac{ch(t)+ch(1)ch(t)-sh(1)sh(t)}{sh(1)} \\ &= ch(t)\int_0^1 k(s)ch(s)ds - \left\{ sh(1)\int_0^1 k(s)ch(s)ds - ch(1)\int_0^1 k(s)sh(s)ds \right\}\frac{ch(t)+ch(1)ch(t)}{sh(1)} \\ &= \frac{ch(t)ch(1)}{sh(1)}\left\{ \int_0^1 k(s)sh(s)ds - \int_0^1 k(s)sh(1-s)ds \right\} = 0 \end{aligned}.$$

It follows that

$$DG_B^H(\theta)\nabla F(\theta) = \int_0^1 \cos\theta(t)f(t)dt = 0.$$

Using the same exact reasoning it follows that

$$DG_B^H(\theta)\nabla G_A^H(\theta) = \int_0^1 \cos\theta(t)g(t)dt = 0,$$

where

$$g(t) = \int_0^t y(s)ch(t-s)ds - \int_0^1 y(s)sh(1-s)ds\frac{ch(t)}{sh(1)},$$

and $y(s)$ has the same symmetry properties as $k(s)$. With these two results it follows that $\lambda_B = 0$. □

3.8 The metric III and the functionals in the constrained case.

Theorem 9. *If $p = 1/2$ in the metric III, then $\nabla_{III}G_A$ preserves rotational symmetries as well as symmetries of reflection and $\nabla_{III}G_B$ preserves rotational symmetries but not symmetries of reflection.*

Proof. The H components are given by

$$\nabla G_A^H(\theta)(t) = \begin{cases} \int_{1/2}^t y(s)ds - B & 0 \le t \le 1/2 \\ \int_{1/2}^t y(s)ds - B(t+1/2) & 1/2 \le t \le 1 \end{cases},$$

$$\nabla G_B^H(\theta)(t) = \begin{cases} -\int_{1/2}^t x(s)ds + A & 0 \le t \le 1/2 \\ -\int_{1/2}^t x(s)ds + A(t+1/2) & 1/2 \le t \le 1 \end{cases}.$$

Suppose $\theta(t) = \theta(1-t)$ so that

$$y(1-t) = L\int_0^{1-t} \sin\theta(s)ds = L\int_t^1 \sin\theta(s)ds = B - y(t)\,,$$

$$x(1-t) = L\int_0^{1-t} \cos\theta(s)ds = L\int_t^1 \cos\theta(s)ds = A - x(t)\,,$$

$$\int_{1/2}^{1-t} y(s)ds = -\int_{1/2}^t y(1-u)du = -B(t-1/2) + \int_{1/2}^t y(s)ds\,, \text{ and}$$

$$\int_{1/2}^{1-t} x(s)ds = -\int_{1/2}^t x(1-u)du = -A(t-1/2) + \int_{1/2}^t x(s)ds\,.$$

It follows that

$$\nabla G_A^H(\theta)(1-t) = \begin{cases} \int_{1/2}^t y(s)ds - B(t+1/2) & 0 \le 1-t \le 1/2 \\ \int_{1/2}^t y(s)ds - B(t-1/2) - B(3/2-t) & 1/2 \le 1-t \le 1 \end{cases},$$

and

$$\nabla G_B^H(\theta)(1-t) = \begin{cases} -\int_{1/2}^t x(s)ds + A(t+1/2) & 0 \le 1-t \le 1/2 \\ -\int_{1/2}^t x(s)ds + A(t-1/2) + A(3/2-t) & 1/2 \le 1-t \le 1 \end{cases}.$$

One more simplification proves that $\nabla G_A^H(\theta)(t) = \nabla G_A^H(\theta)(1-t)$ and

$$\nabla G_B^H(\theta)(t) = \nabla G_B^H(\theta)(1-t)\,.$$

Next, suppose that $\theta(t) = -\theta(1-t)$. As in the proof of Theorem 7 conclude that $\mathrm{B} = 0$ and that $y(1-t) = y(t)$. With

$$\int_{1/2}^{1-t} y(s)ds = -\int_{1/2}^t y(1-u)du = B(t-1/2) - \int_{1/2}^t y(s)ds = -\int_{1/2}^t y(s)ds$$

it follows that $\nabla G_A^H(\theta)(t) = -\nabla G_A^H(\theta)(1-t)$. Similarly

$$\nabla G_B^H(\theta)(1-t) = \begin{cases} -\int_{1/2}^t x(s)ds + A(t-1/2) + A & 0 \le 1-t \le 1/2 \\ -\int_{1/2}^t x(s)ds + A(t-1/2) + A(3/2-t) & 1/2 \le 1-t \le 1 \end{cases},$$

and hence $\nabla G_B^H(\theta)(t) = \nabla G_B^H(\theta)(1-t)$. □

3.9 The metric III and symmetrized curve-straightening. The reasoning is in this case similar to the case of the metric II.

