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Robust and minimum norm partial quadratic **eigenvalue** assignment in vibrating systems: A new optimization approach

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ABSTRACT

The partial quadratic eigenvalue assignment problem (PQEVAP) concerns reassigning a few undesired eigenvalues of a quadratic matrix pencil to suitably chosen locations and keeping the other large number of eigenvalues and eigenvectors unchanged (no spill-over). The problem naturally arises in controlling dangerous vibrations in structures by means of active feedback control design. For practical viability, the design must be robust, which requires that the norms of the feedback matrices and the condition number of the closed-loop eigenvectors are as small as possible. The problem of computing feedback matrices that satisfy the above two practical requirements is known as the Robust Partial Quadratic Eigenvalue Assignment Problem (RPQEVAP). In this paper, we formulate the RPQEVAP as an unconstrained minimization problem with the cost function involving the condition number of the closed-loop eigenvector matrix and two feedback norms. Since only a small number of eigenvalues of the open-loop quadratic pencil are computable using the state-of-the-art matrix computational techniques and/or measurable in a vibration laboratory, it is imperative that the problem is solved using these small number of eigenvalues and the corresponding eigenvectors. To this end, a class of the feedback matrices are obtained in parametric form, parameterized by a single parametric matrix, and the cost function and the required gradient formulas for the optimization problem are developed in terms of the small number of eigenvalues that are reassigned and their corresponding eigenvectors. The problem is solved directly in quadratic setting without transforming it to a standard first-order control problem and most importantly, the significant “no spill-over property” of the closed-loop eigenvalues and eigenvectors is established by means of a mathematical result. These features make the proposed method practically applicable even for very large structures. Results on numerical experiments show that the proposed method considerably reduces both feedback norms and the sensitivity of the closed-loop eigenvalues. A study on robustness of the system responses of the method under small perturbations show that the responses of the perturbed closed-loop system are compatible with perturbations.

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1. Introduction

Vibrating structures are distributed parameter systems [1,2,24]. However, for the sake of computational convenience, these are very often discretized into a system of second-order differential equations of the form [2,24]:

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = 0, \quad (1)$$

where the three n -by- n matrices M , D , and K are, respectively, mass, **damping**, and stiffness matrices. Very often they have special structures, such as, M , D , and K are symmetric, M is positive definite and diagonal or tridiagonal, and K is positive semidefinite and tridiagonal, etc.

By the separation of variables, $\mathbf{x}(t) = \mathbf{x} e^{\lambda t}$, where $\mathbf{x}$ is a constant vector, one can obtain the general solution of (1) in terms of the eigenvalues and eigenvectors $\{(\lambda_k, \mathbf{x}_k)\}_{k=1}^{2n}$ of the quadratic matrix pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K, \quad (2)$$

where $P(\lambda_k)\mathbf{x}_k = 0$ for $k = 1, 2, \dots, 2n$. The eigenvalues are related to natural frequencies and eigenvectors are mode shapes. Thus, the dynamics of (1) is governed by the eigenvalues and eigenvectors of $P(\lambda)$. Dangerous vibrations occur when some of the natural frequencies become close or equal to those of the external frequencies.

Traditionally, vibration of structures, is controlled by using a passive device, such as a shock absorber in a car. Though quite economic and easy to implement, such a passive device has severe practical limitations. These include its limited performance, changing the global dynamics, and others. On the other hand, because of the recent remarkable advances in sensors and actuators, the use of active vibration control force is becoming more popular. The active control devices are capable of overcoming the shortcomings of the passive devices. In active control strategy, vibrations are measured by sensors and then transmitted to a computer system where the required control force is computed in real-time and then applied to the structure by means of actuation.

Thus, the most important and challenging aspect of implementing an active control strategy is to efficiently and effectively compute the required feedback control force. Mathematically, the problem of computing the state feedback control force can be described as follows:

Let a control force in the form $B\mathbf{u}(t)$, where B is an n -by- m control matrix and $\mathbf{u}(t)$ is the associated control vector, be applied to the structure, and suppose

$$\mathbf{u}(t) = F_1^T \dot{\mathbf{x}} + F_2^T \mathbf{x}(t), \quad (3)$$

where the n -by- m matrices F_1 and F_2 are called feedback matrices. Then the dynamics of the closed-loop system is governed by the eigenvalues and eigenvectors of the closed-loop pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D - BF_1^T) + (K - BF_2^T). \quad (4)$$

If M is nonsingular, which is usually the case in practical applications, then there are $2n$ eigenvalues and $2n$ eigenvectors. Therefore, we shall find the two active feedback matrices F_1 and F_2 such that a few undesirable eigenvalues, say $\lambda_1, \dots, \lambda_p$ ($p \ll 2n$), are replaced by suitable chosen ones, $\mu_1, \dots, \mu_p$, while the remaining $(2n - p)$ eigenvalues $\lambda_{p+1}, \dots, \lambda_{2n}$ of $P(\lambda)$ and the associated eigenvectors, remain unchanged; that is, there will be *no spill-over* of the unassigned closed-loop **eigenvalues** and eigenvectors. *This property ensures that those eigenvalues that are not be specifically reassigned will not themselves become resonant or unstable.* The above problem is known as the *partial quadratic eigenvalue assignment problem* (PQEVAP).

Since there exist excellent numerical methods for solving eigenvalue assignment problems in standard first-order control system (see Datta [11]), including some for partial eigenvalue assignment problem, such as by Porter and Crossley [32], Datta and Sarkissian [19], and Datta and Saad [17], it is only natural to think of solving the PQEVAP by transforming the problem to a first-order control problem and then using a suitable first-order technique.

Unfortunately, there are severe computational drawbacks, associated with that approach. These include (i) requirement of inversion of a possibly ill-conditioned mass matrix and (ii) loss of exploitable properties, such as the symmetry, positive definiteness, sparsity, etc. Furthermore, the existing eigenvalue methods are designed for small-order control problems and in order to use an existing method for a **second-order** model, which are usually very large, the model order must be reduced considerably by using techniques of model reduction, such as the Guyan reduction technique (see Inman [23]). However, the existing model reduction techniques, including the popular Guyan reduction technique, may fail to produce even a few natural frequencies and mode shapes accurately (see Friswell and Mottershead [21]). In spite of several attempts by numerical linear algebra researchers, effective numerical model reduction techniques that work exclusively in second-order model are rare. Similarly, the state-of-the-art *independent modal space control* (IMSC) approach (see Inman [23]), Meirovitch et al. [27,28], also is not practical for the large and sparse PQEVAP. For open-loop decoupling, the IMSC approach needs the knowledge of the complete spectrum and eigenvectors of $P(\lambda)$, and for closed-loop decoupling, a stringent requirement of the number of sensors and actuators must be met [23,28]. On the other hand, the state-of-the-art computational technique, the **Jacobi**-Davidson method (see Datta [10] and Tisseur and Meerbergen [38]) for the quadratic eigenproblem, is capable of computing only a few extremal eigenvalues and eigenvectors of a large quadratic pencil.

To meet these practical engineering and computational difficulties, several “*direct, partial-modal and no spill-over*” methods for the PQEVAP have been developed in recent years [12–15,20,34,8].

These methods (i) work directly in the second-order model, (ii) do not require any model reduction, and (iii) can be implemented with the knowledge of only a small number of eigenvalues which are to be reassigned and the corresponding eigenvectors and (iv) above all, *the no spill-over property is established by means of a mathematical theory based on the new orthogonality relations of the eigenvectors of the quadratic matrix pencil $P(\lambda)$, developed in these papers*. Datta and Sarkissian [18] and Datta et al. [16] have also developed a partial eigenvalue assignment method for gyroscopic distributed parameters systems that does not require discretizations of the original model.

Though these methods are suitable to compute a set of feedback matrices in a numerically desirable way, they do not address the problem of *robust control design*.

The robustness of the closed-loop eigenvalues and eigenvectors depends upon (i) norms of the feedback matrices and (ii) the conditioning of the closed-loop eigenvectors (see Datta [11]). The small feedback norms guarantee small signals and therefore low cost, and low conditioning of the closed-loop eigenvalues assures that the closed-loop eigenvalues remain insensitive to small perturbations of the data. The latter is known as *robust partial quadratic eigenvalue assignment problem* (RPQEVAP).

