G-CONTINUED FRACTIONS FOR BASIC HYPERGEOMETRIC FUNCTIONS

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1. Introduction and notation

In previous papers, [1],[2],[3] we have explored the problem of expanding basic hypergeometric functions into continued fractions. There is, however, a limit to what can be done since as one adds free parameters to basic hypergeometric functions, the order of the difference equations they satisfy increases and it soon is not possible to find the second order recurrences necessary for continued fractions.

In this paper we apply a generalization of the continued fraction process which arise from higher order recurrences to find expansions for the quotients of basic hypergeometric functions. This generalization is known as a G-continued fraction. For basic information see the recent book of Lorentzen and Waadeland [6]. Several convergence theorems for G-continued fractions as well as relations with other generalized continued fraction algorithms have been given, but we do not know of any explicit expansions of special functions in terms of G-continued fractions. In this paper we give a general expansion of the quotient of two contiguous basic hypergeometric functions in arbitrarily many variables as a G-continued fraction. We make use of the convergence theorem of Zahar [9] which extends the theorem of Pincherle [7] for ordinary continued fractions.

Our expansion of basic hypergeometric functions will be used in future work on combinatorics of these functions as well as possibly for work on the arithmetic nature of their special values.

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2. Theorem and proof

We consider m-th-order linear homogeneous recurrence relations of the form

$$\sum_{i=0}^{m} a_i(n) y_{n+m-i} = 0,\ a_0(n)a_m(n) \neq 0, n = 0,\ 1, \ldots . \tag{0}$$

Here the $a_i(n)$ are given sequences of complex constants for $0 \leq i \leq m$. A G-continued fraction will be defined shortly. First define the following transformations s_n and S_n from $\mathbb{C}^{m-1}$ into $\mathbb{C} \cup \{\infty\}$ iteratively in terms of the sequences $a_i(n)$ by the following equations.

$$s_n(w_1, \ldots, w_{m-1}) = \frac{-a_m(n)}{a_{m-1}(n) + a_{m-2}(n)w_1 + a_{m-3}(n)w_1w_2 + \cdots + a_0(n)w_1w_2\cdots w_{m-1}},$$

$$S_0(w_1, \ldots, w_{m-1}) = s_0(w_1, \ldots, w_{m-1}),$$

$$S_n(w_1, \ldots, w_{m-1}) = S_{n-1}(s_n(w_1, \ldots, w_{m-1}), w_1, \ldots, w_{m-2}),$$

for $n \geq 1$, and $f_n = S_n(0, 0, \ldots, 0)$. Then the sequence f_n are called the approximants of the G-continued fraction

$$K\left[\frac{-a_m(n)}{a_{m-1}(n); \ldots; a_0(n)}\right].$$

If $f = \lim_{n\to\infty} f_n$ exists, then the G-continued fraction is said to converge to this limiting value.

We define for arbitrary solutions $x_n^{(1)},\ x_n^{(2)}, \cdots, x_n^{(p)}$ of (0)

$$E_N\left(x_n^{(1)}, \cdots x_n^{(p)}\right) = \begin{vmatrix} x_{N+p-m+1}^{(1)} & \cdots & x_{N+p-m+1}^{(m)} \\ \vdots & & \vdots \\ x_{N+p}^{(1)} & \cdots & x_{N+p}^{(m)} \end{vmatrix}.$$

The generalization of Pincherle's Theorem that we will use for giving convergence of our expansion is the following theorem of Zahar [9].

THEOREM 0. *(Zahar) If the recurrence relation* (0) *has a fundamental system of solutions* $X_n^{(1)}, \cdots, X_n^{(m)}$ *with* $X_0^{(m)} \neq 0$ *and for which*

$$\lim_{N\to\infty} \frac{E_N\left(X_n^{(1)}, \cdots, X_n^{(i-1)}, X_n^{(m)}, X_n^{(i+1)}, \cdots, X_n^{(m-1)}\right)}{E_N\left(X_n^{(1)}, \ldots, X_n^{(m-1)}\right)} = 0, \qquad (1)$$

for $i = 1, \ldots, m-1$, *then the G-continued fraction* $K\left[\dfrac{-a_m(n)}{a_{m-1}(n); \ldots; a_0(n)}\right]$ *converges to* $X_1^{(m)}/X_0^{(m)}$.