Theorem 10. *The curve-straightening flow with* $p = 1/2$ *in metric III preserves both rotation symmetries and symmetries of reflection.*

Proof. Use the same notation as in the proof of Theorem 8. Since all the vector fields before the projection preserve rotational symmetries, rotational symmetries are also preserved after the projection. In the case of symmetries of reflection things are more complicated. The trouble is again created by the failure of ∇G_B^H to preserve symmetries of reflection. Recall that $B = 0$ and

$$\lambda_B = \big(DG_A^H(\theta)\nabla G_A^H(\theta) DG_B^H(\theta)\nabla F(\theta) - DG_B^H(\theta)\nabla G_A^H(\theta) DG_A^H(\theta)\nabla F(\theta)\big)/\det .$$

Since $\theta(t) = -\theta(1-t)$ implies $\theta(1/2) = 0$ it must be that $\nabla F(\theta)(t) = \theta(t)/L$. It is also true that

$$\nabla G_A^H(\theta)(t) = \int_{1/2}^{t} y(s)ds .$$

Now

$$DG_B^H(\theta)\nabla F(\theta) = \int_0^1 \theta(s)\cos\theta(s)ds = 0$$

because

$$\int_{1/2}^{1} \theta(s)\cos\theta(s)ds = -\int_0^{1/2} \theta(s)\cos\theta(s)ds .$$

It is also true that

$$DG_B^H(\theta)\nabla G_A^H(\theta) = \int_0^1 \left\{ L\cos\theta(t) \int_{1/2}^{t} y(s)ds \right\} dt = 0$$

because a substitution shows that

$$\int_{1/2}^{1} \left\{ L\cos\theta(t) \int_{1/2}^{t} y(s)ds \right\} dt = -\int_0^{1/2} \left\{ L\cos\theta(t) \int_{1/2}^{t} y(s)ds \right\} dt .$$

With these two results it follows that $\lambda_B = 0$. □

3.10 Examples. In all examples the length is unconstrained and $\nu = 0$ so there is no 'length penalty'. The figures depict flowtime evolution by providing 'snapshots' along the negative gradient trajectory. The timeinterval between consecutive curves is not constant or even monotone. In all cases the initial curve has very high elastic energy. This produces rapid changes along the flow. To capture this a tiny interval is used at the start of the flow. In Figures 5 and 6 the timeinterval is also shortened considerably later as the curves again exhibit significant changes. In the first three figures the initial curve consists of a straight segment followed by a clockwise circle and a finishing straight segment. The Figure 1 illustrates how the traditional curve straightening flow breaks symmetries. This figure familiarizes the reader with a three-dimensional representation of the evolution in time. In Figure 1 metric III is used with $p = 0$, in

Figure 2 metric III is used with $p = 1/2$, and in Figure 3 metric II is used. In both of the last two cases the symmetry is preserved. Figure 4 uses a different initial curve to illustrate more clearly the geometric difference between metric III with $p = 1/2$ and metric II. The initial curve is in both cases the same tiny spiral with no symmetries. Figure 5 illustrates 'spinning' in the traditional curve straightening flow. The initial curve is given by three loops of a 'circular figure eight'. Both metric II and metric III with $p = 1/2$ avoid spinning. The flow in metric III is shown in Figure 6. All the examples considered 'converge' to one loop of the 'Euler figure eight'. A current topic of investigation is to determine exactly which of the 'critical points at infinity' are limits of negative gradient trajectories. The 'double covered figure eight' has emerged as a significant test case since it exposes a lack of longterm stability in current numerical implementations.

4. APPENDIX

As promised, this appendix supplies the proofs of the general gradient formulas derived in [4]. Given any continuously differentiable functional $F : H \to \mathbb{R}$, the directional derivative $DF(x) : H \to \mathbb{R}$ and the gradient $\nabla F(x) \in H$ satisfy $\langle \nabla F(x), v \rangle_H = DF(x)v$. Here $\langle , \rangle_H$ is the inner product in H.

Theorem. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable. Denote the partial derivatives with respect to the second and third argument by f_x, and $f_{\dot{x}}$. Consider only functions $x \in H$ such that $f(t, x(t), \dot{x}(t))$, $f_x(t, x(t), \dot{x}(t))$, $f_{\dot{x}}(t, x(t), \dot{x}(t)) \in L^1[a, b]$. Put*

$$E_x^f(t) = f_{\dot{x}}(t, x(t), \dot{x}(t)) - \int_a^t f_x(s, x(s), \dot{x}(s))\, ds\,, \text{ and } W_x^f = \int_a^b f_x(t, x(t), \dot{x}(t))\, dt\,.$$

