Topology

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Get in shape!

Special topic.

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Linnér

More topology.

Special topic.

More on separation.

Proof of Urysohn's lemma

Special topic.

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Now apply f to the leftmost and the rightmost term.

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To be safe always state the definition you are using.

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Given the disjoint closed subsets A, B it is of interest to produce a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

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One may be even more ambitious and ask for a continuous function that has $f(a) = 0$ if and only if $a \in A$, and $f(b) = 1$ if and only if $b \in B$.

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A year later, Tychonoff replaced normal with regular.

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If X has two points, this is in fact Sierpiński's topology.

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The trouble is that it is not Hausdorff, so Urysohn's theorem is not applicable.

Proof of Urysohn's lemma

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It is well-known that P is countable.

The initial goal is to define a collection of open sets U_p such that $p \in P$ and $U_p \in \tau$ with the property $\overline{U}_p \subset U_{p'}$ whenever $p < p'$ with $p, p' \in P$.

Arrange P as a sequence $\{p_0, p_1, p_2, p_3, \dots\}$ with $p_0 = 0$, $p_1 = 1$, and subsequent entries according to your favorite enumeration scheme.

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Since the space is normal there are some $U, V \in \tau$ such that $A \subset U$ and $B \subset V$ with $U \cap V = \emptyset$.

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Recycle the previous line of reasoning with $\overline{U}_p$ in place of A and $U_{p'}^c$ in place of B.

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This completes the induction argument.

For $p \in \mathbb{Q}$ with $p < 0$ it is expedient to set $U_p = \emptyset$, and when $p > 1$ let $U_p = X$.

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Suppose $x_\lambda \in X$ is a convergent net and $x_\lambda \to x_\infty \in X$.

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Similarly, let $t_k \in P$ satisfy $t_k > r$ and $t_k \to r$.

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This in turn shows that $f(x_\lambda) \to f(x_\infty)$ as a net.

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Armed with Urysohn's lemma, there is for each point $x \in X$ and $U \in \tau$ such that $x \in U$ a continuous function $f : X \to [0, 1]$ so that f is positive at x and zero outside U.

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Suppose there is a countable topological base $\{B_k\}_{k=1}^{\infty}$.

Given a pair $B_k, B_{k'}$ such that $\overline{B_{k'}} \subset B_k$, there are functions $f_{k,k'} : X \to [0,1]$ such that $f(B_k^c) = \{0\}$ and $f(\overline{B_{k'}}) = \{1\}$.

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Now consider the map $F(x) = (f_1(x), f_2(x), \dots)$.

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Is it open, or closed?

Let $A \subset X$ be closed.

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It follows that $f_K(x_{\lambda_k}) = 0$, but then y_λ is not eventually in $\pi_K^{-1}((\frac{1}{2} f_K(x_\infty), 1])$, which is an open set that contains y_∞.

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Since Y is compact there is yet another convergent subnet $y \circ g$ where $g : D_2 \to D_1$ and the limit is y_∞.

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There is a never ending composition.

Recall the topologist's sine curve

$$Z = \left\{ (t, \sin(\frac{\pi}{t})) \mid t \in (0, 1] \right\} \subset \mathbb{R}^2.$$

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Let $Y = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$ and observe that $X = \overline{Z} = Z \cup Y \subset \mathbb{R}^2$.

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Why is X not path-connected?

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Since $\{(x,y) \in \mathbb{R}^2 \mid x > 0\}$ is open in $\mathbb{R}^2$ it follows that Z is open in X and Y is closed in X.

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A sequence $a_k \in A$ such that $a_k \to \alpha$ has $\gamma(a_k) \in Y$ so by the continuity of γ it must be that $\gamma(\alpha) \in Y$ and hence $\alpha \in A$.

Let $B \subset \mathbb{R}$ be given by

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Suppose that $\beta > 0$.

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Observe that both $U = \{(x,y) \in \mathbb{R}^2 \mid x < c\}$, and $V = \{(x,y) \in \mathbb{R}^2 \mid x > c\}$ are open subsets of $\mathbb{R}^2$.

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The last possibility is $\beta = 0$.

Now consider

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As $t_k \to \alpha^+$, oscillation prevents convergence, and hence it is impossible for γ to be continuous.

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A topological space with a countable base is said to be second countable.

If at each point $x \in X$ there is a countable collection $\{U_k\}_{k=1}^{\infty}$ of open sets containing x so that to each open set U containing x there is some U_k such that $U_k \subset U$, then X is said to be first countable.

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What is the relationship between second countable spaces and separable spaces?

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Compact metric spaces with an open cover must have a Lebesgue number $\delta > 0$ such that any subset of diameter less than δ is contained in some member of the open cover.

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When a group structure on a set coexists with a topology in this fashion, the space is said to be a topological group.

