APPROXIMATION OF $\lfloor n\alpha + s \rfloor$ AND THE ZERO OF $\{n\alpha + s\}$

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Abstract

Let α and β be positive real numbers and s a real number satisfying $0 \leq s < 1$. Let $\lfloor x \rfloor$ denote the greatest integer $\leq x$, and $\{x\} = x - \lfloor x \rfloor$. Define $\Psi(\alpha, \beta; s)$ to be the least positive integer n such that $\lfloor n\alpha + s \rfloor \neq \lfloor n\beta + s \rfloor$. When $s = 0$, a simple explicit formula for Ψ is given, and otherwise more complicated formulas are obtained. A natural characterzation of the approximation properties of intermediate convergents (including convergents) without reference to the apparatus of continued fractions is given. When α is irrational, a formula for finding the least $n \in \mathbb{Z}^+$ such that $\{n\alpha + s\} = 0$ is presented. A new characterization of the sequence $\lfloor n\alpha \rfloor$, for $n \geq 1$ is found and $\lfloor n\alpha + s \rfloor$, for $n \geq 1$ is also characterized in a different way.

1 Introduction

When α is a real number it is sometimes useful to replace the sequence $\lfloor n\alpha \rfloor$ by the simpler sequence $\lfloor na/b \rfloor$, where a/b is a rational approximation to α. (Here $\lfloor x \rfloor$ is the greatest integer $\leq x$.) Knowing how well a/b approximates α, the problem is to find how well $\lfloor na/b \rfloor$ approximates $\lfloor n\alpha \rfloor$. For example, in [3], for the purpose of approximating a sum, Davison showed that

$$\left\lfloor n\frac{1+\sqrt{5}}{2} \right\rfloor = \left\lfloor n\frac{F_{i+1}}{F_i} \right\rfloor, \qquad \text{for } 0 \leq n < F_{i+2}, \tag{1}$$

where F_i, $(i \geq 0)$, is the i'th term of the Fibonacci sequence, with $F_0 = 0$ and $F_1 = 1$. Similarly, to approximate a sum in [1], and for different reasons in [11], it was proved that

$$\lfloor n\alpha \rfloor = \left\lfloor n\frac{p_i}{q_i} \right\rfloor, \qquad \text{for } 0 \leq n < q_{i+1}, \tag{2}$$

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where p_i/q_i is the i'th convergent of the simple continued fraction of α. In (1) the interval for n is optimal; at $n = F_{i+2}$, the two sides are not equal. In (2), however, the interval $0 \leq n < q_{i+1}$ is clearly not always optimal, (1) being a counterexample.

This raises the question of what is the largest interval for which the equation in (2) holds? More generally, given any rational a/b, up to which n does $\lfloor na/b \rfloor = \lfloor n\alpha \rfloor$? Still more generally we may ask: For any $\beta \neq \alpha$, up to what $n \in \mathbb{Z}^+$ does $\lfloor n\beta \rfloor = \lfloor n\alpha \rfloor$? In fact it is this latter question we address and answer in this paper. Moreover this will be done for the more general sequence $\lfloor n\alpha + s \rfloor$. Thus we make the following definition. Let α and β be positive real numbers and $0 \leq s < 1$. Define $\Psi(\alpha, \beta; s)$ to be the least $n \in \mathbb{Z}^+$ for which $\lfloor n\alpha + s \rfloor \neq \lfloor n\beta + s \rfloor$. If $\alpha = \beta$, write $\Psi(\alpha, \beta; s) = \infty$. If the value of s is understood, we will write $\Psi(\alpha, \beta)$ for $\Psi(\alpha, \beta; s)$. The central problem of this paper is the evaluation of Ψ in terms of the simple continued fractions of α and β. Theorem 7.2 gives the evaluation.

Another question which looks similar to and is in fact intimately connected to the evaluation of Ψ is the problem of finding the zero of $\{n\alpha + s\}$. Specifically, when α is irrational, find the integer n, when it exists, for which $\{n\alpha + s\} = 0$. When α is rational, this problem reduces to solving a linear congruence, so our problem may be thought of as a generalization of this familiar one. In the course of evaluating Ψ, we will show the connection between these two problems and solve both.

Another natural outcome of this investigation is a new intrinsic characterization of the rational approximation properties of intermediate convergents. Additionally we give a new characterization of the sequence $\lfloor n\alpha \rfloor$, $(n \geq 1)$. This result is contained in §6.

Before going on to the homogeneous case in §6, we give conventions, basic properties and lemmas in sections 2, 3 and 4 respectively. §5 gives the construction of the approximating sequences of intermediate convergents and mediants to be used in the main theorems.

2 Conventions

Let $\alpha > 0$ be a given real number. Write the simple continued fraction for α as $\alpha = [\alpha_0; \alpha_1, \alpha_2, ...]$. Denote the convergents by $C_n = p_n/q_n = [\alpha_0; \alpha_1, \alpha_2, ..., \alpha_n]$, where $p_{-2} = q_{-1} = 0$ and $q_{-2} = p_{-1} = 1$. Recall that $p_n = \alpha_n p_{n-1} + p_{n-2}$ and

fraction may be made even or odd by suitably choosing its last partial quotient to be 1 or > 1.

As usual, $\lfloor x \rfloor$ and $\{x\}$ are the integer and fractional parts of x respectively. Let $\lceil x \rceil$ be the least integer greater than or equal to x. The phrase "$\{nx + y\}$ is never zero" means that $\{nx + y\} \neq 0$ for $n \in \mathbb{Z}^+$. When $y = 0$ this means that x is irrational. Similarly "$\{nx + y\}$ has a zero " means that there exists an $n \in \mathbb{Z}^+$ for which $\{nx + y\} = 0$. In this case we define the function $L(x, y)$ to be the least such n for which this happens. In the event that $\{nx + y\}$ has no zero, we define $L(x, y) = \infty$.

Approximations to β are denoted by putting primes on the corresponding approximations to α. For example, $C'_j = p'_j/q'_j = [\beta_0; \beta_1, \beta_2, \ldots \beta_j]$.

3 Basic properties of Ψ

In this section we establish the basic properties of Ψ. These properties characterize Ψ sufficiently to allow us to evaluate it in trivial cases and identify the difficult cases. Proposition 3.1 is a collection of various results, (d) and (e) being the most important. The characterization (f) is also somewhat noteworthy.

Proposition 3.1.

(a) *If* $\lfloor \alpha \rfloor = \lfloor \beta \rfloor$, *then* $\Psi(\alpha, \beta) = \Psi(\{\alpha\}, \{\beta\})$.

(b) $|\alpha - \beta| \geq 1 \implies \Psi(\alpha, \beta) = 1$.

(c) $\Psi(\alpha, \beta) \leq \lceil 1/(\alpha - \beta) \rceil$, *if* $\alpha > \beta$.

(d) $\Psi(\lambda, \alpha) = \min(\Psi(\lambda, \beta), \Psi(\beta, \alpha))$, *if* $\lambda \leq \beta \leq \alpha$.