The following is true for

$$F(x) = \int_a^b f(t, x(t), \dot{x}(t))\, dt$$

(I) The gradient with respect to

$$\langle w, v \rangle_I = \alpha w(a)v(a) + \beta w(b)v(b) + \int_a^b \dot{w}(t)\dot{v}(t)dt$$

is given by

$$\nabla_I F(x) = \int_a^t E_x^f(s)ds + \left(W_x^f - \beta \int_a^b E_x^f(s)ds \right) \frac{\alpha(t-a)+1}{\alpha\beta(b-a)+\alpha+\beta}\,.$$

(II) The gradient with respect to

$$\langle w, v \rangle_{II} = \int_a^b w(t)v(t)dt + \int_a^b \dot{w}(t)\dot{v}(t)dt$$

is given by

$$\nabla_{II}F(x) = \int_a^t E_x^f(s)\,\mathrm{ch}(t-s)ds + \left(W_x^f - \int_a^b E_x^f(s)\,\mathrm{sh}(b-s)ds \right) \frac{\mathrm{ch}(t-a)}{\mathrm{sh}(b-a)}\,.$$

(III) The gradient with respect to

$$\langle w, v \rangle_{III} = w(p)v(p) + \int_a^b \dot{w}(t)\dot{v}(t)dt$$

is given by

$$\nabla_{III}F(x) = \begin{cases} \displaystyle\int_p^t E_x^f(s)ds + W_x^f, & a \le t \le p \\ \displaystyle\int_p^t E_x^f(s)ds + W_x^f\,(t-p+1) & p \le t \le b \end{cases}.$$

Proof. It suffices to verify the relation $\langle \nabla F(x), v \rangle = DF(x)v$ in the three cases. First, a standard computation using integration by parts shows that

$$DF(x)v = \int_a^b \left(f_x\,(t,x(t),\dot{x}(t))\,v(t) + f_{\dot{x}}\,(t,x(t),\dot{x}(t))\,\dot{v}(t) \right) dt = v(b) \int_a^b f_x\,(t,x(t),\dot{x}(t))\,dt +$$
$$\int_a^b \left(f_{\dot{x}}\,(t,x(t),\dot{x}(t)) - \int_a^t f_x\,(s,x(s),\dot{x}(s))\,ds \right) \dot{v}(t)dt = W_x^f v(b) + \int_a^b E_x^f(t)\dot{v}(t)dt.$$

If the metric is $\langle , \rangle_I$, then

$$\left\langle \nabla_I F(x), v \right\rangle_I = \frac{\alpha}{\alpha\beta(b-a)+\alpha+\beta} \left(W_x^f - \beta \int_a^b E_x^f(s)ds \right) v(a) +$$
$$\beta \left(\int_a^b E_x^f(s)ds + \frac{\alpha(b-a)+1}{\alpha\beta(b-a)+\alpha+\beta} \left(W_x^f - \beta \int_a^b E_x^f(s)ds \right) \right) v(b) +$$
$$\int_a^b \left(E_x^f(t) + \frac{\alpha}{\alpha\beta(b-a)+\alpha+\beta} \left(W_x^f - \beta \int_a^b E_x^f(s)ds \right) \right) \dot{v}(t)dt.$$

The last integral simplifies to

$$\int_a^b E_x^f(t)\dot{v}(t)dt + \frac{\alpha}{\alpha\beta(b-a)+\alpha+\beta} \left(W_x^f - \beta \int_a^b E_x^f(s)ds \right) (v(b)-v(a)).$$

The terms with the factor $v(a)$ disappear. Collect the terms with the factor $v(b)$ and rearrange to see that $v(b)$ is multiplied by

$$\left(\beta - \frac{\alpha\beta^2(b-a)+\beta^2}{\alpha\beta(b-a)+\alpha+\beta} - \frac{\alpha\beta}{\alpha\beta(b-a)+\alpha+\beta} \right) \int_a^b E_x^f(t)dt +$$
$$\left(\frac{\alpha\beta(b-a)+\beta}{\alpha\beta(b-a)+\alpha+\beta} + \frac{\alpha}{\alpha\beta(b-a)+\alpha+\beta} \right) W_x^f = W_x^f.$$

It follows that

$$\left\langle \nabla_I F(x), v \right\rangle_I = W_x^f v(b) + \int_a^b E_x^f(t)\dot{v}(t)dt = DF(x)v\,.$$

When the metric is $\langle , \rangle_{II}$, the first main term of the second integral is rewritten with the help of

Leibniz' rule

$$\left\langle \nabla_{II} F(x), v \right\rangle_{II} = \int_a^b \left(\int_a^t E_x^f(s) \operatorname{ch}(t-s) ds + \left(W_x^f - \int_a^b E_x^f(s) \operatorname{sh}(b-s) ds \right) \frac{\operatorname{ch}(t-a)}{\operatorname{sh}(b-a)} \right) v(t) dt +$$
$$\int_a^b \left(\left(E_x^f(t) + \int_a^t E_x^f(s) \operatorname{sh}(t-s) ds \right) + \left(W_x^f - \int_a^b E_x^f(s) \operatorname{sh}(b-s) ds \right) \frac{\operatorname{sh}(t-a)}{\operatorname{sh}(b-a)} \right) \dot{v}(t) dt.$$