Both the RPQEVAP and the problem of norm minimization are clearly optimization problems. To solve these problems in an optimization setting, some additional challenges must be met. These are:

- The feedback matrices must be expressed in some parametric forms so that the optimization problems can be solved by exploiting this parametric matrix.
- Gradient formulas must be developed using the knowledge of only these few eigenvalues and eigenvectors of $P(\lambda)$ that are computationally available or measurable.

There exist a few papers on robustness for the *complete eigenvalue assignment* in the *first-order* control systems, such as those, by Keel et al. [26], Keel and Bhattacharyya [25], Cavin III and Bhattacharyya [7], Varga [39,40], etc. However, work on robustness for quadratic partial eigenvalue assignment is rare. Only two papers that deals with the robustness of the PQEVAP have been published so far. They are: the paper by Qian and Xu [33] and the recent papers by Brahama and Datta [3,4]. The method proposed by Qian and Xu is not an optimization-based and has limitations. Brahama and Datta, instead of minimizing the condition number of the closed-loop eigenvector matrix Y , $\text{Cond}_F(Y) = \|Y\|_F \|Y^{-1}\|_F$, minimized the expression $\|(Y^T Y - I)^2\|_F$. The rationale is that if the matrix Y is made close to an orthogonal matrix, then Y will be well-conditioned. *But it is desirable that $\text{Cond}_F(Y)$ itself is minimized* [5]. There also exists an earlier paper by Chu and Datta [9] that deals with robustness of the quadratic eigenvalue assignment but complete spectrum assignment is considered there. Another recent paper of Datta et al. [15] considers robust partial assignment in a cubic pencil in an optimization setting that deals with robustness of vibrating structures with aerodynamics effects. The technique proposed there, however, requires knowledge of the complete spectrum and the associated eigenvectors for its implementation. Mottershead et al. [29] have also developed receptance-based method for desensitizing the closed-loop eigenvalues.

In this paper, a mathematically equivalent expression of $\text{Cond}_F(Y)$, namely, $\|Y\|_F^2 + \|Y^{-1}\|_F^2$ is minimized. This minimization problem then is combined with the problem of minimization of two feedback norms to simultaneously minimize the closed-loop condition number and feedback norms.

Numerical experiments are performed to:

- Study the performance of the proposed algorithm.
- Compare the proposed algorithm with Brahama–Datta's algorithm [4] and also with both the single and multi-input versions of Ram–Elhay's algorithm [34].
- Study the robustness of the system.

The results of comparison show that the proposed algorithm is superior in its performances of both aspects of norms and the closed-loop condition number minimization to both Brahama–Datta and Ram–Elhay's algorithms.

Also, a comparison of the system responses of the original system with perturbed system, in which the matrix K goes under several small perturbations while the matrices M , D , B are kept fixed, shows the responses remain insensitive to these perturbations and successfully approach to the steady state.

2. Notations and assumptions

$\{\lambda_1, \lambda_2, \dots, \lambda_p; \lambda_{p+1}, \dots, \lambda_{2n}\}$ — eigenvalues of $P(\lambda)$.

$A_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p)$ — eigenvalue matrix of the eigenvalues that need to be reassigned.

X_1 = matrix of eigenvectors corresponding to the eigenvalues in A_1 .

$A'_1 = \text{diag}(\mu_1, \mu_2, \dots, \mu_p)$ — eigenvalue matrix of the eigenvalues that are to be assigned.

Y = the closed – loop eigenvector matrix.

Y_1 = matrix of eigenvectors corresponding to the eigenvalues in A'_1 .

$A_2 = \text{diag}(\lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_{2n})$ — eigenvalue matrix of the eigenvalues that are left invariant.

X_2 = matrix of eigenvectors corresponding to the eigenvalues in A_2 , which are also left invariant.

$\text{Cond}_F(A) = \|A\|_F \|A^{-1}\|_F$ (condition number of A in Frobenius norm).

A^H — conjugate transpose of A ($A^H = (\bar{A})^T$).

A^+ — the Moore–Penrose inverse of A .

$\text{tr}(A)$ — trace of A .

$\text{vec}(\cdot)$ — the vec operator which vectorizes a matrix by stacking its columns.

$E \otimes F = \begin{bmatrix} e_{11}F & \dots & e_{1n}F \\ \vdots & & \vdots \\ e_{m1}F & \dots & e_{mn}F \end{bmatrix}$ — the Kronecker product of the two matrices $E = (e_{ij})_{m \times n}$ and F of order $s \times t$.

Throughout the paper, the following assumptions will be made:

- M, D , and K are symmetric and M is positive definite ($M > 0$),
- The control matrix B has full column rank,
- $\{\lambda_1, \lambda_2, \dots, \lambda_p\} \cap \{\mu_1, \mu_2, \dots, \mu_p\} = \emptyset$,
- $\{\lambda_1, \lambda_2, \dots, \lambda_p\} \cap \{\lambda_{p+1}, \dots, \lambda_{2n}\} = \emptyset$, and
- $(P(\lambda), B)$ is partially controllable with respect to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_p$; that is $\text{rank}(P(\lambda_i), B) = n$, $i = 1, 2, \dots, p$.

3. Problem statement

The problem of simultaneously minimizing the condition number of the closed-loop eigenvector matrix, Y and two feedback norms, can be stated in an optimization setting as follows:

$$\min J = \frac{1}{2} \alpha (\|Y\|_F^2 + \|Y^{-1}\|_F^2) + \frac{1}{2} (1 - \alpha) (c_1 \|F_1\|_F^2 + c_2 \|F_2\|_F^2), \quad (5)$$

where $0 \leq \alpha \leq 1$ and $c_1, c_2 > 0$. Here c_1 and c_2 are weighting parameters chosen according to the practical requirements. When $c_1 = c_2 = 1$, the equal weights are placed on state and velocity feedbacks.

We note that if $\alpha = 0$, then we have pure norm minimization problem and if $\alpha = 1$, then we have RPQEVAP. We also remark that the feedback matrices F_1 and F_2 stand for different physical quantities. Therefore, one may choose the suitable weighting parameter c_1 and c_2 from the physical point of view.

4. Problem solution

The solution to the problem comes in three sequential steps:

- A family of parametric feedback matrices, parameterized by a single parametric matrix Γ , is obtained.
- The closed-loop eigenvector matrix Y and its inverse are expressed in terms of the matrix Γ .
- Finally, the gradient expressions are derived with knowledge of only a small number of eigenvalues that need to be reassigned and the associated eigenvectors.

4.1. Parametric expressions for feedback matrices F_1 and F_2

The derivation of the parametric expressions of F_1 and F_2 (given below) depends upon the following orthogonality relation of the quadratic pencil $P(\lambda)$.

Lemma 4.1. (i)

$$A_1 X_1^T M X_2 + X_1^T M X_2 A_2 + X_1^T D X_2 = 0;$$

(ii)

$$A_1 X_1^T M X_2 A_2 - X_1^T K X_2 = 0.$$

Proof. We prove here only Part (i). The proof of Part (ii) is similar.