(e) *When* $\beta \leq \alpha$,

$$\begin{aligned} \Psi(\alpha, \beta) &= \textit{the least } n \in \mathbb{Z}^+ \textit{ such that } \quad \beta < \alpha - \frac{\{n\alpha + s\}}{n}, \\ &= \textit{the least } n \in \mathbb{Z}^+ \textit{ such that } \quad \beta + \frac{1 - \{n\beta + s\}}{n} \leq \alpha. \end{aligned}$$

(f) *If* $\beta < \alpha$ *and* $\lfloor \beta \rfloor = \lfloor \alpha \rfloor$, *then* $\Psi(\alpha, \beta)$ *is the least* $n \in \mathbb{Z}^+$ *such that both* $\{n\alpha + s\} < \{\alpha\}$ *and* $\{n\beta + s\} \geq \{\beta\}$.

(g) $\lfloor \Psi\beta + \alpha \rfloor + 1 = \lfloor \Psi\alpha + \alpha \rfloor$ *if* $\lfloor \alpha \rfloor = \lfloor \beta \rfloor$ *and* $\beta < \alpha$.

Proof.

(a), (b) and (c) Immediate from the definition.

(d) $\lfloor n\lambda+s\rfloor = \lfloor n\beta+s\rfloor$ if $0 \le n < \Psi(\lambda,\beta)$, but $\lfloor \Psi(\lambda,\beta)\lambda+s\rfloor \ne \lfloor \Psi(\lambda,\beta)\beta+s\rfloor$. Similarly $\lfloor n\beta+s\rfloor = \lfloor n\alpha+s\rfloor$ if $0 \le n < \Psi(\beta,\alpha)$, but $\lfloor \Psi(\beta,\alpha)\beta+s\rfloor \ne \lfloor \Psi(\beta,\alpha)\alpha+s\rfloor$. Thus $\lfloor n\lambda+s\rfloor = \lfloor n\alpha+s\rfloor$ if $0 \le n < \min(\Psi(\lambda,\beta),\Psi(\beta,\alpha))$. Equality can not hold at $n = \min(\Psi(\lambda,\beta),\Psi(\beta,\alpha))$ since $\lfloor n\lambda+s\rfloor \le \lfloor n\beta+s\rfloor \le \lfloor n\alpha+s\rfloor$.

(e) $\lfloor n\alpha+s\rfloor = \lfloor n\beta+s\rfloor \iff \lfloor n\beta+s-\lfloor n\alpha+s\rfloor\rfloor = 0 \iff \lfloor n\beta+s-n\alpha-s+\{n\alpha+s\}\rfloor = 0$, so

$$\lfloor n\alpha+s\rfloor = \lfloor n\beta+s\rfloor \iff 0 \le n(\beta-\alpha)+\{n\alpha+s\} < 1. \tag{3}$$

Similarly,

$$\lfloor n\alpha+s\rfloor = \lfloor n\beta+s\rfloor \iff 0 \le n(\alpha-\beta)+\{n\beta+s\} < 1. \tag{4}$$

If $\beta < \alpha$ and $\lfloor n\beta+s\rfloor \ne \lfloor n\alpha+s\rfloor$, then (3) implies that $n(\beta-\alpha)+\{n\alpha+s\} < 0$. This is equivalent to the first equation of (e). The second equation follows similarly from (4).

(f) Let $x > 0$. First observe that $\lfloor nx+s\rfloor - \lfloor (n-1)x+s\rfloor = -\lfloor (n-1)x+s-nx-s+\{nx+s\}\rfloor = -\lfloor -x+\{nx+s\}\rfloor = \lceil x-\{nx+s\}\rceil = \lfloor x\rfloor + \lceil \{x\}-\{nx+s\}\rceil = \lfloor x\rfloor$ or $\lfloor x\rfloor+1$ according to whether $\{nx+s\} \ge \{x\}$ or $\{nx+s\} < \{x\}$.

Now $\lfloor n\alpha+s\rfloor = \lfloor n\beta+s\rfloor$ when $n = \Psi-1$, while $\lfloor n\alpha+s\rfloor > \lfloor n\beta+s\rfloor$ when $n = \Psi$. Thus when $x = \alpha$ or β, the difference $\lfloor nx+s\rfloor - \lfloor (n-1)x+s\rfloor$ has only the two possible values $\lfloor\alpha\rfloor$ and $\lfloor\alpha\rfloor+1$. It follows that $\lfloor \Psi\alpha+s\rfloor - \lfloor(\Psi-1)+s\rfloor = \lfloor\alpha\rfloor+1$ and $\lfloor\Psi\beta+s\rfloor - \lfloor(\Psi-1)\beta+s\rfloor = \lfloor\alpha\rfloor = \lfloor\beta\rfloor$, and these conditions occur if and only if $\{\Psi\alpha+s\} < \{\alpha\}$ and $\{\Psi\beta+s\} \ge \{\beta\}$.

(g) Follows from the proof of (f). □

Remark. *If s is fixed and the definition of Ψ is suitably extended, it follows that $1/\Psi$ is a metric on $\mathbb{R}$.*

The following proposition evaluates Ψ in the trivial case $\lfloor\beta\rfloor \ne \lfloor\alpha\rfloor$.

Proposition 3.2.

$$\Psi(\beta, \alpha) = \min\left(\left\lfloor \frac{s}{\lceil\beta\rceil - \beta} \right\rfloor + 1, \left\lceil \frac{1-s}{\alpha - \lceil\beta\rceil} \right\rceil\right) \qquad \textit{if } \lfloor\beta\rfloor < \lfloor\alpha\rfloor.$$

For brevity we omit the proof which is easy.

Hereafter assume $\lfloor\alpha\rfloor = \lfloor\beta\rfloor$ and α, β non-integral.

4 Lemmas

An important result of this section is Lemma 4.2 which will be the main tool for evaluating Ψ in both the homogeneous and inhomogeneous cases. First, however, we clarify the behavior of Ψ by characterizing it in terms of certain approximating sequences for α or β.

For convenience we define the *lower approximating sequence* for α and the *upper approximating sequence* for β.

Definition. *The lower approximating sequence (LAS) for α is defined to be $g_m = (\lfloor J_m\alpha + s\rfloor - s)/J_m, (m \in \mathbb{Z}^+)$, where $J_1 = 1$ and J_{m+1} is the least element of $\mathbb{Z}^+$ such that $\{J_{m+1}\alpha + s\}/J_{m+1} < \{J_m\alpha + s\}/J_m$. If the sequence J is finite, let J_e be its last term. For consistency we define $g_0 = \lfloor\alpha\rfloor$.*

The upper approximating sequence (UAS) for β is defined to be $h_n = (\lfloor K_n\beta + s\rfloor - s + 1)/K_n$, $(n \in \mathbb{Z}^+)$, where $K_1 = 1$ and K_{n+1} is the least $\mathbb{Z}^+$ such that $(1 - \{K_{n+1}\beta + s\})/K_{n+1} < (1 - \{K_n\beta + s\})/K_n$. We define $h_0 = \lceil\beta\rceil$.