Integrate the second main term of the second integral by parts

$$\int_a^b \left(W_x^f - \int_a^b E_x^f(s) \operatorname{sh}(b-s) ds \right) \frac{\operatorname{sh}(t-a)}{\operatorname{sh}(b-a)} dv(t) =$$
$$\left(W_x^f - \int_a^b E_x^f(s) \operatorname{sh}(b-s) ds \right) v(b) - \int_a^b \left(W_x^f - \int_a^b E_x^f(s) \operatorname{sh}(b-s) ds \right) \frac{\operatorname{ch}(t-a)}{\operatorname{sh}(b-a)} v(t) dt.$$

Also integrate the first main term of the first integral by parts again using Leibniz' rule

$$\int_a^b \left(\int_a^t E_x^f(s) \operatorname{ch}(t-s) ds \right) v(t) dt = \int_a^b v(t) d \left(\int_a^t E_x^f(s) \operatorname{sh}(t-s) ds \right) =$$
$$v(b) \int_a^b E_x^f(s) \operatorname{sh}(b-s) ds - \int_a^b \left(\int_a^t E_x^f(s) \operatorname{sh}(t-s) ds \right) \dot{v}(t) dt$$

Once both of these computations are combined, it follows that

$$\left\langle \nabla_{II} F(x), v \right\rangle_{II} = W_x^f v(b) + \int_a^b E_x^f(t) \dot{v}(t) dt = DF(x) v \,.$$

Finally,

$$\left\langle \nabla_{III} F(x), v \right\rangle_{III} = W_x^f v(p) + \int_a^p E_x^f(s) \dot{v}(s) ds + \int_p^b \left(E_x^f(s) + W_x^f \right) \dot{v}(s) ds$$
$$= W_x^f v(b) + \int_a^b E_x^f(s) \dot{v}(s) ds$$

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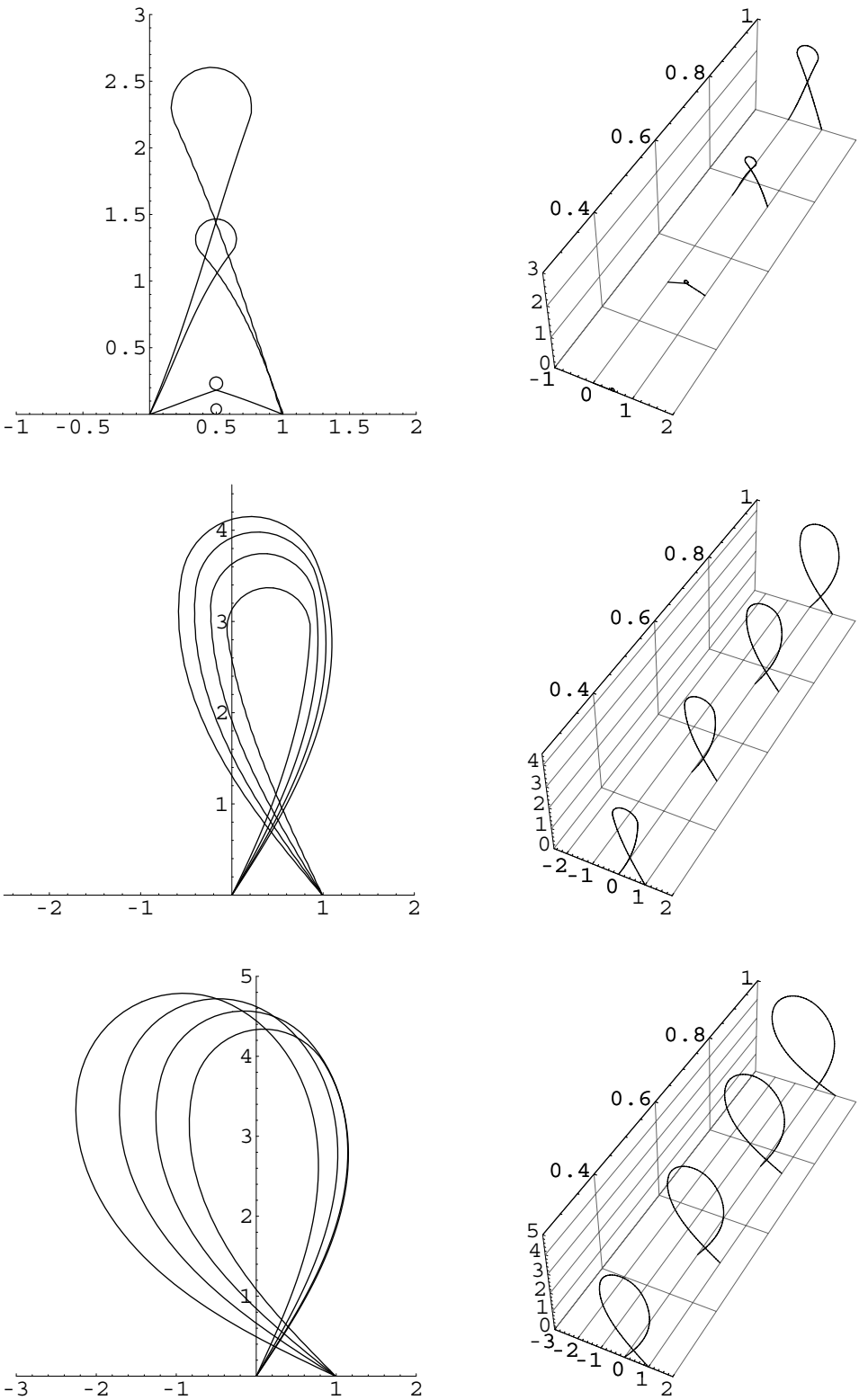
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Figure 1.
Unconstrained length using no length penalty in Metric III with p=0.