Proof of Part (i): Since A_1 and A_2 are eigenvalue matrices of $P(\lambda)$ and X_1 and X_2 are the associate eigenvector matrices, we have

$$M X_1 A_1^2 + D X_1 A_1 + K X_1 = 0, \quad (6)$$

$$M X_2 A_2^2 + D X_2 A_2 + K X_2 = 0. \quad (7)$$

From (6), we get

$$-KX_1 = MX_1A_1^2 + DX_1A_1.$$

Taking the transpose of the both sides gives rise to

$$-X_1^TK = A_1^2X_1^TM + A_1X_1^TD.$$

Post-multiplying both sides of the above equation by X_2 we get

$$-X_1^TKX_2 = A_1^2X_1^TMX_2 + A_1X_1^TDX_2. \quad (8)$$

Also, pre-multiplying both sides of (7) by X_1^T we get

$$X_1^TMX_2A_2^2 + X_1^TDX_2A_2 + X_1^TKX_2 = 0. \quad (9)$$

From (8) and (9) we have

$$A_1^2X_1^TMX_2 + A_1X_1^TDX_2 = X_1^TMX_2A_2^2 + X_1^TDX_2A_2$$

or

$$A_1(A_1X_1^TMX_2 + X_1^TDX_2) = (X_1^TMX_2A_2 + X_1^TDX_2)A_2. \quad (10)$$

Adding the term $A_1X_1^TMX_2A_2$ to both sides of Eq. (10) gives

$$A_1(A_1X_1^TMX_2 + X_1^TMX_2A_2 + X_1^TDX_2) = (A_1X_1^TMX_2 + X_1^TMX_2A_2 + X_1^TDX_2)A_2. \quad (11)$$

Let $\emptyset = (\theta_{ij}) := A_1X_1^TMX_2 + X_1^TMX_2A_2 + X_1^TDX_2$. It then follows from (11) that

$$(\lambda_j - \lambda_{k+p})\theta_{jk} = 0 \quad \text{for } 1 \leq j \leq p \text{ and } 1 \leq k \leq 2n - p. \quad (12)$$

By the assumption that $\{\lambda_1, \dots, \lambda_p\} \cap \{\lambda_{p+1}, \dots, \lambda_{2n}\} = \emptyset$, we have $\theta_{jk} = 0$ for all $1 \leq j \leq p$, $1 \leq k \leq 2n - p$. This completes the proof. $\square$

Remark 4.2. Lemma 4.1 can also be derived from an orthogonality relation proved earlier by Brahama and Datta [3,4]. But there it was assumed that $\lambda_1, \dots, \lambda_p$ are different from zero. No such assumption has been made here. This has a consequence that one does not have to be restricted to be assigning only nonzero eigenvalues. The zero open-loop eigenvalues can also be reassigned, if needed.

Theorem 4.3. Let $\Gamma = [\gamma_1, \dots, \gamma_p] \in \mathbb{C}^{m \times p}$ be a nonzero matrix such that if $\mu_j = \bar{\mu}_k$, then $\gamma_j = \bar{\gamma}_k$.

(i) For any arbitrary choice of Φ , the feedback matrices F_1 and F_2 defined by

$$F_1 = MX_1\Phi^T$$

and

$$F_2 = (MX_1A_1 + DX_1)\Phi^T$$

are such that

$$MX_2A_2^2 + (D - BF_1^T)X_2A_2 + (K - BF_2^T)X_2 = 0$$

(That is, there will be no spill-over).

(ii) If Φ is chosen satisfying the $m \times p$ linear system: $\Phi Z = \Gamma$, where Γ is arbitrary and Z is obtained by solving the Sylvester equation:

$$A_1Z - ZA_1' = -X_1^TB\Gamma, \quad (13)$$

then F_1 and F_2 are real and the closed-loop eigenvalues will include $\mu_1, \mu_2, \dots, \mu_p$ with the corresponding eigenvectors $y_1, y_2, \dots, y_p$.

(iii) Suppose that Z is invertible. Then $S := [F_2^T, F_1^T] = \Gamma Z^{-1}C$ where $C = [A_1X_1^TM + X_1^TD, X_1^TM]$.

Sketch of the proof :

A. Part (i) is proved by using the orthogonality relations given in Lemma 4.1.

B. The proof of Part (ii) amounts to showing that

$$MY_1A_1'2 + (D - BF_1^T)Y_1A_1' + (K - BF_2^T)Y_1 = 0,$$

which is equivalent to showing that

$$MY_1A_1'2 + DY_1A_1' + KY_1 = B\Phi Z,$$

where

$$Z = A_1 X_1^T M Y_1 A_1' - X_1^T K Y_1.$$

Proof of Part (ii) now follows by setting $\Phi Z = \Gamma$ and noting that Z solves the Sylvester equation (13).

C. Proof of Part (iii) is easy.

Details of the proof are omitted, because they are similar to the proof given in Brahma and Datta [4].

4.2. Expressing the matrix Y and Y^{-1} in terms of the parametric matrix Γ

To express Y in terms of Γ , we first note that Y can be expressed as [4,33]

$$Y = \begin{bmatrix} Y_1 & X_2 \\ Y_1 A_1' & X_2 A_2 \end{bmatrix}. \quad (14)$$

Let $Y_1 = (y_1, y_2, \dots, y_p)$. Then each y_k , $k = 1, \dots, p$ is uniquely determined by the system of equations:

$$(\mu_k^2 M + \mu_k D + K)y_k = B\gamma_k. \quad (15)$$

Thus, Y_1 is a function of the parametric matrix Γ . Also, the (unknown) matrices X_2 and A_2 are fixed and do not take part in feedback computations, so they are independent of Γ .

We now show how to express Y^{-1} as a function of Γ . To do so, we first state the following preliminary lemma.

Lemma 4.4 (Sun [37, Lemma 1.3]). Let $A_1 \in \mathbb{C}^{p \times m}$, $A_2 \in \mathbb{C}^{n \times q}$, $A_3 \in \mathbb{C}^{p \times q}$, $E \in \mathbb{C}^{m \times n}$ be given. Define

$$\Omega := \{E \in \mathbb{C}^{m \times n} : A_1 E A_2 = A_3\}.$$

Then $\Omega \neq \emptyset$ if and only if A_1, A_2, A_3 satisfy

$$A_1 A_1^+ A_3 A_2^+ A_2 = A_3$$

and in case of $\Omega \neq \emptyset$, any $E \in \Omega$ can be expressed as

$$E = A_1^+ A_3 A_2^+ + T - A_1^+ A_1 T A_2 A_2^+,$$

where $T \in \mathbb{C}^{m \times n}$. Moreover, there is a unique matrix $E^{(0)} \in \Omega$ given by

$$E^{(0)} = A_1^+ A_3 A_2^+ + E^* - A_1^+ A_1 E^* A_2 A_2^+$$

such that for any unitarily invariant norm $\|\cdot\|$,

$$\|E^{(0)} - E^*\| = \min_{E \in \Omega} \|E - E^*\|.$$

The next theorem shows how the inverse of the nonsingular matrix Y is determined.

Theorem 4.5. Suppose that the nontrivial matrix $\Gamma = \{\gamma_1, \dots, \gamma_p\} \in \mathbb{C}^{m \times p}$ is such that if $\mu_j = \bar{\mu}_k$, then $\gamma_j = \bar{\gamma}_k$. Let Z satisfy (13) and suppose that Z is invertible. Also, let the matrices $F_1 = M X_1 \Phi^T$, $F_2 = (M X_1 A_1 + D A_1) \Phi^T$, where Φ is determined by $\Phi Z = \Gamma$, and let the matrices S and C be defined as in Theorem 4.3 and the $2n$ -by- $2n$ complex matrix Y defined in (14) be nonsingular. Let

$$Y := [\tilde{Y}_1, \tilde{X}_2], \quad \tilde{Y}_1 \in \mathbb{C}^{2n \times p}, \quad \tilde{X}_2 \in \mathbb{C}^{2n \times (2n-p)}.$$

Then the PQEVAP is solvable if Y^{-1} is given by

$$Y^{-1} = \begin{bmatrix} Z^{-1} C \\ \tilde{X}_2^+ (I - \tilde{Y}_1 Z^{-1} C) \end{bmatrix}. \quad (16)$$

Proof. It easily follows that the PQEVAP is solvable if and only if

$$M[Y_1 A_1' 2, X_2 A_2'] + (D - B F_1^T)[Y_1 A_1', X_2 A_2] + (K - B F_2^T)[Y_1, X_2] = 0$$

or

$$B F_1^T[Y_1 A_1', X_2 A_2] + B F_2^T[Y_1, X_2] = M[Y_1 A_1' 2, X_2 A_2'] + D[Y_1 A_1', X_2 A_2] + K[Y_1, X_2]. \quad (17)$$

