Exam 1

1. Use the simplex algorithm to solve the problem:

$$\max 2x_1 + 3x_2 + 6x_3 \text{ when } \begin{cases} 2x_1 + 3x_2 + x_3 \le 10 \\ x_1 + x_2 + 2x_3 \le 8 \\ 2x_2 + 3x_3 \le 6 \\ x_1, x_2, x_3 \ge 0 \end{cases}.$$

Make sure that each step employed is the step suggested by the simplex algorithm.

The initial tableau is given by

$$\begin{array}{cccccc|c} 2 & 3 & 1 & 1 & 0 & 0 & 10 \\ 1 & 1 & 2 & 0 & 1 & 0 & 8 \\ 0 & 2 & 3 & 0 & 0 & 1 & 6 \\ \hline 2 & 3 & 6 & 0 & 0 & 0 & 0 \end{array}.$$

The next tableau is given by

$$\begin{array}{cccccc|c} 2 & 4/3 & 0 & 1 & 0 & -1/3 & 8 \\ 1 & -1/3 & 0 & 0 & 1 & -2/3 & 4 \\ 0 & 2/3 & 1 & 0 & 0 & 1/3 & 2 \\ \hline 2 & -1 & 0 & 0 & 0 & -2 & -12 \end{array}.$$

The final tableau is given by

$$\begin{array}{cccccc|c} 0 & 2 & 0 & 1 & -2 & 1 & 0 \\ 1 & -1/3 & 0 & 0 & 1 & -2/3 & 4 \\ 0 & 2/3 & 1 & 0 & 0 & 1/3 & 2 \\ \hline 0 & -1/3 & 0 & 0 & -2 & -2/3 & -20 \end{array}.$$

The solution is given by $\hat{x}_1 = 4, \hat{x}_2 = 0, \hat{x}_3 = 2$.

2. **Convert** the following problem to the form of a standard dual problem. Apply the simplex method to the corresponding primal problem. Get the minimizing $\hat{\lambda}^T = \begin{bmatrix} \hat{\lambda}_1 & \hat{\lambda}_2 & \hat{\lambda}_3 \end{bmatrix}$ from the final tableau.

The converted problem is given by

$$\min 18\lambda_1 + 20\lambda_2 + 2\lambda_3 \text{ when } \begin{cases} -3\lambda_1 + 5\lambda_2 + 2\lambda_3 \ge -4 \\ 6\lambda_1 - 8\lambda_3 \ge 9 \\ \lambda_1, \lambda_2, \lambda_3 \ge 0 \end{cases}.$$

The corresponding primal problem is given by

$$\max -4x_1 + 9x_2 \text{ when } \begin{cases} -3x_1 + 6x_2 \le 18 \\ 5x_1 \le 20 \\ 2x_1 - 8x_2 \le 2 \\ x_1, x_2 \ge 0 \end{cases}.$$

The initial tableau is given by

$$\begin{array}{ccccc|c} -3 & 6 & 1 & 0 & 0 & 18 \\ 5 & 0 & 0 & 1 & 0 & 20 \\ 2 & -8 & 0 & 0 & 1 & 2 \\ \hline -4 & 9 & 0 & 0 & 0 & 0 \end{array}.$$

The next tableau is given by

$$\begin{array}{ccccc|c} -1/2 & 1 & 1/6 & 0 & 0 & 3 \\ 5 & 0 & 0 & 1 & 0 & 20 \\ -2 & 0 & 4/3 & 0 & 1 & 26 \\ \hline 1/2 & 0 & -3/2 & 0 & 0 & -27 \end{array}.$$

The final tableau is given by

$$\begin{array}{ccccc|c} 0 & 1 & 1/6 & 1/10 & 0 & 5 \\ 1 & 0 & 0 & 1/5 & 0 & 4 \\ 0 & 0 & 4/3 & 2/5 & 1 & 34 \\ \hline 0 & 0 & -3/2 & -1/10 & 0 & -29 \end{array}.$$

The solution is given by $\hat{\lambda}_1 = 3/2, \hat{\lambda}_2 = 1/10, \hat{\lambda}_3 = 0$.

3. **Complete** the **first phase** of the two-phase method to determine a feasible point of a problem with constraints:

$$\begin{cases} x_1 - x_2 + x_3 \le -1 \\ -2x_1 + x_2 + x_3 \le -2 \\ x_1, x_2, x_3 \ge 0 \end{cases}.$$

Make sure that each step employed is the step suggested by the first phase of the two-phase method.

The augmented system is given by

$$\begin{aligned} x_1 - x_2 + x_3 + y_1 &= -1 \\ -2x_1 + x_2 + x_3 + y_2 &= -2 \end{aligned}.$$

The auxiliary system is given by

$$\begin{aligned} -x_1 + x_2 - x_3 - y_1 + z_1 &= 1 \\ 2x_1 - x_2 - x_3 - y_2 + z_2 &= 2 \end{aligned}.$$

The initial tableau is given by

$$\begin{array}{ccccccc|c} -1 & 1 & -1 & -1 & 0 & 1 & 0 & 1 \\ 2 & -1 & -1 & 0 & -1 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 0 & -2 & -1 & -1 & 0 & 0 & 3 \end{array}.$$

The next tableau is given by

$$\begin{array}{ccccccc|c} 0 & 1/2 & -3/2 & -1 & -1/2 & 1 & 1/2 & 2 \\ 1 & -1/2 & -1/2 & 0 & -1/2 & 0 & 1/2 & 1 \\ \hline 0 & 1/2 & -3/2 & -1 & -1/2 & 0 & -1/2 & 2 \end{array}.$$

The final tableau is given by

$$\begin{array}{ccccccc|c} 0 & 1 & -3 & -2 & -1 & 2 & 1 & 4 \\ 1 & 0 & -2 & -1 & -1 & 1 & 1 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{array}.$$

The feasible point is given by $x_1 = 3, x_2 = 4, x_3 = 0$.