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The maps $L_g : G \to G$ given by $L_g(x) = gx$, and $R_g : G \to G$ given by $R_g(x) = xg$ are homeomorphisms.

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The word 'homogeneous' is often used to indicate this.

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$$\begin{bmatrix} 1-2t & \sqrt{2t(1-t)} \\ -\sqrt{2t(1-t)} & 1-2t \end{bmatrix}$$

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On the one hand, H is a subgroup of G since $(g_0 g_1)^{-1} = g_1^{-1} g_0^{-1} = g_1^T g_0^T = (g_0 g_1)^T$.

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It follows that H is closed but then G is not connected so it must be that $G = H$.

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Let $\epsilon > 0$ be such that $B_{x_\infty}(\epsilon) \subset U_{\lambda_\infty}$.

If $a_k \in A_k$, then it is possible to arrange so that

$$d(a_k, x_\infty) \leq d(a_k, x_{k'}) + d(x_{k'}, x_\infty) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

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The second term is controlled by the demands of convergence, and the convergent subsequence guarantees an endless supply of indices.

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It is possible to define θ on $[0, \frac{\delta}{2})$ as a continuous function with values in $(-\pi, \pi)$.

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Proceeding in this fashion it is possible to define θ on all of $[0, 1]$.

Category Theory

Topology

Linnér

More topology.

Special topic.

More on separation.

Proof of Urysohn's lemma

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There is also an arrow $1_A \in [A, A]$ such that $f \circ 1_A = f$ and $1_A \circ g = g$ for $f \in [A, B]$ and $g \in [C, A]$, respectively.

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A contravariant functor S reverses the direction of arrows so that $Sf \in [SA', SA]$ and $S(g \circ f) = Sf \circ Sg$.

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The homotopy functors bring groups into play and the homology functors engage abelian groups.

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Given two loops γ and η, there is a third loop σ based at p defined by

$$\sigma(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \eta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

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This convention is opposite the one used for composition, so $\gamma * \eta$ means first traverse γ then η, whereas $f \circ g$ means first evaluate g then evaluate f.

The fact that σ is continuous is a consequence of a more general result known as the glueing lemma.

The relationship between σ and the other two loops is sometimes written more succinctly as $\sigma = \gamma * \eta$, or using some other operator symbol in place of $*$.

This convention is opposite the one used for composition, so $\gamma * \eta$ means first traverse γ then η, whereas $f \circ g$ means first evaluate g then evaluate f.

The 'operator' $*$ lacks basic algebraic properties primarily due to differences in the parameterization.

A homotopy between two continuous functions $f, g : [0,1] \to X$ is a continuous function

$$H : [0,1] \times [0,1] \to X$$

such that $H(s,0) = f(s)$ and $H(s,1) = g(s)$ for all $s \in [0,1]$.

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The choice to let the second component of $[0,1] \times [0,1]$ correspond to the evolution of the homotopy so that $H(s,t)$ as a function of s is an 'intermediate function between' f and g is optional, and some authors prefer the first component instead.

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A base-point preserving homotopy H satisfies $H(0,t) = H(1,t) = p$ for all $t \in [0,1]$.

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For each $s \in [0,1]$ let $I = G * H$, and conclude that $f \sim h$.

It follows that $\sim$ is an equivalence relation on the collection of paths in X defined on $[0,1]$.

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The last crucial piece is the fact that $([\gamma][\eta])[\varphi] = [\gamma]([\eta][\varphi])$, and therefore the associative law is satisfied.

Putting all of this together one has a group, the fundamental group, which is denoted by $\pi_1(X, p)$.

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The stereo-graphic projection from $S^2 - \{(0,0,1)\}$ to $\mathbb{R}^2$ nonetheless shows that S^2 and $\mathbb{R}^2$ are not homeomorphic since $\mathbb{R}^2$ with one point removed will have non-trivial fundamental group.

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Let $C(X; Y)$ be space all continuous map from X to Y with the topology induced by the sub-base generated by compact K in X and open V in Y

$$W_{K,V} = \{f \in C(X; Y) \mid f(K) \subset V\}.$$

Now $f \sim g$ if and only if f, g belong to the same path component in $C(X; Y)$ when X is metrizable, or locally compact and Hausdorff.

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In odd-dimensional spheres define $v(x) = (-y_1, \ldots, -y_k, x_1, \ldots, x_k)$.

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It follows that a non-vanishing continuous vectorfield exists if and only if the dimension is odd.

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If either X or Y are contractible, then a continuous f is homotopic to a constant map.

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Now define $H : S^n \times [0,1] \to X$ by $H(s,t) = \overline{H}(s,t)$.

Conversely, observe that $b = (1-t)s$ for some $s \in S^n$ and $t \in [0,1]$, so define $\overline{f}((1-t)s) = \overline{\overline{H}}(s,t)$ using the homotopy to the constant.