It is clear that the sequence J and hence the LAS for α is infinite if and only if $\{n\alpha + s\}$ is never zero. It follows from the definition that if $\{n\alpha + s\}$ has a zero, then $L(\alpha, s) = J_e$. Notice that when $s = 0$, both sequences J and K are infinite if and only if α is irrational. In any case the sequence K, and hence the UAS for β is infinite. Notice that the approximating sequences are rational if and only if s is rational.

By Proposition 3.1(e), $\alpha - \{J_m\alpha + s\}/J_m \le \beta < \alpha - \{J_{m+1}\alpha + s\}/J_{m+1} \implies \Psi(\beta, \alpha) = J_{m+1}$, and $\beta + (1 - \{K_{n+1}\beta + s\})/K_{n+1} \le \alpha < \beta + (\{K_n\beta + s\})/K_n \implies \Psi(\beta, \alpha) = K_{n+1}$, where $m, n \in \mathbb{Z}^+$. But $\alpha - \{J_m\alpha + s\}/J_m = (\lfloor J_m\alpha + s\rfloor - s)/J_m = g_m$ and $\beta + (1 - \{K_n\beta + s\})/K_n = (\lfloor K_n\beta + s\rfloor - s + 1)/K_n = h_n$. Hence the interval $[\lfloor\alpha\rfloor, \alpha)$ can be broken up into the (possibly finite) sequence of subintervals:

$[g_0, g_1)$, $[g_1, g_2)$, $[g_2, g_3)$, ..., where if β is in the m'th subinterval, $\Psi(\beta, \alpha) = J_m$. Notice that $\Psi((\lfloor J_{m-1}\alpha + s \rfloor - s)/J_{m-1}, \alpha) = J_m$, so that the sequence J can be recursively determined when Ψ is known. Further, when α and s are fixed and $\beta < \alpha$ varies, Ψ takes values from the sequence J depending on which subinterval contains β.

Similarly, the interval $(\beta, \lceil \beta \rceil)$ can be broken up into the infinite sequence of subintervals: $...[h_3, h_2)$, $[h_2, h_1)$, $[h_1, h_0)$. From the above, if α is in the n'th subinterval (counting from the right), then $\Psi(\beta, \alpha) = K_n$. Unfortunately it does not seem easy to determine K recursively from Ψ. We have $\Psi(\beta, (\lfloor K_{n-1}\beta + s \rfloor - s + 1)/K_{n-1} - \epsilon) = K_n$, for some $\epsilon > 0$, but the size of ϵ does not seem obvious. On the other hand, in this case the estimate $K_n \leq 1 + \lfloor K_{n-1}/(1 - \{K_{n-1}\beta + s\}) \rfloor$ holds. Continuing the analysis, it follows that when β and s are fixed and $\alpha > \beta$ varies, Ψ takes values from the sequence K depending on which subinterval contains α.

Now let β be in the m'th subinterval of $[\lfloor \alpha \rfloor, \alpha)$, and α in the n'th subinterval of $(\beta, \lceil \beta \rceil]$. Then it follows from the above remarks that $\Psi(\beta, \alpha) = J_m = K_n$. Since $\lfloor \Psi\beta + s \rfloor + 1 = \lfloor \Psi\alpha + s \rfloor$, it follows that $(\lfloor J_m\alpha + s \rfloor - s)/J_m = (\lfloor K_n\beta + s \rfloor - s + 1)/K_n$. Hence the UAS for β and the LAS for α have a common term: $(\lfloor \Psi\alpha + s \rfloor - s)/\Psi$. This term is unique since J and K are strictly increasing. This motivates the following definition.

Definition. *Let $\beta < \alpha$, and $0 \leq s < 1$. The <u>coconvergent</u> of β and α is defined to be $(\lfloor \Psi\alpha + s \rfloor - s)/\Psi$.*

When $s = 0$, the coconvergent is easy to find (Corollary 6.1); it will play a central role in evaluating Ψ for $s \geq 0$.

Lemma 4.1.

(a) *For any $N \in \mathbb{Z}^+$, there exists an $M \in \mathbb{Z}^+$ such that $n \geq M \implies \Psi(\alpha, \beta_n; s) \geq N$ if $\lim_{n\to\infty} \beta_n = \alpha$, and*

(1) β_n is increasing and $\{n\alpha + s\}$ is never zero, or

(2) β_n is decreasing.

(b) *If $L(\alpha, s) < \infty$, then there exists a $\beta < \alpha$ such that $\beta \leq \lambda < \alpha \implies$*

Proof.

(a) Both parts follow directly from the above discussion and the fact that if some $\beta_k = \alpha$, then $\Psi(\beta_n, \alpha) = \infty$ for all $n \geq k$.

(b) By the above discussion J is finite and its last term J_e is equal to $L(\alpha, s)$. Moreover, if $\beta = \alpha - \{J_{e-1}\alpha + s\}/J_{e-1} \leq \lambda < \alpha - \{J_e\alpha + s\}/J_e$, then $\Psi(\lambda, \alpha) = J_e$. Now if $\lambda < \beta$, then $\Psi(\lambda, \alpha) = \min(\Psi(\lambda, \beta), \Psi(\beta, \alpha)) = \min(\Psi(\lambda, \beta), J_e)$. Hence $\Psi(\lambda, \alpha) \leq J_e$. Thus

$$L(\alpha, s) = \max_{0 \leq \lambda < \alpha} \Psi(\lambda, \alpha; s).$$

□

The next lemma evaluates Ψ when α and β are rational and properly related.

Lemma 4.2.

$$\Psi\left(\frac{a}{b}, \frac{c}{d}\right) = d + d\lfloor bs \rfloor - b\lfloor ds \rfloor$$

if $a, b, c, d \in \mathbb{Z}^+$, $a/b + 1/db = c/d$, *and* $s \geq 0$.

Proof.

We first note that the solution of $an \equiv r \pmod{b}$, with $r \in \mathbb{Z}$, and $L \leq n < L + b$, $L \in \mathbb{Z}^+$, is given by $n = b\lceil (L + rd)/b \rceil - rd$. To see this, observe that $a/b + 1/bd = c/d \implies ad + 1 = bc \implies ad \equiv -1p \pmod{b} \implies a(-rd) \equiv r \pmod{b} \implies n = Eb - rd$, for some $E \in \mathbb{Z}^+$. To find E, we require $L \leq n < L + b$, but $L \leq n < L + b \iff L \leq Eb - rd < L + b$. This yields $E = \lceil (L + rd)/b \rceil$, giving the above solution.