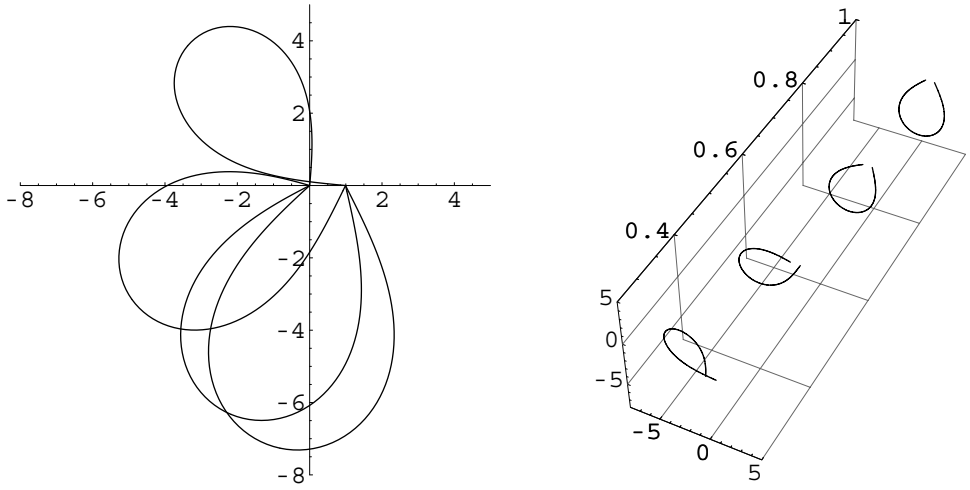


Figure 2.
Jnconstrained length using no length penalty in Metric III with p=1/2

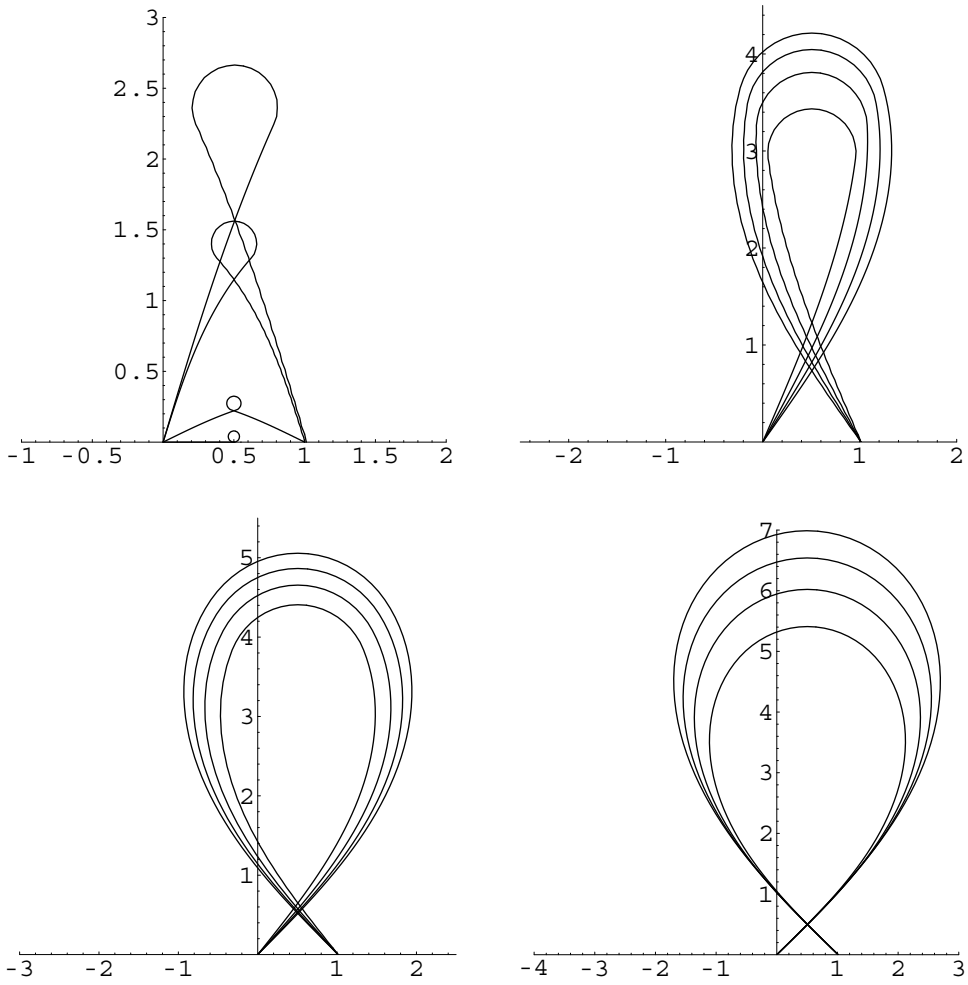


Figure 3.
Unconstrained length using no length penalty in Metric II.