It is easy to know that (17) holds if and only if Y satisfies

$$B S Y = [M Y_1 A_1' 2 + D Y_1 A_1' + K Y_1, M X_2 A_2' 2 + D X_2 A_2 + K X_2]. \quad (18)$$

By (15), we have

$$M Y_1 A_1' 2 + D Y_1 A_1' + K Y_1 = B \Gamma. \quad (19)$$

Using the notations X_2 and A_2 , we get

$$MX_2A_2^2 + DX_2A_2 + KX_2 = 0. \quad (20)$$

Substituting (19) and (20) into (18) gives rise to

$$BSY = [B\Gamma, 0] = B[\Gamma, 0].$$

Since the matrix B is full column rank and the eigenvector matrix Y is nonsingular, we have

$$SY = [\Gamma, 0]. \quad (21)$$

By (iii) of Theorem 4.3, we know that $S = \Gamma Z^{-1}C$. Then (21) is equivalent to

$$\Gamma Z^{-1}C = \Gamma[I_p, 0]Y^{-1}. \quad (22)$$

It is observed that (22) holds if

$$[I_p, 0]Y^{-1} = Z^{-1}C. \quad (23)$$

By Lemma 4.4, the equation $[I_p, 0]E = Z^{-1}C$ has a solution as follows:

$$E_Y^{(0)} = [I_p, 0]^+ Z^{-1}C + (I - [I_p, 0]^+ [I_p, 0])Y^{-1}, \quad (24)$$

which is closest to Y^{-1} in the sense of any unitarily invariant norm. By using (23), it is easy to check that $E_Y^{(0)}Y = I$. Thus, $Y^{-1} = E_Y^{(0)}$. Let

$$Y^{-1} := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad W_1 \in \mathbb{R}^{p \times 2n}, \quad W_2 \in \mathbb{R}^{(2n-p) \times 2n}.$$

Then, by (24), we obtain that

$$W_1 = Z^{-1}C. \quad (25)$$

Next, we compute the expression of W_2 . By $YY^{-1} = I$, we have

$$\tilde{X}_2 W_2 = I - \tilde{Y}_1 W_1. \quad (26)$$

Let the SVD (cf. [22]) of the matrix $\tilde{X}_2$ be given by

$$\tilde{X}_2 = \tilde{U} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \tilde{V}, \quad \tilde{U} = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{U}_1 \in \mathbb{C}^{2n \times (2n-p)},$$

$$\Sigma = \text{diag}(\sigma_1, \dots, \sigma_{2n-p}), \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{2n-p} > 0.$$

Premultiplying (26) by $\tilde{U}^H$ gives

$$\begin{cases} \Sigma \tilde{V}^H W_2 = \tilde{U}_1^H (I - \tilde{Y}_1 W_1), \\ 0 = \tilde{U}_2^H (I - \tilde{Y}_1 W_1). \end{cases}$$

Thus,

$$W_2 = \tilde{V} \Sigma^{-1} \tilde{U}_1^H (I - \tilde{Y}_1 W_1).$$

This, together with (25) leads to (16). $\square$

Expressing the cost function J as a function of Γ : In view of Theorem 4.5 and using (25) and (26), we can further reduce the cost function J as follows:

$$J = \frac{1}{2} \alpha (\|Y_1\|_F^2 + \|Y_1 A_1\|_F^2 + \|W_1\|_F^2 + \|W_2\|_F^2) + \frac{1}{2} \alpha (\|X_2\|_F^2 + \|X_2 A_2\|_F^2) + \frac{1}{2} (1 - \alpha) (c_1 \|F_1\|_F^2 + c_2 \|F_2\|_F^2). \quad (27)$$

We note that A_2 is a fixed matrix, and X_2 is independent of Γ . Also, from Theorem 4.3 and (15), we see that the matrices F_1 , F_2 , Y_1 , W_1 , and W_2 are the functions of Γ .

The expression (27) still contains the term W_2 which involves $\tilde{X}_2^+$. However, we note that

$$\|W_2\|_F \leq \|\tilde{X}_2^+\| \|I - \tilde{Y}_1 W_1\|_F = 1/\sigma_{\min}(\tilde{X}_2) \|I - \tilde{Y}_1 W_1\|_F,$$

where $\sigma_{\min}(\tilde{X}_2)$ denotes the smallest singular value of $\tilde{X}_2$. By assumption that $\tilde{X}_2$ has linearly independent columns, so we may expect that $\sigma_{\min}(\tilde{X}_2)$ is not too small. So, instead of minimizing J given by (27), we will minimize the following expression, which also for simplicity, we write as J :

$$J = \alpha J_1 + (1 - \alpha) J_2 + \alpha J_3, \quad (28)$$

where

$$\begin{cases} J_1 = \frac{1}{2}(\|Y_1\|_F^2 + \|Y_1 A_1'\|_F^2 + \|W_1\|_F^2 + \beta\|I - \tilde{Y}_1 W_1\|_F^2), \\ J_2 = \frac{1}{2}(c_1\|F_1\|_F^2 + c_2\|F_2\|_F^2), \\ J_3 = \frac{1}{2}(\|X_2\|_F^2 + \|X_2 A_2\|_F^2), \end{cases}$$

with the parameter $\beta > 0$ being a-priori estimate for the upper bound of $1/\sigma_{\min}^2(\tilde{X}_2)$.

Since J_3 is independent of Γ , we can now focus on the following unconstrained minimization problem:

$$\min_{\Gamma} \alpha J_1 + (1 - \alpha) J_2 = \frac{1}{2} \alpha (\|Y_1\|_F^2 + \|Y_1 A_1'\|_F^2 + \|W_1\|_F^2 + \beta\|I - \tilde{Y}_1 W_1\|_F^2) + \frac{1}{2} (1 - \alpha) (c_1\|F_1\|_F^2 + c_2\|F_2\|_F^2). \quad (29)$$

4.3. Gradient expressions in terms of A_1 , X_1 and Y_1

The next theorem gives the gradient formula of J_2 in term of the known quantities A_1 , A_1' , X_1 , and B .

Theorem 4.6. Let $\Gamma = [\gamma_1, \dots, \gamma_p] \in \mathbb{C}^{m \times p}$ be such that if $\mu_j = \bar{\mu}_k$, then $\gamma_j = \bar{\gamma}_k$. Let Z be the unique solution of the Sylvester equation:

$$A_1 Z - Z A_1' = -X_1^T B \Gamma.$$

Let $F_1 = M X_1 \Phi^T$ and $F_2 = (M X_1 A_1 + D X_1) \Phi^T$, where Φ is determined by $\Phi Z = \Gamma$. Define $G := [\sqrt{c_2} F_2^T, \sqrt{c_1} F_1^T]$ and $Q := [\sqrt{c_2} (A_1 X_1^T M + X_1^T D), \sqrt{c_1} X_1^T M]$. Suppose that Z is nonsingular and Γ satisfies the Sylvester equation:

$$A_1' \Gamma - \Gamma A_1 = -Z^{-1} Q G^H \Phi.$$

Then:

$$\begin{aligned} \text{(i)} \quad & G = \Gamma Z^{-1} Q; \\ \text{(ii)} \quad & \nabla_{\Gamma} J_2 = \frac{1}{2} [Z^{-1} Q G^H - \Gamma X_1^T B]^T. \end{aligned}$$

Proof. The result can be proved similarly as in Theorem 2 in [4]. We omit the details here. $\square$

In the following, we shall provide an explicit formula of the gradient of J_1 with respect to the parameter Γ via the Sylvester equation-based parameterization. By Theorem 4.3, we prove the following result on the gradient of J_1 with respect to Γ .