Next we note that

$$\left\lfloor n\frac{c}{d} + s \right\rfloor = \left\lfloor n\frac{a}{b} + s \right\rfloor \iff \left\lfloor n\frac{c}{d} + s - n\frac{a}{b} - s + \left\{n\frac{a}{b} + s\right\} \right\rfloor = 0$$

$$\iff 0 \leq \frac{n}{db} + \left\{n\frac{a}{b} + s\right\} < 1 \iff 0 \leq \frac{n}{db} + \frac{\{bs\}}{b} + \left\{\frac{na + \lfloor bs \rfloor}{b}\right\} < 1$$

$$\iff 0 \leq \frac{n + d\{bs\}}{db} + \left\{\frac{na + \lfloor bs \rfloor}{b}\right\} < 1.$$

Now when $n > 0$ both terms in the middle of the last inequality are ≥ 0; consequently Ψ is the least $n \in \mathbb{Z}^+$ such that

$$\frac{n + d\{bs\}}{db} + \left\{ \frac{na + \lfloor bs \rfloor}{b} \right\} \geq 1. \tag{5}$$

Consider any $n \in \mathbb{Z}^+$ for which (5) holds. For some positive integer K,

$$\frac{n + d\{bs\}}{db} \geq \frac{K}{b} \text{ and } \left\{ \frac{na + \lfloor bs \rfloor}{b} \right\} \geq \frac{b - K}{b}.$$

Thus $Kd - d\{bs\} \leq n$ and $\{(na + \lfloor bs \rfloor)/b\} \geq (b - K)/b$, with $K \in \mathbb{Z}^+$. Further,

$$\left\{ \frac{na + \lfloor bs \rfloor}{b} \right\} \geq \frac{b - K}{b} \iff \left\{ \frac{na + \lfloor bs \rfloor}{b} \right\} = \frac{b - M}{b},$$

where $0 < M \leq \min(b, K)$. But this is the same as $an \equiv -(M + \lfloor bs \rfloor) \pmod{b}$ with $0 < M \leq \min(b, K)$. Using the condition $Kd - d\{bs\} \leq n$, and remembering that solutions of the congruence are periodic modulo b, (5) implies that $an \equiv -(M + \lfloor bs \rfloor) \pmod{b}$, $Kd - d\{bs\} \leq n < Kd - d\{bs\} + b$ and $0 < M \leq \min(b, K)$. It is clear that (5) holds if and only if $n \in \mathbb{Z}^+$ satisfies these three conditions. Using the first two conditions and the above remark, we obtain

$$n = b \left\lceil \frac{Kd - d\{bs\} - d(M + \lfloor bs \rfloor)}{b} \right\rceil + d(M + \lfloor bs \rfloor)$$

$$= dM + d\lfloor bs \rfloor - b \left\lfloor (M - K)\frac{d}{b} + ds \right\rfloor = f(M, K).$$

The problem is to minimize $f(M, K)$ subject to $0 < M \leq \min(b, K)$. The minimum occurs when $M = K = 1$, and the lemma follows. $\square$

Two corollaries are now given. The first will be employed in evaluating Ψ in the homogeneous case. The second, which will not be used, is included because of its intrinsic interest.

Corollary 4.1. *$\Psi(a/b, c/d; 0) = d$, when $a/b + 1/bd = c/d$, and $a, b, c, d \in \mathbb{Z}^+$*

Proof. Put $s = 0$ in Lemma 4.2. $\square$

Corollary 4.2. *Ψ satisfies the following functional equation:*

$$\begin{vmatrix} \Psi\left(\frac{a}{b}, \frac{c}{d}\right) & \Psi\left(\frac{a}{c}, \frac{b}{d}\right) \\ \Psi\left(\frac{c}{a}, \frac{d}{b}\right) & \Psi\left(\frac{b}{a}, \frac{d}{c}\right) \end{vmatrix} = \left(\Psi\left(\frac{d'}{d}, \frac{a'}{a}\right) - a\right)\left(\Psi\left(\frac{b'}{b}, \frac{c'}{c}\right) - b\right),$$

where $s \geq 0$, and $a, b, c, d, a', b', c', d' \in \mathbb{Z}^+$ satisfy

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} d' & a' \\ d & a \end{vmatrix} = \begin{vmatrix} b' & c' \\ b & c \end{vmatrix} = -1.$$

Proof. Assuming only the condition

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = -1,$$

it is easily verified by Lemma 4.2 that the determinant in the functional equation is equal to $(a\lfloor ds \rfloor - d\lfloor as \rfloor)(c(1 + \lfloor bs \rfloor) - b(1 + \lfloor ca \rfloor))$. Now $(a, b) = (b, c) = 1$ if and only if there exist a', b', c', d' satisfying the other two determinant conditions. The functions $\Psi(d'/d, a'/a)$ and $\Psi(b'/b, c'/c)$ can be evaluated by Lemma 4.2 to yield $\Psi(d'/d, a'/a) = a + (a\lfloor ds \rfloor - d\lfloor as \rfloor)$, and $\Psi(b'/b, c'/c) = b + (c(1 + \lfloor bs \rfloor) - b(1 + \lfloor ca \rfloor))$. This completes the proof. □

The following lemma will be used in proving Theorem 6.1.

Lemma 4.3. *Let $s = 0$. Then $\Psi(\beta_n, \alpha) = \Psi(\beta_n, \beta_{n+1})$ for $n \geq 0$, where β_n is a monotone sequence of positive real numbers such that $\lim_{n\to\infty} \beta_n = \alpha$,*

$$\Psi(\beta_n, \beta_{n+1}) \leq \Psi(\beta_{n+1}, \beta_{n+2}), \qquad \text{for } n \geq 0, \tag{6}$$

and if α is rational, then $\beta_k = \alpha$ for some k.

Proof. There are two cases: β_n increasing or β_n decreasing. Suppose $\beta_n \leq \beta_{n+1}$ for $n \geq 0$. Let $n \geq 0$ be fixed. Applying Proposition 3.1(d) inductively gives $\Psi(\beta_n, \alpha) = \min(\Psi(\beta_n, \beta_{n+1}), \Psi(\beta_{n+1}, \beta_{n+2}), ..., \Psi(\beta_{n+r-1}, \beta_{n+r}))$ for any $r \in \mathbb{Z}^+$. Using (6) gives $\Psi(\beta_n, \alpha) = \min(\Psi(\beta_n, \beta_{n+1}), \Psi(\beta_{n+r-1}, \beta_{n+r}))$. By Lemma 4.1(a), if α is irrational, $\Psi(\beta_n, \alpha)$ can be made $> \Psi(\beta_n, \beta_{n+1})$ by making β_{n+r-1} sufficiently

close to α; the lemma follows. If α is rational β_{n+r} can be chosen $= \alpha$, and the lemma follows.

The second case is similar and even simpler. $\square$

If β_n, $(n \geq 0)$, is a sequence of rational numbers approximating α from below or above and satisfying the conditions of Corollary 4.1 and Lemma 4.3, then $\Psi(\beta_n, \alpha; 0)$ can be evaluated exactly. It is easy to construct such approximating sequences; then $\Psi(\beta, \alpha; 0)$ is easily evaluated in terms of them. From this we can express $\Psi(\beta, \alpha; 0)$ as a function of the convergents of the simple continued fraction for α or β. This is Theorem 6.1. The approximating sequences are constructed in the next section.