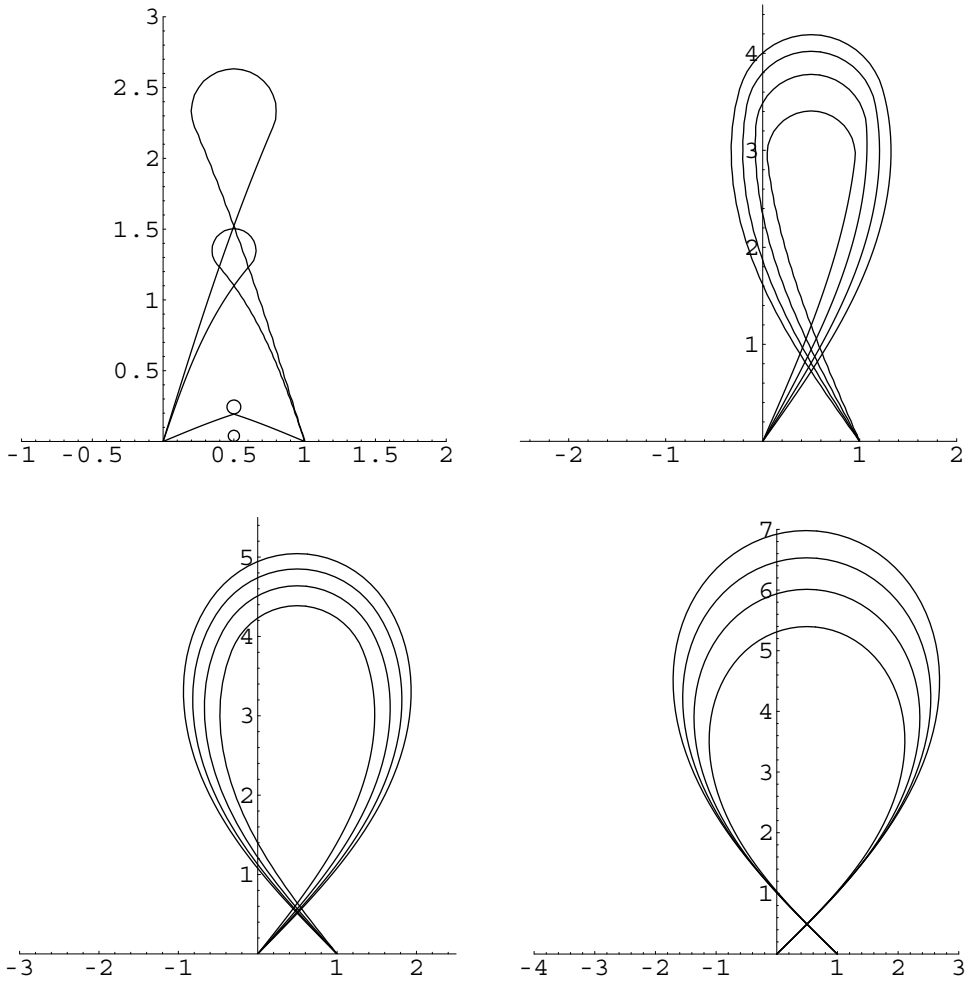
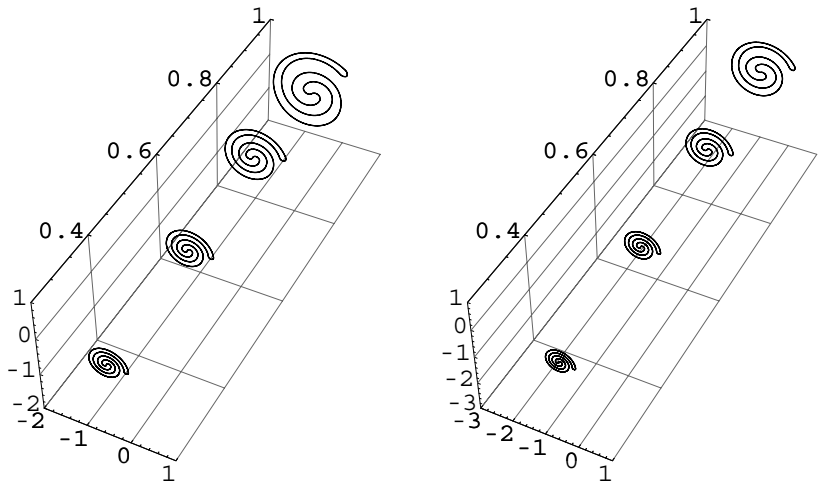
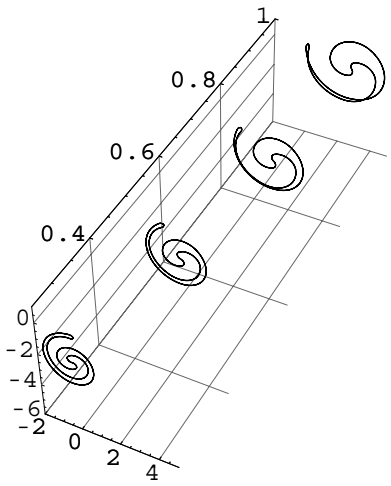
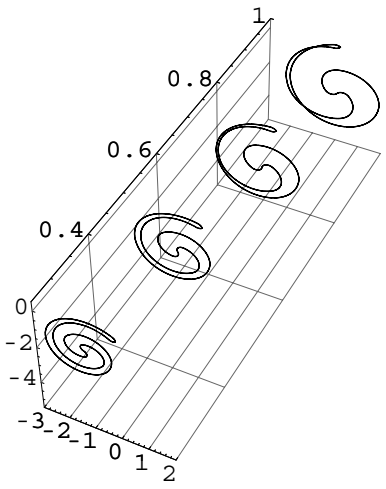
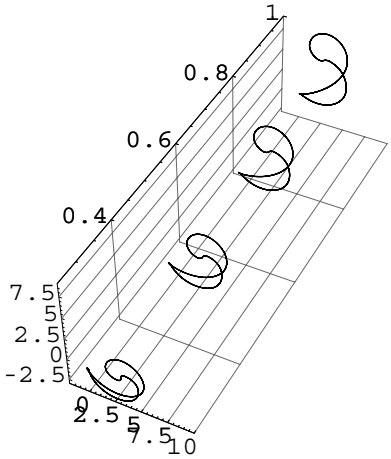