Theorem 4.7. Let the nontrivial matrix $\Gamma = \{\gamma_1, \dots, \gamma_p\} \in \mathbb{C}^{m \times p}$ be such that if $\mu_j = \bar{\mu}_k$, then $\gamma_j = \bar{\gamma}_k$. Let Z be the unique solution of the Sylvester equation (13). Let $F_1 = M X_1 \Phi^T$ and $F_2 = (M X_1 A_1 + D X_1) \Phi^T$, where Φ is determined by $\Phi Z = \Gamma$. Define the matrices W_1 and W_2 by (25) and (26). Suppose that Z is invertible and Y_1 , U , U_1 , U_2 , V , V_1 , and V_2 , respectively, satisfy the following equations:

$$M Y_1 A_1' + D Y_1 A_1' + K Y_1 = B \Gamma, \quad (30)$$

$$\begin{cases} M U A_1' + D U A_1' + K U = [(I + A_1' \bar{A}_1') Y_1^H]^T, \\ M U_1 A_1' + D U_1 A_1' + K U_1 = [W_1 W_{11}^H]^T, \\ M U_2 A_1' + D U_2 A_1' + K U_2 = [A_1' W_1 W_{21}^H]^T \end{cases} \quad (31)$$

and

$$\begin{cases} A_1' V - V A_1 = -W_1 W_{11}^H Z^{-1}, \\ A_1' V_1 - V_1 A_1 = -W_1 W_{11}^H Y_1 Z^{-1}, \\ A_1' V_2 - V_2 A_1 = -W_1 W_{21}^H Y_1 A_1' Z^{-1}, \end{cases} \quad (32)$$

where

$$W_{11} = E_1 - Y_1 W_1 \quad \text{and} \quad W_{21} = E_2 - Y_1 A_1' W_1, \quad (33)$$

with $E_1 := [I_n, 0] \in \mathbb{R}^{n \times 2n}$ and $E_2 := [0, I_n] \in \mathbb{R}^{n \times 2n}$. Then, the gradient $\nabla_{\Gamma} J_1$ of J_1 with respect to Γ is given by

$$\nabla_{\Gamma} J_1 = \frac{1}{2} [U^T B - V X_1^T B]^T - \frac{1}{2} \beta [(U_1 + U_2)^T B - (V_1 + V_2) X_1^T B]^T. \quad (34)$$

Proof. We rewrite the function J_1 defined in (29) in the form:

$$J_1 = J_{11} + J_{12} + \beta J_{13}, \quad (35)$$

where $J_{11} = \frac{1}{2} \|Y_1\|_F^2 + \frac{1}{2} \|Y_1 A_1'\|_F^2$ and $J_{12} = \frac{1}{2} \|W_1\|_F^2$, and $J_{13} = \|I - \tilde{Y}_1 W_1\|_F^2$. It follows that

$$\nabla_{\Gamma} J_{11} = \nabla_{\Gamma} J_{11} + \nabla_{\Gamma} J_{12} + \beta \nabla_{\Gamma} J_{13}. \quad (36)$$

We first give the formula for the gradient $\nabla_{\Gamma} J_{11}$ of J_{11} . For $J_{11} = \frac{1}{2} \text{tr}([I + A_1' \bar{A}_1'] Y_1^H Y_1)$, we have the gradient $\nabla_{\Gamma} J_{11}$ from the **first-order** variation as

$$\begin{aligned} \Delta J_{11} &= \frac{1}{2} \text{tr}([I + A_1' \bar{A}_1'] Y_1^H \Delta Y_1) + \frac{1}{2} \text{tr}([I + A_1' \bar{A}_1'] \Delta Y_1^H Y_1) = \frac{1}{2} \text{tr}([I + A_1' \bar{A}_1'] Y_1^H \Delta Y_1) + \frac{1}{2} \text{tr}(Y_1 [I + A_1' \bar{A}_1'] \Delta Y_1^H) \\ &= \frac{1}{2} \text{tr}(Z_1 \Delta Y_1) + \frac{1}{2} \text{tr}(Z_1^H \Delta Y_1^H), \end{aligned} \quad (37)$$

where $Z_1 = [I + A_1' \bar{A}_1'] Y_1^H$.

We now express $\text{tr}(Z_1 \Delta Y_1)$ in terms of $\Delta \Gamma$. From (15), we know that ΔY_1 is uniquely determined by

$$M \Delta Y_1 A_1' 2 + D \Delta Y_1 A_1' + K \Delta Y_1 = B \Delta \Gamma \quad (38)$$

or

$$\text{vec}(\Delta Y_1) = Z_2^{-1} Z_3 \text{vec}(\Delta \Gamma),$$

where $Z_2 = A_1' 2 \otimes M + A_1' \otimes D + I_p \otimes K$ and $Z_3 = I_p \otimes B$. From the fact that if $A_1 \in \mathbb{C}^{m \times n}$ and $A_2 \in \mathbb{C}^{n \times m}$, then $\text{tr}(A_1 A_2) = \text{vec}(A_1^T)^T \text{vec}(A_2)$, we get

$$\text{tr}(Z_1 \Delta Y_1) = \text{vec}(Z_1^T)^T \text{vec}(\Delta Y_1) = \text{vec}(Z_1^T)^T Z_2^{-1} Z_3 \text{vec}(\Delta \Gamma) = (Z_2^{-T} \text{vec}(Z_1^T))^T \text{vec}(B \Delta \Gamma) = \text{tr}(U^T B \Delta \Gamma), \quad (39)$$

where U is determined by (31).

Next, we derive an expression for the term $\text{tr}(Z_1^H \Delta Y_1^H)$ in (37). Taking the conjugate transpose of (38) gives

$$\bar{A}_1' 2 \Delta Y_1^H M + \bar{A}_1' \Delta Y_1^H D + \Delta Y_1^H K = \Delta \Gamma^H B^T$$

or

$$\text{vec}(\Delta Y_1^H) = Z_4^{-1} Z_5 \text{vec}(\Delta \Gamma^H),$$

where $Z_4 = M \otimes \bar{A}_1' 2 + D \otimes \bar{A}_1' + K \otimes I_p$ and $Z_5 = B \otimes I_p$. Then we have

$$\text{tr}(Z_1^H \Delta Y_1^H) = \text{vec}(\bar{Z}_1)^T \text{vec}(\Delta Y_1^H) = \text{vec}(\bar{Z}_1)^T Z_4^{-1} Z_5 \text{vec}(\Delta \Gamma^H) = (Z_4^{-T} \text{vec}(\bar{Z}_1))^T \text{vec}(\Delta \Gamma^H B^T) = \text{tr}(B^T \bar{U} \Delta \Gamma^H). \quad (40)$$

Substituting (39) and (40) into (37) gives rise to

$$\Delta J_{11} = \frac{1}{2} \text{tr}(U^T B \Delta \Gamma) + \frac{1}{2} \text{tr}(B^T \bar{U} \Delta \Gamma^H)$$

and from which the gradient of J_{11} is given by

$$\nabla_{\Gamma} J_{11} = \frac{1}{2} [U^T B]^T. \quad (41)$$

Next, we establish the gradient $\nabla_{\Gamma} J_{12}$. For $J_{12} = \frac{1}{2} \text{tr}(W_1^H W_1)$, we have

$$\Delta J_{12} = \frac{1}{2} \text{tr}(W_1^H \Delta W_1) + \frac{1}{2} \text{tr}(\Delta W_1^H W_1) = \frac{1}{2} \text{tr}(W_1^H \Delta W_1) + \frac{1}{2} \text{tr}(W_1 \Delta W_1^H). \quad (42)$$

We first deduce the expression of $\text{tr}(W_1^H \Delta W_1)$. By (25) we get

$$\Delta W_1 = -Z^{-1} \Delta Z Z^{-1} C = -Z^{-1} \Delta Z W_1. \quad (43)$$

Then

$$\text{tr}(W_1^H \Delta W_1) = -\text{tr}(W_1^H Z^{-1} \Delta Z W_1) = -\text{tr}(W_1 W_1^H Z^{-1} \Delta Z). \quad (44)$$

Now, we express $\text{tr}(W_1 W_1^H Z^{-1} \Delta Z)$ in terms of $\Delta \Gamma$. First, it follows from (13) that

$$A_1 \Delta Z - \Delta Z A_1' = -X_1^T B \Delta \Gamma. \quad (45)$$

It is well known that ΔZ of the Sylvester equation can be written as [36]

$$\Delta Z = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \phi_{jk} (A_1)^j (-X_1^T B \Delta \Gamma) (A_1')^k,$$

where ϕ_{jk} are scalars.