5 Definition of semiconvergents

For $n \geq 0$ define the sequences a_n, b_n, ε_n, and the numbers c_0, d_0, δ_0 by

$$\varepsilon_0 = \frac{a_0}{b_0} = \frac{p_{-2} + 0p_{-1}}{q_{-2} + 0q_{-1}}, \qquad \delta_0 = \frac{c_0}{d_0} = \frac{p_{-1} + 0p_0}{q_{-1} + 0q_0} = \infty,$$

and if

$$\varepsilon_k = \frac{a_k}{b_k} = \frac{p_v + ap_{v+1}}{q_v + aq_{v+1}} \qquad \text{with } 0 \leq a < \alpha_{v+2}, \quad v \text{ even}, \quad k \geq 0, \text{ then}$$

$$\varepsilon_{k+1} = \frac{a_{k+1}}{b_{k+1}} = \frac{p_v + (1+a)p_{v+1}}{q_v + (1+a)q_{v+1}}.$$

In particular when $a = \alpha_{v+2} - 1$,

$$\varepsilon_{k+1} = \frac{p_v + \alpha_{v+2}p_{v+1}}{q_v + \alpha_{v+2}q_{v+1}} = \frac{p_{v+2} + 0p_{v+3}}{q_{v+2} + 0q_{v+3}}$$

and the process continues. If α is rational and $\varepsilon_k = \alpha$ define $\varepsilon_{k+n} = \alpha$, for $n \in \mathbb{Z}^+$. Then $\{\varepsilon_k\}$ is the union of the sequences of convergents and intermediate convergents less than α arranged in non-decreasing order.

If α is rational, let the last partial quotient be chosen so that $C_{e-1} < \alpha = C_e$, i.e. so as to make e odd. In this case, define δ_n, c_n, d_n for $n \in \mathbb{Z}^+$ as follows. If

(7) $$\delta_k = \frac{c_k}{} = \frac{p_v + ap_{v+1}}{} > \frac{p_{e-2} + (\alpha_e - 1)p_{e-1}}{},$$

with $0 \le a < \alpha_{v+2}$, v odd, $k \ge 0$, then

$$\delta_{k+1} = \frac{c_{k+1}}{d_{k+1}} = \frac{p_v + (1+a)p_{v+1}}{q_v + (1+a)q_{v+1}}.$$

Otherwise, if

$$\delta_k = \frac{c_k}{d_k} = \frac{p_{e-2} + (\alpha_e - 1)p_{e-1}}{q_{e-2} + (\alpha_e - 1)q_{e-1}}, \qquad \text{then}$$

$$\delta_{k+n} = \frac{c_{k+n}}{d_{k+n}} = \frac{(n+1)p_{e-2} + (\alpha_e - 1)p_{e-1}}{(n+1)q_{e-2} + (\alpha_e - 1)q_{e-1}}, \qquad \text{for } n \in \mathbb{Z}^+.$$

In this case δ_n $(n \ge k+1)$ are called the special mediants of α. If α is irrational there will be no e for which $\alpha = C_e$, and δ_n, c_n, d_n are defined by (7) for $n \in \mathbb{Z}^+$. Thus when α is irrational, $\{\delta_n\}$ is the sequence of convergents and intermediate convergents greater than α arranged in decreasing order. When α is rational, δ_n is the same sequence with the special mediants adjoined. The special mediants are the simplest decreasing infinite sequence of mediants between α and the last intermediate convergent greater than α.

The following facts about ε_n and δ_n can easily be proved directly; some can be found in [2], [8], [9], [10], [14]:

$$\frac{a_n}{b_n} + \frac{1}{b_n b_{n+1}} = \frac{a_{n+1}}{b_{n+1}} \text{ for } n \ge 0, \tag{8}$$

$$\frac{c_{n+1}}{d_{n+1}} + \frac{1}{d_n d_{n+1}} = \frac{c_n}{d_n} \text{ for } n \ge 1; \tag{9}$$

$$d_{n+1} > d_n \text{ for } n \ge 0,$$

$$b_{n+1} > b_n \text{ for } n > \alpha_0.$$

It is easy to show from these and other well-known properties of convergents, intermediate convergents and mediants (see the above references) that $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \delta_n = \alpha$. For brevity, ε_n and δ_n $(n \ge 0)$ will be called the semiconvergents of α.

6 The homogeneous case

The subject of this section is the homogeneous sequence $\lfloor n\alpha \rfloor$ $(n \geq 1)$. The questions posed in the introduction will be answered and a new characterization of $\lfloor n\alpha \rfloor$ $(n \geq 1)$ will be given. Theorem 1 shows that the LAS and UAS for α are simply the sequences of semiconvergents below and above α respectively. It follows as a corollary that when $s = 0$, the coconvergent is the common term of the semiconvergents for β greater than β and the semiconvergents for α less than α. The observation that arbitrary pairs of irrational numbers in $(0, 1)$ have a common semiconvergent seems to be new. This observation will prove important in evaluating Ψ in the inhomogeneous case. Throughout this section $s = 0$.

Theorem 6.1. *If $s = 0$, then*

(a) $\Psi(\beta, \alpha) = b_{n+1}$, *where* $\varepsilon_n \leq \beta < \varepsilon_{n+1} \leq \alpha$ *for* $n \geq 0$.

(a′) $\Psi(\beta, \alpha) = q_n + q_{n+1}\lfloor 1 + (\beta q_n - p_n)/(p_{n+1} - \beta q_{n+1}) \rfloor$, *where* $C_n \leq \beta < C_{n+2} \leq \alpha$.

(b) $\Psi(\beta, \alpha) = d_{n+1}$, *where* $\alpha < \delta_{n+1} \leq \beta < \delta_n$ *for* $n \geq 0$.

(b′) $\Psi(\beta, \alpha) = q_n + q_{n+1}\lceil (p_n - \beta q_n)/(\beta q_{n+1} - p_{n+1}) \rceil$, *where* $\alpha < C_{n+2} \leq \beta < C_n$ *n odd* ≥ 1.

(b″) $\Psi(\beta, \alpha) = q_{n+1}\lceil (\beta q_n - p_n)/(\beta q_{n+1} - p_{n+1}) \rceil - q_n$, *where* $\alpha = C_{n+1} < \beta < (p_{n+1} - p_n)/(q_{n+1} - q_n)$, *and n is made even by suitably choosing the last partial quotient* $= 1$ *or* $\neq 1$.

Proof.

(a′), (b′) and (b″) are derived from (a) and (b) by solving for the semiconvergents in terms of the convergents. For example to derive (b″) from (b) we let M be the least positive integer such that

$$\frac{Mp_{n+1} + p_{n+1} - p_n}{Mq_{n+1} + q_{n+1} - q_n} \leq \beta.$$

The fraction on the left is the special mediant sought. It follows easily that

$$M = \left\lceil \frac{\beta q_n - p_n}{\quad} \right\rceil - 1.$$

Using this in (b) gives (b''). Properties (a') and (b') are derived similarly. Thus it suffices to prove (a) and (b).

