Exam 2

1. Consider a primal problem with

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix},$$

and a final tableau given by

$$\left[\begin{array}{cccccc|c} 0 & 3 & 0 & 1 & -2 & 1 & 0 \\ 1 & -1/3 & 0 & 0 & 1 & -2/3 & 4 \\ 0 & 2/3 & 1 & 0 & 0 & 1/3 & 2 \\ \hline 0 & -1/3 & 0 & 0 & -2 & -2/3 & -20 \end{array}\right].$$

(a) Determine b.

The basic variables are y_1, x_1, x_3 and it follows that

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since

$$B^{-1}b = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix},$$

it must be that

$$b = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}.$$

(b) Determine c.

Since $c^T - c_B^T B^{-1} A = \begin{bmatrix} 0 & -1/3 & 0 \end{bmatrix}$ and $c_B^T B^{-1} = \begin{bmatrix} 0 & 2 & 2/3 \end{bmatrix}$, it follows that

$$c^T = \begin{bmatrix} 0 & 2 & 2/3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1/3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 10/3 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1/3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \end{bmatrix}$$

2. Consider the problem

$$\max\ 3x_1 + x_2 + 3x_3 \text{ when } \begin{cases} 2x_1 + x_2 + x_3 \le 2 \\ x_1 + 2x_2 + 3x_3 \le 5 \\ 2x_1 + 2x_2 + x_3 \le 6 \\ x_1, x_2, x_3 \ge 0 \end{cases},$$

with a final tableau given by

$$\begin{array}{cccccc|c} 1 & 1/5 & 0 & 3/5 & -1/5 & 0 & 1/5 \\ 0 & 3/5 & 1 & -1/5 & 2/5 & 0 & 8/5 \\ 0 & 1 & 0 & -1 & 0 & 1 & 4 \\ \hline 0 & -7/5 & 0 & -6/5 & -3/5 & 0 & -27/5 \end{array}.$$

(a) The third slack variable has value 4. It follows that b_3 can be reduced from 6 to 2 with the same optimal solution. Is it possible to reduce b_3 by more than 4 units and still have the same basic variables in the optimal solution? If so, by how much?

Since

$$B^{-1}(b + \Delta b) = B^{-1}b + B^{-1}\begin{bmatrix} 0 \\ 0 \\ \Delta b_3 \end{bmatrix} = B^{-1}b + \begin{bmatrix} 0 \\ 0 \\ \Delta b_3 \end{bmatrix},$$

so only the third entry changes, it follows that $4 + \Delta b_3 \ge 0$ and hence $\Delta b_3 \ge -4$. The answer is no.

(b) The first slack variable has value 0. Is it possible to reduce b_1 and still have the same basic variables in the optimal solution? If so, by how much?

This time

$$B^{-1}(b + \Delta b) = B^{-1}b + B^{-1}\begin{bmatrix} \Delta b_1 \\ 0 \\ 0 \end{bmatrix} = B^{-1}b + \begin{bmatrix} \frac{3}{5}\Delta b_1 \\ -\frac{1}{5}\Delta b_1 \\ -\Delta b_1 \end{bmatrix},$$

and

$$\begin{aligned} &\tfrac{1}{5} + \tfrac{3}{5}\Delta b_1 \ge 0 \\ &\tfrac{8}{5} - \tfrac{1}{5}\Delta b_1 \ge 0 \\ &4 - \Delta b_1 \ge 0 \end{aligned}.$$

It follows that $-\frac{1}{3} \le \Delta b_1 \le 4$, and hence the answer is yes, by a third of a unit.

(c) The dual solution is unique. By how much should c_2 increase to make it worthwhile to start producing the product represented by x_2.

With the objective $3x_1 + c_2x_2 + 3x_3$ the second constraint in the dual changes to $\lambda_1 + 2\lambda_2 + 2\lambda_3 \ge c_2$. With the current dual solution this gives $\frac{6}{5} + 2\frac{3}{5} + 0 = \frac{12}{5} \ge c_2$. The increase must be bigger than $\frac{12}{5} - 1 = \frac{7}{5}$.

3. Consider the problem

$$\max\ 3x_1 + x_2 + 3x_3 \text{ when } \begin{cases} 2x_1 + x_2 + x_3 \le 2 \\ x_1 + 2x_2 + 3x_3 \le 5 \\ 2x_1 + 2x_2 + x_3 \le 6 \\ x_1, x_2, x_3 \ge 0 \end{cases},$$

with a final tableau given by

$$\left.\begin{array}{cccccc|c} 1 & 1/5 & 0 & 3/5 & -1/5 & 0 & 1/5 \\ 0 & 3/5 & 1 & -1/5 & 2/5 & 0 & 8/5 \\ 0 & 1 & 0 & -1 & 0 & 1 & 4 \\ \hline 0 & -7/5 & 0 & -6/5 & -3/5 & 0 & -27/5 \end{array}\right. .$$

(a) What is the admissible range of variation in c_2 that retains the same optimal solution?

Since x_2 is a non-basic variable the only concern is the inequality

$$\left(c^T + \begin{bmatrix} 0 & \Delta c_2 & 0 \end{bmatrix}\right) - c_B^T B^{-1} A \le \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

This simplifies to $\begin{bmatrix} 0 & \Delta c_2 & 0 \end{bmatrix} \le \begin{bmatrix} 0 & 7/5 & 0 \end{bmatrix}$. In other words, c_2 can be increased by no more than 7/5.