This theorem will be applied to basic hypergeometric series by making use of the theorem of Thomae [8], which states that the analytic solutions (analytic up to a multiple of z^γ, $\gamma \in \mathbb{R}$) at the origin of the q-difference equation

$$(1-\alpha_0 zq^n)f(zq^n) + (\beta_1 - \alpha_1 zq^n)f(zq^{n+1}) + \cdots + (\beta_m - \alpha_m zq^n)f(zq^{n+m}) = 0 \quad (1)$$

are given for $i = 1,2, \ldots, m$ by

$$f_i(zq^n) = (zq^n)^{\gamma_i} \sum_{k\geq 0} \frac{(c_1 d_i^{-1})_k \cdots (c_{k'-l'} d_i^{-1})_k}{(d_1 q d_i^{-1})_k \cdots (d_m q d_i^{-1})_k} q^{\binom{k}{2} l'} \left(\frac{\alpha_{l'}}{d_i^{l'}}\right)^k (zq^n)^k,$$

where $|q| < 1$, $q^{-\gamma_i} = d_i$,

$$\sum_{i=0}^{m} \alpha_i z^i = \alpha_{l'} z^{l'} (1 - c_1 z) \cdots (1 - c_{k'-l'} z)$$

and

$$\sum_{i=0}^{m} \beta_i z^i = (1 - d_1 z) \cdots (1 - d_m z) \text{ with } \beta_0 = 1, \quad \beta_m \neq 0.$$

We are now ready to state our theorem. This G-continued fraction generalizes the continued fraction given earlier by the first author in [2]. It reduces to this result in the event that $m = 2$. Here we have arbitrarily many free parameters, while before we had only three, not counting the independant variable and q.

THEOREM 1. *If d_i for $i = 1, \cdots, m$ are distinct and $|d_i| < |d_m|$ for $i = 1, \cdots, m$, the G-continued fraction*

$$K\left[\frac{(\alpha_0 z q^n - 1)}{(\beta_1 - \alpha_1 z q^n); \cdots; (\beta_{m-1} - \alpha_{m-1} z q^n); (\beta_m - \alpha_m z q^n)}\right]$$

converges to $\dfrac{f_m(zq)}{f_m(z)}$.

Before we give our proof we introduce some notation.

DEFINITION 1. *The Casorati determinant for q-difference equations is defined by*

$$C(f_1, \cdots, f_n) = \begin{vmatrix} f_1 & \cdots & f_n \\ \eta f_1 & \cdots & \eta f_n \\ \vdots & & \vdots \\ \eta^{n-1} f_1 & \cdots & \eta^{n-1} f_n \end{vmatrix},$$

where $\eta^k f(z) = f(zq^k)$.

Note that the non-vanishing of the Casorati implies the linear independance of the solutions of the q-difference equation over the coefficient field [4], which in this case is $\mathbb{C}(z)$.

Proof. Write $f_i(zq^n) = z^{\gamma_i} d_i^{-n} (1 + g_i(zq^n))$, where

$$g_i(zq^n) = \sum_{k \geq 1} \frac{(c_1 d_i^{-1})_k \cdots (c_{k'-l'} d_i^{-1})_k}{(d_1 q d_i^{-1})_k \cdots (d_m q d_i^{-1})_k} q^{\binom{k}{2} l'} \left(\frac{\alpha_{l'}}{d_i^{l'}} z q^n\right)^k.$$

Then

$$C\left(f_1(zq^n), \cdots, f_m(zq^n)\right)$$

$$= \prod_{j=1}^{m} z^{\gamma_j} d_j^{-n} \begin{vmatrix} 1+g_1(zq^n) & \dots & 1+g_m(zq^n) \\ \vdots & & \vdots \\ d_1^{-(m-1)}\left(1+g_1(zq^{n+m-1})\right) & \dots & d_m^{-(m-1)}\left(1+g_m(zq^{n+m-1})\right) \end{vmatrix}$$

$$= \prod_{j=1}^{m} z^{\gamma_j} d_j^{-n} \sum_{\sigma \in S_m} sgn(\sigma) a_{1\sigma(1)} \cdots a_{m\sigma(m)}$$

$$= \prod_{j=1}^{m} z^{\gamma_j} d_j^{-n} \left[\sum_{\sigma \in S_m} sgn(\sigma) d_{\sigma(2)}^{-1} \cdots d_{\sigma(m)}^{-(m-1)} + R_n \right],$$

with $a_{ij} = d_j^{-(i-1)}\left(1+g_j(zq^{n+i-1})\right)$, where S_n is the symmetric group on n letters, $sgn(\sigma)$ is the sign of the permutation σ, and

$$R_n = \sum_{\sigma \in S_m} sgn(\sigma) d_{\sigma(2)}^{-1} \cdots d_{\sigma_m}^{-(m-1)} \left[\left(1+g_{\sigma(1)}(zq^n)\right) \cdots \left(1+g_{\sigma(m)}(zq^{n+m-1})\right) - 1 \right].$$