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Let S^3 be $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and consider the map $(z, w) \mapsto (2z\bar{w}, |z|^2 - |w|^2)$.

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This is a map into S^2 as a subset of $\mathbb{C} \times \mathbb{R}$.

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Examples of finite groups include $\pi_{14}(S^2)$, which is isomorphic to $\mathbb{Z}_{84} \times \mathbb{Z}_2^2$, $\pi_{10}(S^4)$, which is isomorphic $\mathbb{Z}_{24} \times \mathbb{Z}_3$, and $\pi_{59}(S^3)$, which is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_2^2 \times \mathbb{Z}_9 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{31}$.

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The stable groups are better understood than the 'unstable' ones, but their general pattern is still 'illusive'.

Freudenthal showed that $\pi_{k+n}(S^n)$ is independent of n when $n > k+1$.

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π_{59}^S is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_2^2 \times \mathbb{Z}_9 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{31}$ seen already.

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This may be written as $aba^{-1}b^{-1} = 1$, where 1 represents the unit in the group.

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Presentations are not unique and it is in general difficult to tell if two presentations correspond to non-isomorphic groups.

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The result is a group known as the free product $G * H$ of G and H.

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The group

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If $U \cap V$ is path-connected and the base-point is in the intersection, then

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V).$$

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Let V be an open disk that contains the center point.

Observe that $\pi_1(V) \cong \{0\}$, $\pi_1(U \cap V) \cong \mathbb{Z}$, and $\pi_1(U)$ is free on two generators a, b.

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Since one factor in the amalgamated product is trivial and the other is free on two symbols, it is only necessary to add the relation $aba^{-1}b^{-1} = 1$.

Since the relation implies the group is abelian, one concludes that the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, as expected.

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The fundamental group of the projective plane is $\mathbb{Z}_2$.

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Observe how both the projective plane and the Klein bottle have a copy of $\mathbb{Z}_2$, which is an indication that there is a Möbius strip present.

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$$B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

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If $A \subset X$ and there is some continuous $r : X \to A = r(X)$ such that $r(a) = a$ when $a \in A$, then A is said to be a retract of X.

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Schauder Fixed Point Theorem

If X is a nonempty closed convex subset of a Banach space, then every continuous $f : X \to X$ with compact image has a fixed point.

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$\pi_1\left(X \times Y, (x, y)\right) \cong \pi_1\left(X, x\right) \times \pi_1\left(Y, y\right)$, so for the n-dimensional torus $\pi_1\left(\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}\right) \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}}$.

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If there is a homotopy from a continuous $f : S^{2n} \to S^{2n}$ to the identity map, then f has a fixed point.

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Suppose a continuous $f : S^n \to \mathbb{R}^n$ satisfies $f(-x) = -f(x)$, then $f(\hat{x}) = 0$ for some $\hat{x} \in S^n$.

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Schoenflies Theorem

If $A^1 \subset \mathbb{R}^2$ is homeomorphic to S^1, then $\mathbb{R}^2 - A^1$ is not connected. In fact, $\mathbb{R}^2 - A^1$ has exactly two components one bounded and one unbounded. A^1 is their common boundary.

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Suppose $U \subset S^n$ and $V \subset S^n$ are homeomorphic. If U is open, then V is open. The same is true if S is replaced by $\mathbb{R}$.

Suppose $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open and homeomorphic, then $n = m$.

If $n < m$, then every continuous function $f : S^n \to S^m$ is homotopic to a constant.

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The homology of the two-dimensional torus is given by $H_0\left(S^1 \times S^1\right) \cong \mathbb{Z}$, $H_1\left(S^1 \times S^1\right) \cong \mathbb{Z} \times \mathbb{Z}$, $H_2\left(S^1 \times S^1\right) \cong \mathbb{Z}$, and $H_k\left(S^1 \times S^1\right) \cong \{0\}$ if $k > 2$.

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The homology of the Klein bottle is given by $H_0\left(K\right) \cong \mathbb{Z}$, $H_1\left(K\right) \cong \mathbb{Z}_2 \times \mathbb{Z}$, and $H_k\left(K\right) \cong \{0\}$ if $k > 1$.

The homology of complex projective space is given by

$$H_k\left(\mathbb{C}P^n\right) \cong \begin{cases} \mathbb{Z}, & k = 0, 2, 4, \cdots, 2n \\ 0, & \text{otherwise} \end{cases}.$$

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The homology of quaternionic projective space is given by

$$H_k\left(QP^n\right) \cong \begin{cases} \mathbb{Z}, & k = 0, 4, 8, \cdots, 4n \\ 0, & \text{otherwise} \end{cases}.$$

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If $n \geq 1$, then $H_k(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z} \times \mathbb{Z} & k = n \\ \{0\} & \text{otherwise} \end{cases}$.

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