(b) Determine the admissible region for $\Delta c_1, \Delta c_3$ that retains the same optimal solution.

Both variables are basic and the system of inequalities

$$\begin{aligned} &\left(c^T + \begin{bmatrix} \Delta c_1 & 0 & \Delta c_3 \end{bmatrix}\right) - \left(c_B^T + \begin{bmatrix} \Delta c_1 & \Delta c_3 & 0 \end{bmatrix}\right) B^{-1} A \le \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \\ &-\left(c_B^T + \begin{bmatrix} \Delta c_1 & \Delta c_3 & 0 \end{bmatrix}\right) B^{-1} \le \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{aligned},$$

must be satisfied. This simplifies as

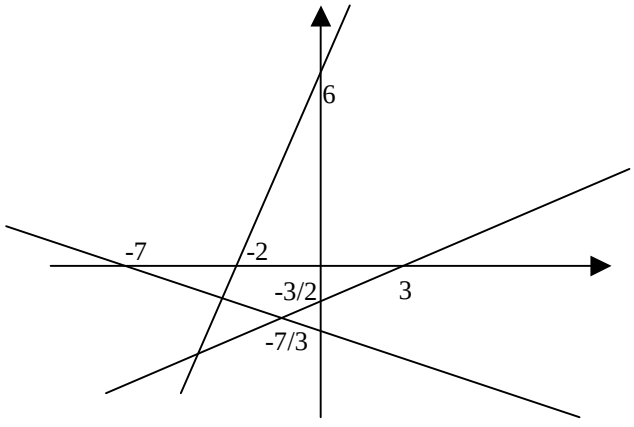
$$\begin{aligned} &\begin{bmatrix} \Delta c_1 & 0 & \Delta c_3 \end{bmatrix} - \begin{bmatrix} \Delta c_1 & \Delta c_3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/5 & 0 \\ 0 & 3/5 & 1 \\ 0 & 1 & 0 \end{bmatrix} \le \begin{bmatrix} 0 & 7/5 & 0 \end{bmatrix} \\ &-\begin{bmatrix} \Delta c_1 & \Delta c_3 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \le \begin{bmatrix} 6/5 & 3/5 & 0 \end{bmatrix} \end{aligned},$$

or

$$\begin{aligned} &\begin{bmatrix} \Delta c_1 & 0 & \Delta c_3 \end{bmatrix} - \begin{bmatrix} \Delta c_1 & \frac{1}{5}\Delta c_1 + \frac{3}{5}\Delta c_3 & \Delta c_3 \end{bmatrix} \le \begin{bmatrix} 0 & 7/5 & 0 \end{bmatrix} \\ &\begin{bmatrix} -\frac{3}{5}\Delta c_1 + \frac{1}{5}\Delta c_3 & \frac{1}{5}\Delta c_1 - \frac{2}{5}\Delta c_3 & 0 \end{bmatrix} \le \begin{bmatrix} 6/5 & 3/5 & 0 \end{bmatrix} \end{aligned}.$$

The admissible region is given by

$$\begin{aligned} -\Delta c_1 - 3\Delta c_3 &\le 7 \\ -3\Delta c_1 + \Delta c_3 &\le 6 \\ \Delta c_1 - 2\Delta c_3 &\le 3 \end{aligned}.$$



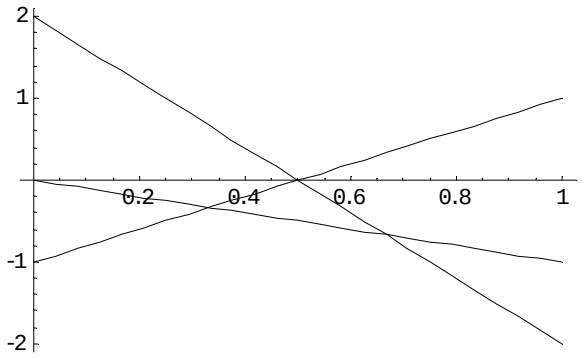
(c) Neither c_1 nor c_3 is sensitive.

Let the payoff matrix of a game be given by

$$\begin{bmatrix} 1 & 2 & -2 & -1 & 2 & 1 \\ -1 & 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 1 & -1 \end{bmatrix},$$

(a) Determine both players' optimal strategy.

The row-player's third plan dominates the second plan. The column player eliminates options 1,2 and 5 since they are dominated by option 6. Assume the row-player chooses option 1 with probability x. The expected payoffs for the column-player are $-2x + 2(1-x) = 2 - 4x$, $-x$, $2x - 1$ for plan 3,4,6 respectively.



The best worst-case scenario is if the row-player chooses x such that $-x = 2x - 1$, i.e., $x = 1/3$. The row-player should play according to $\begin{bmatrix} 1/3 & 0 & 2/3 \end{bmatrix}$ and the value of the game is $-1/3$.

(b) Determine the column-player's optimal strategy.

The column-player only considers plan 4 and 6. Let y be the probability to use option 4. It must be that $0 \cdot y - 1 \cdot (1-y) = -1/3$ and hence $y = 2/3$. The column-player should play according to $\begin{bmatrix} 0 & 0 & 0 & 2/3 & 0 & 1/3 \end{bmatrix}$.

(c) Who does the game favor? The column-player.