Since

$$\sum_{\sigma \in S_m} sgn(\sigma) d_{\sigma(2)}^{-1} \cdots d_{\sigma(m)}^{-(m-1)} = \begin{vmatrix} 1 & \dots & 1 \\ d_1^{-1} & \dots & d_m^{-1} \\ \vdots & & \vdots \\ d_1^{-(m-1)} & \dots & d_m^{-(m-1)} \end{vmatrix} = \prod_{1 \le p < q \le m} (d_q^{-1} - d_p^{-1}),$$

it follows that $R_n = O(q^n)$. Hence

$$C(f_1, \cdots, f_m) = \prod_{j=1}^{m} z^{\gamma_j} d_j^{-n} \left[\prod_{1 \le p < q \le m} (d_q^{-1} - d_p^{-1}) + O(q^n) \right].$$

Since the d_i $(i = 1, \cdots, m)$ are distinct, we see that $\prod_{1 \le p < q \le m} (d_q^{-1} - d_p^{-1}) + R_n \neq 0$; so $C(f_1, \cdots, f_m) \neq 0$. Hence $f_1(zq^n), \cdots, f_m(zq^n)$ are linearly independent over $\mathbb{C}(z)$. Also clearly $f_m(z) \neq 0$.

Similarly we have

$$C\left(f_1(zq^{N+2}), \cdots, f_{m-1}(zq^{N+2})\right)$$

$$= \prod_{j=1}^{m-1} z^{\gamma_j} d_j^{-(N+2)} \left[\prod_{1 \le p < q \le m-1} (d_q^{-1} - d_p^{-1}) + O(q^{N+2}) \right],$$

and

$$C\left(f_1(zq^{N+2}), \cdots, f_{i-1}(zq^{N+2}), f_m(zq^{N+2}), f_{i+1}(zq^{N+2}), \cdots, f_{m-1}(zq^{N+2})\right)$$

$$= \prod_{\substack{j=1 \\ j \ne i}}^{m} z^{\gamma_j} d_j^{-(N+2)} \left[\prod_{1 \le p < q \le m-1} (e_q^{-1} - e_p^{-1}) + O(q^{N+2}) \right],$$

where $e_p = d_p$ $(p = 1, \cdots, i-1, i+1, \cdots, m-1)$ and $e_i = d_m$. Therefore,

$$\frac{E_N\left(f_1(zq^n), \cdots, f_{i-1}(zq^n), f_m(zq^n), \cdots, f_{m-1}(zq^n)\right)}{E_N\left(f_1(zq^n), \cdots, f_{m-1}(zq^n)\right)}$$

$$= \frac{C\left(f_1(zq^{N+2}), \cdots, f_{i-1}(zq^{N+2}), f_m(zq^{N+2}) f_{i+1}(zq^{N+2}), \cdots, fm-1(zq^{N+2})\right)}{C\left(f_1(zq^{N+2}), \cdots, f_{m-1}(zq^{N+2})\right)}$$

$$= z^{\gamma_m - \gamma_i} \left(\frac{d_m}{d_i}\right)^{-(N+2)} \frac{\prod_{1 \le p < q \le m-1} (d_q^{-1} - d_p^{-1}) + O(q^{N+2})}{\prod_{1 \le p < q \le m-1} (e_q^{-1} - e_p^{-1}) + O(q^{N+2})}.$$

So (1) is satisfied if $|d_i| < |d_m|$ for $i = 1, \cdots, m-1$. Thus by Theorem 0 the G-continued fraction converges and its value is given by $\dfrac{f_m(zq)}{f_m(z)}$.

□

We were able to obtain our theorem by using the fact that for the basic hypergeometric functions considered, the full vector space of solutions of their q-difference equations was known. In general this is not the case.

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