Topology

Anders Linnér

alinner@math.niu.edu

Get in shape!

Special topic.

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A norm is a map $|| \; || : X \to [0, \infty)$.

It coexists with scalar multiplication, $||ax|| = |a|||x||$, vector addition, $||x + y|| \leq ||x|| + ||y||$, and $||x|| = 0$ if and only if $x = 0$.

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With this in mind, let X^* be the collection of all continuous linear maps from X to $\mathbb{R}$.

The members of X^* are known as continuous linear functionals.

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An attempted remedy is to introduce the weak topology associated with the collection X^*.

The norm-continuous linear functionals will still be continuous in the weak topology, and this new topology may have 'fewer' open sets.

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Write $x_k \to^w x_\infty$ when $\langle x_k, y \rangle \to \langle x_\infty, y \rangle$ for each $y \in X$.

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This provides a more concrete description of the weak topology.

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Since $||x_k|| = 1$, it is not true that $x_k \to 0$, so weak convergence does not imply norm-convergence.

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The linear map $(x_1, x_2, x_3, \dots) \mapsto (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots) : l^2 \to l^2$, has non-closed image.

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Nonetheless, they are the same for convex sets, so in particular so are affine subsets, which includes the linear subspaces.

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Now $\overline{B_x(r)} \subset B_x[r]$, but the discrete topology shows that the inclusion may be strict!

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It follows that $B_x[r]$ is both weakly closed and weakly bounded in a normed space.

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The weak topology is induced by a metric if and only if the space is finite-dimensional.

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The open sets are unions of finite intersections of sets $\phi_\alpha^{-1}(U_\alpha)$, where U_α is open in Y_α.

A net $x_\lambda \in X$ converges to $x_\infty \in X$ if and only if $\phi_\alpha(x_\lambda)$ converges to $\phi_\alpha(x_\infty)$ in Y_α for each α.

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The weak topology on X is the topology of pointwise convergence on A.

Given $\epsilon > 0$ and $a \in A$, let
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Suppose $A = [0, 1]$ and consider the set containing only the zero function.

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The topology of pointwise convergence on $[0, 1]$ is not given by a metric.

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Observe that $B_x(r)$ is open and convex.

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It follows that $x \in \phi^{-1}((-\infty, a))$ and $y \in \phi^{-1}((a, \infty))$, with both sets weakly open.

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There is an associated norm, the operator norm, defined on X^* given by $||\phi|| = \sup_{B_0[1]} |\phi(x)|$.

One aspect of the Hahn-Banach theorem asserts the existence of a continuous linear map $\phi : X \to \mathbb{R}$ such that $\phi^{-1}(\{a\})$ separates $B_x(r)$ and $\{y\}$ for some $a \in \mathbb{R}$.

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Look at $|\phi(x_k) - \phi(x_\infty)| = |\phi(x_k - x_\infty)| \leq ||\phi||||x_k - x_\infty||$.

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Apparently, norms are weakly lower semi-continuous.

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The lesson is that it matters when a closure is applied.

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As seen, the map Φ is in fact an isometric isomorphism.

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Now

$$\overline{\Phi(B_X)}^{\text{weak}^*} = B_{X^{**}},$$

where the weak* topology has been put to work in X^{**}.

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Eberlein-Šmulian

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Let X be a normed space and consider the subspace topology on B_{X^*} induced by the weak* topology on X^*.

This topology is metrizable if and only if X is separable.

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This is particularly remarkable in light of the fact that the weak* topology on X^* is not metrizable when X is infinite-dimensional.

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The book ‘An introduction to Banach Space Theory’ by Robert E. Megginson, Springer 1998, describes these topologies, and a few others, particularly well.

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Assume the boundary $\partial\Omega$ is a surface on which there is electrical charge described by a function $f : \partial\Omega \to \mathbb{R}$.

The question is if there is an extension $\bar{u}$ of u so that $\bar{u} \mid_{\partial\Omega} = f$ and $\Delta u = 0$ in Ω, the celebrated classical Dirichlet problem.

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Consider the greatest lower bound, which here is non-negative.

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The general existence question is much harder and still the subject of much research.

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Of the three that specifically relate to the calculus of variations, the 19th and 20th are now consider resolved but the 23d is not.