We are given

$$\varepsilon_k \leq \beta < \varepsilon_{k+1} \leq \alpha, \text{ with } k \geq 0. \tag{10}$$

By Corollary 4.1, (11) and Lemma 4.3 we have $\Psi(\varepsilon_k, \alpha) = b_{k+1}$. Proposition 3.1(d) gives $\Psi(\varepsilon_k, \alpha) = \min(\Psi(\varepsilon_k, \beta), \Psi(\beta, \alpha))$. Hence $\Psi(\beta, \alpha) \geq b_{k+1}$. From (10), $b_{k+1}\beta < a_{k+1} \leq b_{k+1}\alpha$. Thus $\lfloor b_{k+1}\alpha \rfloor \geq a_{k+1} > b_{k+1}\beta \geq \lfloor b_{k+1}\beta \rfloor \implies \Psi(\beta, \alpha) \leq b_{k+1}$. The only possibility is $\Psi(\beta, \alpha) = b_{k+1}$.

(b) Since $\alpha < \delta_{k+1} \leq \beta < \delta_k$, we have $\Psi(\beta, \alpha) = \min(\Psi(\alpha, \delta_{k+1}), \Psi(\delta_{k+1}, \beta))$. Corollary 4.1, (9) and Lemma 4.3 give $\Psi(\alpha, \delta_{k+1}) = d_{k+1}$. Hence $\Psi(\beta, \alpha) = \min(d_{k+1}, \Psi(\delta_{k+1}, \beta))$. So it suffices to show that $\Psi(\delta_{k+1}, \beta) \geq d_{k+1} + d_k$. There exists an integer $N \in \mathbb{Z}^+$ such that

$$\frac{c_{k+1}}{c_{k+1}} < \frac{c_{k+1} + c_k}{d_{k+1} + d_k} < \ldots < \frac{c_{k+1} + (N-1)c_k}{d_{k+1} + (N-1)d_k} \leq \beta < \frac{c_{k+1} + Nc_k}{d_{k+1} + Nd_k}.$$

It is easy to show that any two consecutive fractions in this sequence satisfy the conditions of Corollary 4.1. Using Proposition 3.1(d) inductively and simplifying with Lemma 4.3 we obtain

$$\Psi\left(\frac{c_{k+1}}{d_{k+1}}, \frac{c_{k+1} + Nc_k}{d_{k+1} + Nd_k}\right) = d_{k+1} + d_k = \min\left(\Psi\left(\frac{c_{k+1}}{d_{k+1}}, \beta\right), \Psi\left(\beta, \frac{c_{k+1} + Nc_k}{d_{k+1} + Nd_k}\right)\right).$$

Hence $\Psi(c_{k+1}/d_{k+1}, \beta) \geq d_{k+1} + d_k$, and the proof is complete. $\square$

Theorem 6.1 completely answers the questions posed in the introduction for the homogeneous case. In addition, the semiconvergents are easily seen to be the rational numbers with the smallest denominators that make Ψ exceed a given integer. Also it follows that $J_n = b_w$, where $w = \alpha_0 + n - 1$ and $K_m = d_m$ for $m, n \geq 1$.

The following corollary shows how $\Psi(\beta, \alpha, 0)$ is fundamentally connected to the rational approximations of both α and β. In this light it seems somewhat surprising that Ψ appears not to have been studied before. It should also clarify the choice of the term "semiconvergent".

Corollary 6.1. *Write* $\Psi = \Psi(\beta, \alpha; 0)$, *where* $\beta < \alpha$. *Then* $\lfloor \Psi\alpha \rfloor / \Psi$ *is a semiconvergent of both* α *and* β.

Proof. This follows directly from Theorem 6.1 and the properties of the approximating sequences. □

It is easy to see that real numbers (say in $[0, 1]$) have a common semiconvergent, but it is not as clear that its denominator is Ψ. Notice that the coconvergent is not necessarily the rational number between α and β with the smallest denominator.

It is well known that the convergents of a simple continued fraction can be characterized intrinsically in terms of their rational approximation properties; see for example [7], [8], [9], [10] or [14]. In particular, the property of a rational number of being a "best approximation of the second kind" (see [7] for a definition of this) characterizes convergents. It is also well known [7] that every best approximation of the first kind of α is a semiconvergent of α. But since the converse does not hold [7], this result does not give a characterization of semiconvergents in terms of their rational approximation properties [14]. However, for the subsequence of semiconvergents obtained by allowing only one element to be an integer, that is to say, by omiting the terms $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{\alpha_0 - 1}$, when $\alpha_0 > 1$, there exists a classical intrinsic characterization [2]:

Let $a/b < \alpha$ *(respectively* $a/b > \alpha$*). Then a necessary and sufficient condition that* a/b *be a semiconvergent of* α *is that the inequalities* $c/d < \alpha$ *(respectively* $c/d > \alpha$*),* $0 < d \leq b$, *and* $a/b \neq c/d$ *imply* $|\alpha - a/b| < |\alpha - c/d|$.

Our intrinsic characterization is identical to the above except that the metric $|x - y|$ is replaced by the metric $1/\Psi(x, y)$. Since Ψ is constant on the intervals between the semiconvergents, it follows that $1/\Psi$ is the coarsest metric for which a characterization of this type holds. This provides yet another characterization of Ψ.

Corollary 6.2. *Let $a/b < \alpha$ (respectively $a/b > \alpha$). Then a necessary and sufficient condition that a/b be a semiconvergent of α is that the inequalities $c/d < \alpha$ (respectively $c/d > \alpha$), $0 < d \leq b$, and $a/b \neq c/d$ imply $\Psi(a/b, \alpha) > \Psi(c/d, \alpha)$.*

Proof. Similar to the last proof. □

The next theorem is analogous to the characterization of $\lfloor n\alpha \rfloor$ in [4], except that here semiconvergents are used instead of convergents. We recall the following well-known result:

Let $1 = u_0 < u_1 < u_2 < \ldots$ be any finite or infinite sequence of integers. Any non-negative integer N has precisely one representation in the system $S = \{u_0, u_1, u_2, \ldots\}$ of the form $N = \sum_{i=0}^{n} D_i u_i$, where the D_i are non-negative integers satisfying $\sum_{i=0}^{k} D_i u_i < u_{k+1}$, $(k \geq 0)$.