Let $Z_6 = W_1 W_1^H Z^{-1}$ and $Z_7 = X_1^T B \Delta \Gamma$. Then we have

$$\text{tr}(Z_6 \Delta Z) = \text{tr} \left(\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} Z_6 \phi_{jk} (A_1)^j (-Z_7) (A_1')^k \right) = \text{tr} \left(\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \phi_{jk} (A_1')^k (-Z_6) (A_1)^j Z_7 \right) = \text{tr}(V X_1^T B \Delta \Gamma), \quad (46)$$

where V is the unique solution to (32). Substituting (46) into (44) gives rise to

$$\text{tr}(W_1^H \Delta W_1) = -\text{tr}(V X_1^T B \Delta \Gamma). \quad (47)$$

Next, we deduce an expression for the term $\text{tr}(W_1 \Delta W_1^H)$ in (42). By (43), we have

$$\text{tr}(W_1 \Delta W_1^H) = -\text{tr}(W_1 W_1^H \Delta Z^H Z^{-H}) = -\text{tr}(Z^{-H} W_1 W_1^H \Delta Z^H). \quad (48)$$

Similar to the proof of (47), we can show that

$$\text{tr}(W_1 \Delta W_1^H) = -\text{tr}([V X_1^T B]^H \Delta \Gamma^H). \quad (49)$$

It follows from (42), (47), and (49) that

$$\Delta J_{12} = -\frac{1}{2} \text{tr}(V X_1^T B \Delta \Gamma) - \frac{1}{2} \text{tr}([V X_1^T B]^H \Delta \Gamma^H). \quad (50)$$

Thus the gradient $\nabla_{\Gamma} J_{12}$ of J_{12} is given by

$$\nabla_{\Gamma} J_{12} = -\frac{1}{2} [V X_1^T B]^T. \quad (51)$$

Finally, we derive the expression of $\nabla_{\Gamma} J_{13}$ for J_{13} . By (33), we get

$$J_{13} = \frac{1}{2} \|I - \tilde{Y}_1 W_1\|_F^2 = \frac{1}{2} \text{tr}(W_{11}^H W_{11}) + \frac{1}{2} \text{tr}(W_{21}^H W_{21}).$$

Thus,

$$\Delta J_{13} = \frac{1}{2} \text{tr}(W_{11}^H \Delta W_{11}) + \frac{1}{2} \text{tr}(W_{11} \Delta W_{11}^H) + \frac{1}{2} \text{tr}(W_{21}^H \Delta W_{21}) + \frac{1}{2} \text{tr}(W_{21} \Delta W_{21}^H). \quad (52)$$

We first give an expression for $\text{tr}(W_{11}^H \Delta W_{11})$. By (33) and (43), we have

$$\Delta W_{11} = -\Delta Y_1 W_1 - Y_1 \Delta W_1 = -\Delta Y_1 W_1 + Y_1 Z^{-1} \Delta Z W_1.$$

Thus

$$\text{tr}(W_{11}^H \Delta W_{11}) = -\text{tr}(W_1 W_{11}^H \Delta Y_1) + \text{tr}(W_1 W_{11}^H Y_1 Z^{-1} \Delta Z). \quad (53)$$

We now express $\text{tr}(W_1 W_{11}^H \Delta Y_1)$ in terms of $\Delta \Gamma$. Similarly, for $\text{tr}(Z_1 \Delta Y_1)$ in (39), we can show that

$$\text{tr}(W_1 W_{11}^H \Delta Y_1) = \text{tr}(U_1^T B \Delta \Gamma), \quad (54)$$

where U_1 is determined by (31). By following the similar proof lines of (46), we obtain

$$\text{tr}(W_1 W_{11}^H Y_1 Z^{-1} \Delta Z) = \text{tr}(V_1 X_1^T B \Delta \Gamma), \quad (55)$$

where V_1 is determined by (32). Substituting (54) and (55) into (53) gives

$$\text{tr}(W_{11}^H \Delta W_{11}) = -\text{tr}([U_1^T B - V_1 X_1^T B] \Delta \Gamma). \quad (56)$$

Using a proof similar to (56), we can show that

$$\text{tr}(W_{11} \Delta W_{11}^H) = -\text{tr}([U_1^T B - V_1 X_1^T B]^H \Delta \Gamma^H). \quad (57)$$

Next, we express $\frac{1}{2} \text{tr}(W_{21}^H \Delta W_{21})$ in terms of $\Delta \Gamma$. By (33) and (43), we have

$$\Delta W_{21} = -\Delta Y_1 A_1' W_1 - Y_1 A_1' \Delta W_1 = -\Delta Y_1 A_1' W_1 + Y_1 A_1' Z^{-1} \Delta Z W_1.$$

Therefore,

$$\text{tr}(W_{21}^H \Delta W_{21}) = -\text{tr}(A_1' W_1 W_{21}^H \Delta Y_1) + \text{tr}(W_1 W_{21}^H Y_1 A_1' Z^{-1} \Delta Z). \quad (58)$$

We now express $\text{tr}(A_1' W_1 W_{21}^H \Delta Y_1)$ in terms of $\Delta \Gamma$. By the similar proof for $\text{tr}(Z_1 \Delta Y_1)$ in (39), we can show that

$$\text{tr}(A_1' W_1 W_{21}^H \Delta Y_1) = \text{tr}(U_2^T B \Delta \Gamma), \quad (59)$$

where U_2 is determined by (31). By following the similar proof lines of (46), we obtain

$$\text{tr}(W_1 W_{21}^H Y_1 A_1' Z^{-1} \Delta Z) = \text{tr}(V_2 X_1^T B \Delta \Gamma), \quad (60)$$

where V_2 is determined by (32). Substituting (59) and (60) into (58) gives

$$\text{tr}(W_{21}^H \Delta W_{21}) = -\text{tr}([U_2^T B - V_2 X_1^T B] \Delta \Gamma). \quad (61)$$

As with proof of (61), we can show that

$$\text{tr}(W_{21} \Delta W_{21}^H) = -\text{tr}([U_2^T B - V_2 X_1^T B]^H \Delta \Gamma^H). \quad (62)$$

Substituting (56), (57), (61), and (62) into (52), we get

$$\Delta J_{13} = -\frac{1}{2} \text{tr}([(U_1 + U_2)^T B - (V_1 + V_2) X_1^T B] \Delta \Gamma) - \frac{1}{2} \text{tr}([(U_1 + U_2)^T B - (V_1 + V_2) X_1^T B]^H \Delta \Gamma^H). \quad (63)$$

Thus the gradient $\nabla_{\Gamma} J_{13}$ of J_{13} is given by

$$\nabla_{\Gamma} J_{13} = -\frac{1}{2} [(U_1 + U_2)^T B - (V_1 + V_2) X_1^T B]^T. \quad (64)$$

The proof of the required gradient formula now follows from (41), (51), (64), and (36). $\square$

4.4. An optimization algorithm for simultaneous feedback norm and closed-loop condition minimization

Based on above analysis, we write down the following algorithm for simultaneously minimizing feedback norms and the closed-loop condition number.

Algorithm 4.1.

Inputs:

1. The matrices $M, D, K \in \mathbb{R}^{n \times n}$, where $M^T = M > 0$, $K^T = K$, and $D^T = D$.
2. The control matrix $B \in \mathbb{R}^{n \times m}$ ($m \leq n$).
3. A self-conjugate subset $\{\lambda_1, \dots, \lambda_p\}$ of the spectrum of $P(\lambda)$ and the corresponding eigenvectors.
4. A suitably chosen self-conjugate subset $\{\mu_1, \dots, \mu_p\}$.
5. $\alpha \in [0, 1]$, $c_1, c_2 > 0$, and $\beta \geq 0$.
6. ε = Tolerance limit for gradient.
7. Max_{iter} = Maximum number of iterations.

Outputs:

The real feedback matrices F_1 and F_2 such that the spectrum of the closed-loop pencil $P_c(\lambda)$ is the set $\{\mu_1, \dots, \mu_p, \lambda_{p+1}, \dots, \lambda_{2n}\}$ and the objective function $\alpha J_1 + (1 - \alpha)J_2$ defined in (29) is minimized.

Step 1. Form the matrices A_1 , A_1^* , X_1 , from the given eigenvalues and eigenvectors, and the matrix C defined in Theorem 4.3 and the matrix Q in Theorem 4.6. Set $k = 1$.

Step 2. Choose a matrix $\Gamma = [\gamma_1, \dots, \gamma_p] \in \mathbb{C}^{m \times p}$ ($m \leq p$) such that if $\bar{\mu}_j = \mu_k$, then $\bar{\gamma}_j = \gamma_k$.

Step 3. Compute the solution Z of the Sylvester equation $A_1 Z - Z A_1^* = -X_1^T B \Gamma$. (This requires $O(npm + p^3)$ flops. It is done by the MATLAB function `lyap`.) If the condition number of Z is large, choose another Γ .