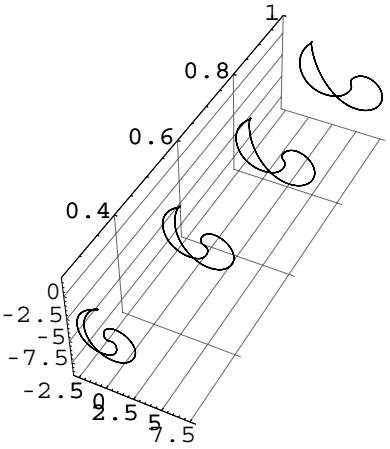
Figure 4.
Compare metric III with p=1/2 and metric II.

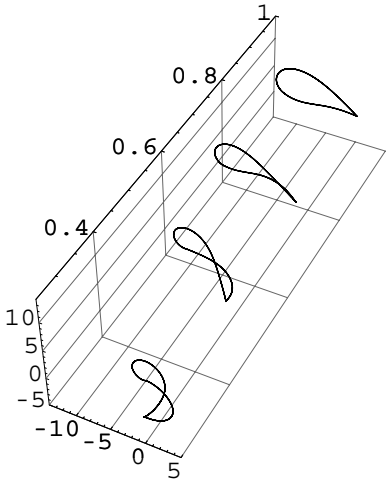



1
0.8
0.6
0.4
0
-2
-4
-6
-2
0
2
4


1
0.8
0.6
0.4
0
-2
-4
-3
-2
-1
0
1
2


1
0.8
0.6
0.4
7.5
5
2.5
0
-2.5
0
2.5
5
7.5
10


1
0.8
0.6
0.4
0
-2.5
-5
-7.5