Theorem 6.2. *Let α have the lower semiconvergents a_n/b_n $(n \geq 0)$. Assume $1 \leq \alpha < 2$ and consider the systems of representation $S^b = \{1 = b_0 < b_1 < b_2 < \ldots\}$ and $S^a = \{1 = a_0 < a_1 < a_2 < \ldots\}$. In the system S^b, if N has the representation*

$$N = \sum_{i=0}^{n} D_i b_i \text{ with } N_k = \sum_{i=0}^{k} D_i b_i < b_{k+1}, \tag{11}$$

for $0 \leq k \leq n$, then in the representation system S^a, $\lfloor N\alpha \rfloor$ has the representation

$$\lfloor N\alpha \rfloor = \sum_{i=0}^{n} D_i a_i \qquad \text{subject to} \tag{12}$$

$$\sum_{i=0}^{k} D_i a_i < a_{k+1} \qquad \text{for } 0 \leq k \leq n \tag{13}$$

Proof. In S^b let N have the representation (11). Thus $N = N_n < b_{n+1} = \Psi(\alpha, \varepsilon_n)$ by Theorem 6.1. Hence $\lfloor N\alpha \rfloor = \lfloor N_n \varepsilon_n \rfloor = \lfloor N_{n-1} \varepsilon_n \rfloor + D_n a_n$. Now by (11) and Corollary 6.1, $N_{n-1} < b_n = \Psi(\varepsilon_n, \varepsilon_{n-1})$, so $\lfloor N_{n-1}\varepsilon \rfloor = \lfloor N_{n-1}\varepsilon_{n-1} \rfloor$, giving $\lfloor N\alpha \rfloor = \lfloor N_{n-1}\varepsilon_{n-1} \rfloor + D_n a_n = \lfloor N_{n-2}\varepsilon_{n-1} \rfloor + D_{n-1}a_{n-1} + D_n a_n$. In general from (11) and Corollary 6.1, $N_k < b_{k+1} = \Psi(\varepsilon_{k+1}, \varepsilon_k)$; thus $\lfloor N_k \varepsilon_{k+1} \rfloor = \lfloor N_k \varepsilon_k \rfloor$

$\lfloor N_{j-1}\varepsilon_j \rfloor + D_j a_j$, for $0 \le j \le n$. Hence (12) follows by induction. To prove (13) observe that (11) implies that $N_j\varepsilon_{j+1} < a_{j+1}$ for $1 \le j \le n$; hence$\lfloor N_j\varepsilon_{j+1} \rfloor < a_{j+1}$, for $1 \le j \le n$. Now $N_j < b_{j+1} < b_{j+2} = \Psi(\varepsilon_{j+1}, \alpha)$ by Theorem 6.1; thus $\lfloor N_j\varepsilon_j \rfloor = \lfloor N_j\varepsilon_{j+1} \rfloor < a_{j+1}$. But from the above, $\lfloor N_j\alpha \rfloor = \sum_{i=0}^{j} D_i a_i$, and so $\sum_{i=0}^{j} D_i a_i < a_{j+1}$, for $0 \le j \le n$, which is (13). $\square$

Theorem 6.2 can easily be extended to α outside the interval $[1, 2)$, but its statement becomes slightly more complicated. Comparison of Theorem 6.2 with the characterization of $\lfloor n\alpha \rfloor$ in [4] raises the following question. What is the representation system with basis elements of smallest growth rate which will provide such a characterization for $\lfloor n\alpha \rfloor$?

7 The inhomogeneous case

In this section $\lfloor n\alpha + s \rfloor$ is characterized, Ψ is evaluated when $s \ge 0$ and $L(\alpha, s)$ is evaluated for irrational α.

Theorem 6.2 gives a new characterization of $\lfloor n\alpha \rfloor$ analogous to that in [4]. Other characterizations of $\lfloor n\alpha \rfloor$ can be found in [5], [6], [12] and [13]. [5] and [12] give extensive bibliographies of the work on $\lfloor n\alpha \rfloor$ and $\lfloor n\alpha + s \rfloor$ in general. However, $\lfloor n\alpha + s \rfloor$ does not appear to have been characterized, (see the references given in [5] and [12].) Here we would like to point out that the method of [13] easily generalizes to yield the following characterization of $\lfloor n\alpha + s \rfloor$, $\quad (n \in \mathbb{Z})$.

Proposition 7.1. *If $\alpha > 1$, the integer k occurs in the sequence $\lfloor n\alpha + s \rfloor$ $(n \in \mathbb{Z})$ if and only if $\{(s-k)/\alpha\} < 1/\alpha$.*

Proof. We have $\lfloor n\alpha + s \rfloor = k \iff k \le n\alpha + s < k + 1 \iff (k-s)/\alpha \le n < (k+1-s)/\alpha \implies \lceil (k-s)/\alpha \rceil \le n < \lceil (k+1-s)/\alpha \rceil$. There can be such an n if and only if $\lceil (k+1-s)/\alpha \rceil - \lceil (k-s)/\alpha \rceil \ge 1 \iff \lceil (k+1-s)/\alpha - (k-s)/\alpha - \{-(k-s)/\alpha\} \rceil \ge 1 \iff \lceil 1/\alpha - \{(s-k)/\alpha\} \rceil \ge 1 \iff 0 < 1/\alpha - \{(s-k)/\alpha\} \iff \{(s-k)/\alpha\} < 1/\alpha$. $\square$

It is now important to distinguish whether $\{n\alpha + s\}$ has a zero or not, and it is of interest to find the least zero if there is one. The next theorem gives a solution to this problem for irrational α. When α is rational the problem reduces to solving a linear congruence.

Theorem 7.1. *If α is irrational, $L(\alpha, s) = \max_{n\geq 0}(\Psi_n)$, where $\Psi_n = q_{2n+1} + q_{2n+1}\lfloor q_{2n}s \rfloor - q_{2n}\lfloor q_{2n+1}s \rfloor$. Moreover $\Psi_{n+1} \geq \Psi_n$.*

Proof. By Lemma 4.2, $\Psi_n = \Psi(C_{2n}, C_{2n+1}) = \min(\Psi(C_{2n}, \alpha), \Psi(\alpha, C_{2n+1}))$. By Lemma 4.1(a), $\Psi(\alpha, C_{2n+1})$ is unbounded. However if $\{n\alpha+s\}$ has a zero, then from Lemma 4.1(b) $\Psi(C_{2n}, \alpha) = L$ from some n onwards. Hence from some n onwards $\Psi_n = L$. If $\{n\alpha + s\}$ has no zero, by Lemma 4.1(a), $\Psi(C_{2n}, \alpha)$ is unbounded, and the theorem is also true.

Finally $\Psi(C_{2n}, C_{2n+1}) = \min(\Psi(C_{2n}, C_{2n+2}), \Psi(C_{2n+2}, C_{2n+3}), \Psi(C_{2n+3}, C_{2n+1}))$ $\implies \Psi(C_{2n}, C_{2n+1}) \leq \Psi(C_{2n+2}, C_{2n+3})$, that is $\Psi_n \leq \Psi_{n+1}$. $\square$

The next theorem evaluates Ψ for $s \geq 0$.

Theorem 7.2. *Assume $\beta < \alpha$. Then*

$$\Psi(\beta, \alpha) = \min_{n\geq 0}\big(d'_{l+n} + d'_{l+n}\lfloor d'_{l+n+1}s \rfloor - d'_{l+n+1}\lfloor d'_{l+n}s \rfloor,$$
$$b_{k+n+1} + b_{k+n+1}\lfloor b_{k+n}s \rfloor - b_{k+n}\lfloor b_{k+n+1}s \rfloor, L(\alpha, s)\big),$$

and $\delta'_l = \varepsilon_k$ is the coconvergent (for $s = 0$).