Step 4. Solve the linear system $\Phi Z = \Gamma$ for Φ . (This requires $O(p^3)$ operations.)

Step 5. Form the matrix $G = \Phi Q$, $W_1 = Z^{-1}C$, $W_{11} = E_1 - Y_1 W_1$, and $W_{21} = E_2 - Y_1 A_1^* W_1$. (This requires $O(nmp + p^3 + np^2 + n^2 p)$ flops.)

Step 6. Solve the equation $M Y_1 A_1^* Z + D Y_1 A_1^* + K Y_1 = B \Gamma$ for Y_1 . (This requires $O(n^3 p)$ operations.)

Step 7. Solve for the matrices U , U_1 , and U_2 as follows:

7.1. Solve for U : $M U A_1^* Z + D U A_1^* + K U = (I + A_1^* A_1) Y_1^H \Gamma^T$,

7.2. Solve for U_1 : $M U_1 A_1^* Z + D U_1 A_1^* + K U_1 = [W_1 W_{11}^H]^T$,

7.3. Solve for U_2 : $M U_2 A_1^* Z + D U_2 A_1^* + K U_2 = [A_1^* W_1 W_{21}^H]^T$.

(These computations require $O(n^3 p)$ flops.)

Step 8. Compute Y , V , V_1 , and V_2 by solving the following Sylvester equations:

8.1. Solve for Y : $A_1^* Y - Y A_1 = -Z^{-1} Q G^H \Phi$,

8.2. Solve for V : $A_1^* V - V A_1 = -W_1 W_{11}^H Z^{-1}$,

8.3. Solve for V_1 : $A_1^* V_1 - V_1 A_1 = -W_1 W_{11}^H Y_1 Z^{-1}$,

8.4. Solve for V_2 : $A_1^* V_2 - V_2 A_1 = -W_1 W_{21}^H Y_1 A_1^* Z^{-1}$.

(These steps require $O(p^3 + np^2)$ flops.)

Step 9. Compute the gradient $Grad := \alpha \nabla_{\Gamma} J_1 + (1 - \alpha) \nabla_{\Gamma} J_2$, using results from Theorems 4.6 and 4.7, respectively.

(This requires $O(p^3 + np^2)$ operations.) If $\|Grad\|_F \leq \varepsilon$ or if the number of iterations exceeds Max_{iter} , go to Step 6; Otherwise, go to Step 10.

Step 10. Compute a new Γ (as shown below) using a gradient-based optimization method (we use the Broyden–Fletcher–Goldfarb–Shanno (BFGS) method, which is described in details in [31, Chap. 8]). Set $k = k + 1$ and return to Step 2.

Step 11. For the Γ which gives the minimum value of $\alpha J_1 + (1 - \alpha)J_2$, compute the matrices F_1 and F_2 as in Theorem 4.3. Stop.

Computation of new Γ in Step 10: The function to be minimized is $\alpha J_1(\Gamma) + (1 - \alpha)J_2(\Gamma) = \frac{1}{2} \alpha (\|Y(\Gamma)\|_F^2 + \|Y^{-1}(\Gamma)\|_F^2) + \frac{1}{2} (1 - \alpha) (c_1 \|F_1(\Gamma)\|_F^2 + c_2 \|F_2(\Gamma)\|_F^2)$. We denote the current value of Γ by Γ_{old} and the new value of Γ by Γ_{new} . Then Γ_{new} is obtained as follows:

- (a) Replace Γ_{old} by $\hat{\Gamma} = \Gamma_{old} + \delta d_j$, where d_j is the search direction which, in the BFGS method, is given by $d_j = -D_j Grad$. Here $Grad$ represents the current value of the gradient and D_j is the metric obtained as in the BFGS method if $j > 1$ and equal to D_1 if $j = 1$.
- (b) Let $\hat{Z}$ be the solution of the Sylvester equation: $A_1 \hat{Z} - \hat{Z} A_1^* = -X_1^T B \hat{\Gamma}$. Let $\hat{\Phi} = \hat{\Gamma} \hat{Z}^{-1}$ be the solution of the linear system: $\hat{\Phi} \hat{Z} = \hat{\Gamma}$. Compute the value $\hat{Y}_1$ of Y_1 corresponding to $\hat{\Gamma}$ as follows: Let $\hat{Y}_1 = [\hat{y}_1, \dots, \hat{y}_p]$ and $\hat{\Gamma} = [\hat{\gamma}_1, \dots, \hat{\gamma}_p]$. Then $\hat{y}_k$ is uniquely determined by $(\mu_k^2 M + \mu_k D + K) \hat{y}_k = \hat{y}_k$ for $k = 1, \dots, p$. Form $\hat{G} = \hat{\Phi} Q$, $\hat{W}_1 = \hat{Z}^{-1} C$, $\hat{W}_{11} = E_1 - \hat{Y}_1 \hat{W}_1$, and $\hat{W}_{21} = E_2 - \hat{Y}_1 A_1^* \hat{W}_1$.
- (c) Find $l = \min_{\delta} \{ \frac{1}{2} \alpha (\|\hat{Y}_1\|_F^2 + \|\hat{Y}_1 A_1^* \hat{W}_1\|_F^2 + \|\hat{W}_1\|_F^2 + \beta \|\hat{W}_{11}\|_F^2 + \beta \|\hat{W}_{21}\|_F^2) + \frac{1}{2} (1 - \alpha) \|\hat{G}\|_F^2 \}$. This is done by using the MATLAB function `fminbnd`.
- (d) $\Gamma_{new} = \Gamma_{old} + l d_j$.

Remark 4.8. In order to implement the control law (3) with the feedback matrices computed by Algorithm 4.1, knowledge of this displacement vector $x(t)$ and the velocity vector $\dot{x}(t)$ is needed. To this end, an algorithm to estimate these quantities

by means of an observer has been recently proposed by Carvalho and Datta [6]. Experiments are currently underway to implement the control law (3) on real-life structures by integrating this technique with our feedback formulas.

5. Numerical experiments

In this section, we will state some numerical results for solving the RPQVEAP using Algorithm 4.1. The algorithm was implemented in MATLAB 7.0 and run on a PC Intel Pentium IV of 3.00 GHz CPU. In our computational experiments, we use the tolerance $\varepsilon = 1.0 \times 10^{-6}$ and the maximum number of iterations, $Max_{iter} = 500$. The algorithm is applied to the following several different test problems.

Problem 5.1. Here the three matrices M, D, K are randomly generated correlation matrices by MATLAB routine function `gallery('randcorr', n)` and B is random matrix, i.e.,

$$M = \begin{bmatrix} 1.0000 & -0.1067 & -0.1895 & -0.5170 & 0.1696 \\ -0.1067 & 1.0000 & 0.3893 & 0.1415 & 0.2020 \\ -0.1895 & 0.3893 & 1.0000 & -0.1305 & 0.2989 \\ -0.5170 & 0.1415 & -0.1305 & 1.0000 & 0.1928 \\ 0.1696 & 0.2020 & 0.2989 & 0.1928 & 1.0000 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.0000 & -0.3784 & 0.2015 & 0.0388 & 0.1476 \\ -0.3784 & 1.0000 & -0.0072 & -0.1233 & 0.2483 \\ 0.2015 & -0.0072 & 1.0000 & 0.3487 & -0.2794 \\ 0.0388 & -0.1233 & 0.3487 & 1.0000 & 0.1081 \\ 0.1476 & 0.2483 & -0.2794 & 0.1081 & 1.0000 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.0000 & 0.6591 & -0.4258 & 0.1630 & 0.6081 \\ 0.6591 & 1.0000 & -0.4137 & 0.0308 & 0.0804 \\ -0.4258 & -0.4137 & 1.0000 & -0.4927 & -0.2494 \\ 0.1630 & 0.0308 & -0.4927 & 1.0000 & 0.4947 \\ 0.6081 & 0.0804 & -0.2494 & 0.4947 & 1.0000 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3357 & 0.8105 \\ 0.8484 & 0.9999 \\ 0.2499 & 0.8417 \\ 0.0228 & 0.7662 \\ 0.3405 & 0.8373 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{-6.4634, -0.3743 \pm 1.6463i, -0.4769 \pm 1.0401i, -0.2714 \pm 0.7579i, -0.0855, -0.3804 \pm 0.2495i\}$, where $i := \sqrt{-1}$. The eigenvalue $\{-0.0855\}$ was reassigned to $\{-0.1\}$ and the other eigenvalues were kept unchanged.