-2.5
0
2.5
5
7.5


1
0.8
0.6
0.4
10
5
0
-5
-10
-5
0
5

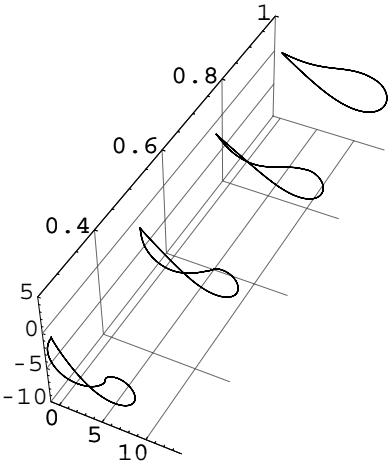

1
0.8
0.6
0.4
5
0
-5
-10
0
5
10

Figure 5.
Metric III with p=0 illustrating spinning.

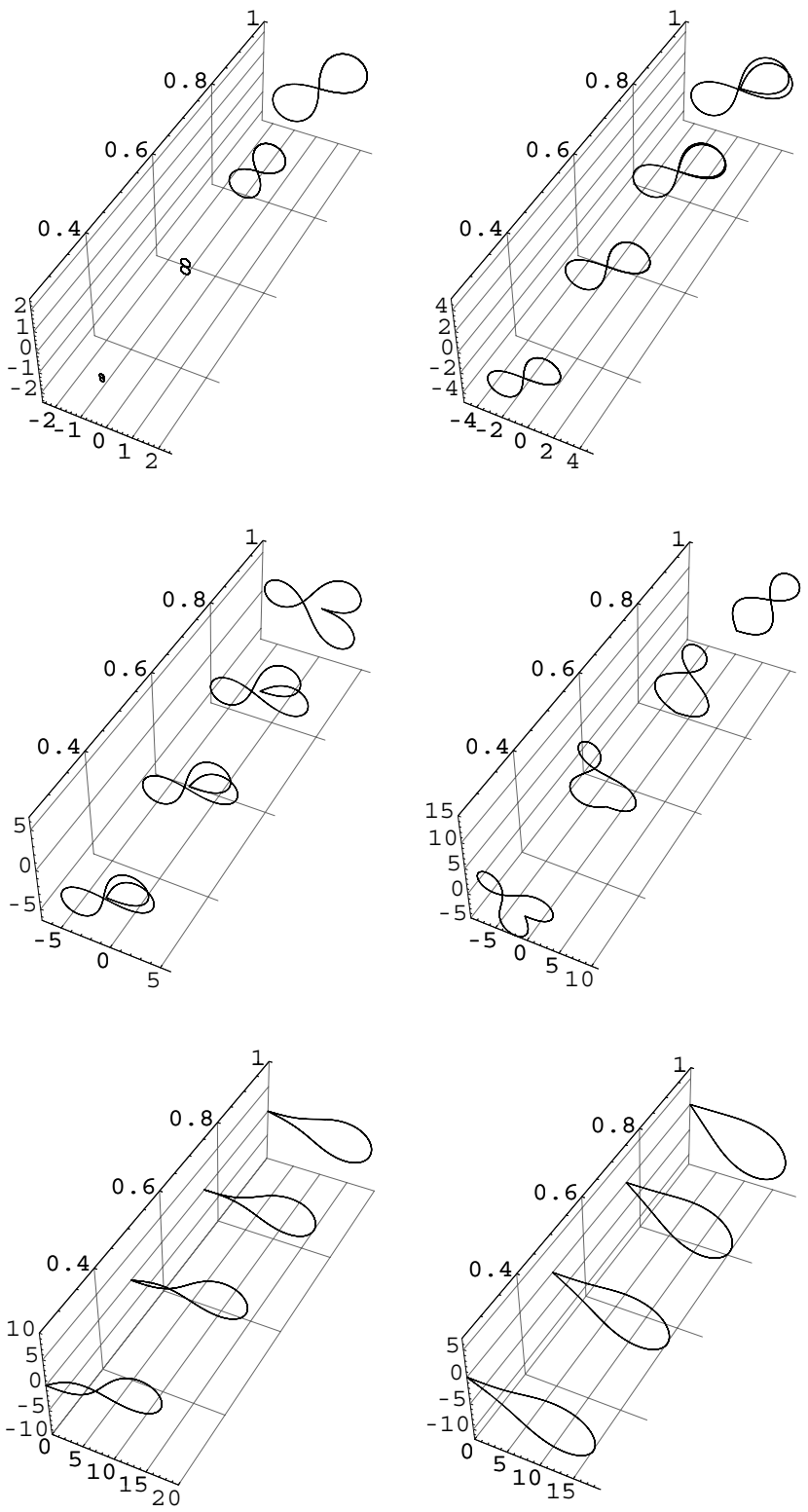


Figure 6.
Metric III with p=1/2 illustrating removal of spinning.