Proof. By Proposition 4.3(d), $\Psi(\beta, \alpha) = \min(A, B)$ where $A = \Psi(\beta, \delta'_l)$, $B = \Psi(\varepsilon_k, \alpha)$ and $\delta'_l = \varepsilon_k$ is the coconvergent (for $s = 0$). Now

$$\Psi(\beta, \delta'_l) = \min(\Psi(\beta, \delta'_{l+i}), ..., \Psi(\delta'_{l+1}, \delta'_l))$$

for any $i \in \mathbb{Z}^+$. By the construction of the sequence δ' and by Lemma 4.1(a) there exists $N \in \mathbb{Z}^+$ such that $\Psi(\beta, \delta'_{l+i}) > \Psi(\delta'_{l+1}, \delta'_l)$ for all $i \geq N$. Thus $A = \Psi(\beta, \delta'_l) = \min_{n\geq 0}(\Psi(\delta'_{l+n+1}, \delta'_{l+n}))$, and Lemma 4.2 gives the evaluation of A. For B, $\Psi(\varepsilon_k, \alpha) = \min(\Psi(\varepsilon_k, \varepsilon_{k+1}), \ldots, \Psi(\varepsilon_{m-1}, \varepsilon_m), \Psi(\varepsilon_m, \alpha))$ for any $m > k$. Now if $\{n\alpha + s\}$ is never zero, by Lemma 4.1(a) there is an $M \in \mathbb{Z}^+$ such that $m \geq M \implies \Psi(\varepsilon_m, \alpha) > \Psi(\varepsilon_k, \varepsilon_{k+1})$, and so $\Psi(\varepsilon_k, \alpha) = \min_{n\geq 0}(\Psi(\varepsilon_{k+n}, \varepsilon_{k+n+1}))$. Otherwise, by Lemma 4.1(b) there exists an $M \in \mathbb{Z}^+$ such that $m \geq M \implies \Psi(\varepsilon_m, \alpha) = L(\alpha, s)$. Thus since $\Psi(\varepsilon_k, \alpha) \leq L(\alpha, s)$ for $m \geq M$, we have

$$\Psi(\varepsilon_k, \alpha) = \min(\Psi(\varepsilon_k, \varepsilon_{k+1}), ..., \Psi(\varepsilon_{k+M-1}, \varepsilon_{k+M}), L(\alpha, s)).$$

Therefore $\Psi(\varepsilon_k, \alpha) = \min_{n\geq 0}(\Psi(\varepsilon_{k+n}, \varepsilon_{k+n+1}), L(\alpha, s))$, and Lemma 4.2 gives the theorem. $\square$

Although this theorem is a nice theoretical solution of our problem, it suffers the defect in that it is computationally ineffective since the min has an undetermined number of terms. This problem is remedied by following theorem.

Theorem 7.3. *Observe that $\Psi(\alpha, \beta) = \min(A, B)$, where the A and B are as in the above proof. Then*

(a) *If $\{n\beta + s\}$ is never zero, then*

$$A = \min_{0\leq n<m-l}(d'_{l+n} + d'_{l+n}\lfloor d'_{l+n+1}s\rfloor - d'_{l+n+1}\lfloor d'_{l+n}s\rfloor),$$

where $m \geq l$ is such that $\delta'_m = C'_j$ (a convergent of β), and $q'_j + q'_j\lfloor q'_{j+1}s\rfloor - q'_{j+1}\lfloor q'_j s\rfloor \geq \min_{0\leq n<m-l}(d'_{l+n} + d'_{l+n}\lfloor d'_{l+n+1}s\rfloor - d'_{l+n+1}\lfloor d'_{l+n}s\rfloor)$.

(b) *If $\{n\alpha + s\}$ is never zero, then*

$$B = \min_{0\leq n<v-k}(b_{k+n+1} + b_{k+n+1}\lfloor b_{k+n}s\rfloor - b_{k+n}\lfloor b_{k+n+1}s\rfloor),$$

where $v \geq k$ is such that $\varepsilon_v = C_w$, and $q_{w+1} + q_{w+1}\lfloor q_w s\rfloor - q_w\lfloor q_{w+1}s\rfloor \geq \min_{0\leq n<v-k}(b_{k+n+1} + b_{k+n+1}\lfloor b_{k+n}s\rfloor - b_{k+n}\lfloor b_{k+n+1}s\rfloor)$.

Proof. (a) Because all convergents are semiconvergents, there is a convergent $C'_j = \delta'_m$ for some $m \geq l$. Thus $\Psi(\beta, \delta'_l) = \min(\Psi(\beta, C'_j), \Psi(\delta'_m, \delta'_{m-1}), \ldots, \Psi(\delta'_{l+1}, \delta'_l))$. Now for any m, this min is bounded by $\Psi(\delta'_{l+1}, \delta'_l)$. On the other hand, $\Psi(\beta, C'_j)$ $\geq \min(\Psi(C'_{j+1}, \beta), \Psi(\beta, C'_j)) = q'_j + q'_j\lfloor q'_{j+1}s\rfloor - q'_{j+1}\lfloor q'_j s\rfloor$, and this can be made arbitrarily large by Lemma 4.1(a). Thus when a j is chosen so that

$$q'_j + q'_j\lfloor q'_{j+1}s\rfloor - q'_{j+1}\lfloor q'_j s\rfloor \geq \min_{0\leq n<m-l}(\Psi(\delta'_{l+n}, \delta'_{l+n+1}))$$

$$= \min_{0\leq n<m-l}(d'_{l+n} + d'_{l+n}\lfloor d'_{l+n+1}s\rfloor - d'_{l+n+1}\lfloor d'_{l+n}s\rfloor),$$

A is equal to this last min.

(b) There is a convergent $C_w = \varepsilon_v$, where $v \geq k$. Again

Note that for any $v \geq k$, this last min is bounded by $\Psi(\varepsilon_k, \varepsilon_{k+1})$. Now $\Psi(C_w, \alpha) \geq \min(\Psi(C_w, \alpha), \Psi(\alpha, C_{w+1})) = \Psi(C_w, C_{w+1}) = q_{w+1} + q_{w+1}\lfloor q_w s \rfloor - q_w \lfloor q_{w+1} s \rfloor$, and both terms in the min (and hence the last expression), can be made arbitrarily large by Lemma 4.1(a). Hence when a w is chosen so that

$$q_{w+1}+q_{w+1}\lfloor q_w s\rfloor - q_w\lfloor q_{w+1}s\rfloor \geq \min_{o\leq n<v-k}(b_{k+n+1}+b_{k+n+1}\lfloor b_{k+n}s\rfloor - b_{k+n}\lfloor b_{k+n+1}s\rfloor),$$

B is equal to the last min. $\square$

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