4. The problem:

$$\max -700x_1 - 900x_2 + 500x_3 + 1000x_4 \text{ when } \begin{cases} -x_1 + x_3 \le 6 \\ -x_1 + x_4 \le 5 \\ -x_2 + x_3 \le 4 \\ -x_2 + x_4 \le 8 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

has a maximum when $\hat{x}_1 = 3, \hat{x}_2 = 0, \hat{x}_3 = 4, \hat{x}_4 = 8$. Use **nothing but complementary slackness** to find the optimal solution $\hat{\lambda}$ of the dual problem.

The dual problem is given by

$$\min 6\lambda_1 + 5\lambda_2 + 4\lambda_3 + 8\lambda_4 \text{ when } \begin{cases} -\lambda_1 - \lambda_2 \ge -700 \\ -\lambda_3 - \lambda_4 \ge -900 \\ \lambda_1 + \lambda_3 \ge 500 \\ \lambda_2 + \lambda_4 \ge 1000 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0 \end{cases}.$$

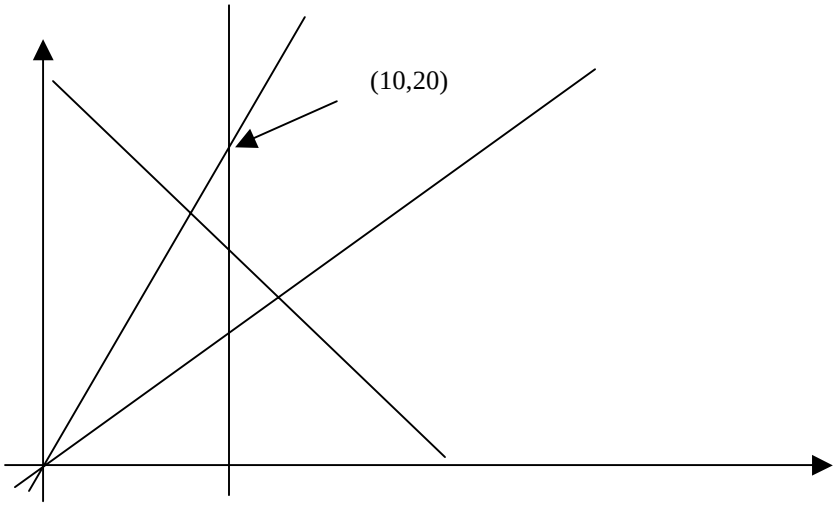
Complementary slackness: $\hat{x}_1 > 0 \Rightarrow \hat{\lambda}_1 + \hat{\lambda}_2 = 700$, $\hat{x}_3 > 0 \Rightarrow \hat{\lambda}_1 + \hat{\lambda}_3 = 500$, $\hat{x}_4 > 0 \Rightarrow \hat{\lambda}_2 + \hat{\lambda}_4 = 1000$, $\hat{y}_1 > 0 \Rightarrow \hat{\lambda}_1 = 0$.

It follows that the dual solution is given by $\hat{\lambda}_1 = 0, \hat{\lambda}_2 = 700, \hat{\lambda}_3 = 500, \hat{\lambda}_4 = 300$.

5. Consider the problem:

$$\max x_1 + 2x_2 \text{ when } \begin{cases} x_1 \le 10 \\ x_1 \le x_2 \\ 2x_1 \ge x_2 \\ x_1 \ge 27 - x_2 \\ x_1, x_2 \ge 0 \end{cases}.$$

(a) Use the **graphical method** to determine the solution.



(b) Convert the problem to standard form and use complementary slackness to get the solution to the dual problem.

The converted problem is given by

$$\max x_1 + 2x_2 \text{ when } \begin{cases} x_1 \le 10 \\ x_1 - x_2 \le 0 \\ -2x_1 + x_2 \le 0 \\ -x_1 - x_2 \le -27 \\ x_1, x_2 \ge 0 \end{cases}.$$

The corresponding dual problem is given by

$$\min 10\lambda_1 - 27\lambda_4 \text{ when } \begin{cases} \lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 \ge 1 \\ -\lambda_2 + \lambda_3 - \lambda_4 \ge 2 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0 \end{cases}.$$

With $\hat{x}_1 = 10, \hat{x}_2 = 20$ there is slack in the second and fourth inequality. It follows that $\hat{\lambda}_2 = \hat{\lambda}_4 = 0$. Both inequalities in the dual problem must in fact be satisfied with equality, and it follows that $\hat{\lambda}_3 = 2$ and also $\hat{\lambda}_1 = 5$.

(c) Use the Verification Theorem to prove that your solution is indeed optimal.

Both objectives evaluate to 50 so both solutions are optimal.