Problem 5.2 (See [30]). The matrices M, D, K , and B here are given by

$$M = 10I_3, \quad D = 0, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{\pm 3.6039i, \pm 2.4940i, \pm 0.8901i\}$. In this example, the first two eigenvalues $\{\pm 3.6039i\}$ were reassigned to $\{-1, -2\}$, the other eigenvalues were kept unchanged.

Problem 5.3. The matrices M, D, K , and B here are taken as follows:

$$M = I_3, \quad D = \begin{bmatrix} 2.5 & 2.0 & 0 \\ 2.0 & 1.7 & 0.4 \\ 0 & 0.4 & 2.5 \end{bmatrix}, \quad K = \begin{bmatrix} 16 & 12 & 0 \\ 12 & 13 & 4 \\ 0 & 4 & 29 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{-0.0129 \pm 1.4389i, -1.3342 \pm 5.2311i, -2.0030 \pm 4.7437i\}$. We change the first two open-loop eigenvalues: $\{-0.0129 \pm 1.4389i\}$ to $\{-0.1 \pm 1.4389i\}$. The other eigenvalues are kept unchanged.

Problem 5.4 (See [4, Problem 1]). In this example, M, D, K , and B are taken as follows:

$$M = 4I_n, \quad D = 4I_n, \quad K = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

We take $n = 10$. Then there are 20 open-loop eigenvalues. The sixteenth and the seventeenth eigenvalues $\{0.0000, -0.0251\}$ were reassigned to $\{-0.1, -0.2\}$, and the other eigenvalues were kept unchanged.

5.1. Performance of Algorithm 4.1

From the physical point of view, the weighting parameters c_1 and c_2 should be chosen according the practical requirements. Here, for simplicity of demonstration, we display in the following two tables the performance of Algorithm 4.1 with $c_1 = c_2 = 1.0$ and $c_1 = 1.0, c_2 = 0.01$, respectively. In each table, we show the numerical results with three different values of α , $\alpha = 0, 0.5, 1$; and with three different values of β , $\beta = 1, 10, 100$. Note that

- $\alpha = 0$ corresponds to the minimum feedback norm problem and of interests there are the quantities $\|F_1\|_F$ and $\|F_2\|_F$.
- $\alpha = 1$ corresponds to RPQPEVAP and of interest there is the condition number $\kappa_2(Y)$.
- $\alpha = 0.5$, corresponds to simultaneous minimization of feedback norms and the sensibility of the closed-loop eigenvalues with equal weights. The quantities of interests here are both $\|F_1\|_F$, $\|F_2\|_F$, and the condition number $\kappa_2(Y)$.

Interpretation: The results in Tables 1 and 2 below, show that the proposed algorithm worked well in all cases:

For $\alpha = 0$, $\|F_1\|_F$ and $\|F_2\|_F$ are small, for $\alpha = 1$, $\kappa_2(Y)$ is small and for $\alpha = 0.5$, $\|F_1\|_F$, $\|F_2\|_F$ and $\kappa_2(Y)$ are all small.

Table 1
Numerical results for Algorithm 4.1 with $c_1 = c_2 = 1.0$.

Problem	$\alpha = 1$			$\alpha = 0.5$			$\alpha = 0$		
	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$
$\beta = 1$									
5.1	0.0502	0.0819	20.67	0.0501	0.0818	20.66	0.0488	0.0797	963.0424
5.2	26.58	99.68	10.19	21.31	72.23	177.05	19.36	70.89	4.86×10^3
5.3	0.1364	0.0339	9.25	0.1342	0.0245	9.25	0.1343	0.0224	1.236×10^4
5.4	1.8593	1.8560	21.35	1.8586	1.8554	21.24	1.4333	1.4114	1.419×10^5
$\beta = 10$									
5.1	0.0502	0.0819	20.67	0.0502	0.0819	20.67	0.0488	0.0797	963.0424
5.2	26.81	99.55	10.21	31.34	84.08	35.52	19.36	70.89	4.86×10^3
5.3	0.1359	0.0327	9.25	0.1347	0.0284	9.25	0.1343	0.0224	1.236×10^4
5.4	1.8710	1.8674	23.82	1.8708	1.8673	23.81	1.4333	1.4114	1.419×10^5
$\beta = 100$									
5.1	0.0502	0.0819	20.67	0.0502	0.0819	20.67	0.0488	0.0797	963.0424
5.2	26.87	99.52	10.22	30.11	95.67	11.41	19.36	70.89	4.86×10^3
5.3	0.1358	0.0324	9.25	0.1356	0.0317	9.25	0.1343	0.0224	1.236×10^4

Table 2
Numerical results for Algorithm 4.1 with $c_1 = 1.0$ and $c_2 = 0.01$.

Problem	$\alpha = 1$			$\alpha = 0.5$			$\alpha = 0$		
	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$
$\beta = 1$									
5.1	0.0502	0.0819	20.67	0.0502	0.0819	20.67	0.0488	0.0797	963.0396
5.2	26.58	99.68	10.19	19.42	106.56	15.21	19.36	70.89	6.53×10^3
5.3	0.1364	0.0339	9.25	0.1343	0.0265	9.25	0.1342	0.0240	1.242×10^4
5.4	1.8593	1.8560	21.35	1.8589	1.8557	21.29	1.4333	1.4114	1.420×10^5
$\beta = 10$									
5.1	0.0502	0.0819	20.67	0.0502	0.0819	20.67	0.0488	0.0797	963.0396
5.2	26.81	99.55	10.21	20.04	105.08	13.18	19.36	70.89	6.53×10^3
5.3	0.1359	0.0327	9.25	0.1350	0.0298	9.25	0.1342	0.0240	1.242×10^4
5.4	1.8710	1.8674	23.82	1.8709	1.8673	23.81	1.4333	1.4114	1.420×10^5
$\beta = 100$									
5.1	0.0502	0.0819	20.67	0.0502	0.0819	20.67	0.0488	0.0797	963.0396
5.2	26.87	99.52	10.22	24.06	101.28	10.43	19.36	70.89	6.53×10^3
5.3	0.1358	0.0324	9.25	0.1357	0.0320	9.25	0.1342	0.0240	1.242×10^4
5.4	1.8721	1.8685	24.01	1.8721	1.8685	24.01	1.4333	1.4114	1.420×10^5

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Table 3
Comparison of Algorithm 4.1 with Brahma–Datta's method.

Problem		Brahma–Datta's method			Alg. 4.1 with $\alpha = 1$ and $\beta = 1$		
		$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$
5.1	IV.	0.0554	0.0905	4.83×10^3	0.0554	0.0905	4.83×10^3
	FV.	2.5451	4.1538	263.15	0.0502	0.0819	20.67
	RE.	–4490.50	–4490.50	94.55	9.47	9.47	99.57
5.2	IV.	74.98	130.71	6.98×10^3	74.98	130.71	6.98×10^3
	FV.	59.44	161.54	232.99	26.58	99.68	10.19
	RE.	20.72	–23.58	96.66	64.55	23.74	99.85
5.3	IV.	0.2621	0.1075	2.91×10^3	0.2621	0.1075	2.91×10^3
	FV.	0.2799	0.2187	2.22×10^9	0.1364	0.0339	9.25
	RE.	–6.80	–103.39	-7.62×10^7	47.97	68.47	99.68
5.4	IV.	2.12×10^8	6.45×10^8	2.76×10^{10}	2.12×10^8	6.45×10^8	2.76×10^{10}
	FV.	5.3×10^5	1.19×10^6	1.10×10^9	4.0×10^5	9.1×10^5	2.56×10^8
	RE.	99.75	99.81	96.02	99.81	99.86	99.07

Table 4
Numerical results for Problem 5.5.

n	Multi-step method			Single-step method			Alg. 4.1 with $\alpha = 1$ and $\beta = 1$		
	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$	$\ F_1\ _F$	$\ F_2\ _F$	$\kappa_2(Y)$
10	436	1024	–	556	2721	6.81×10^3	99	528	738
20	4090	10167	–	6083	29821	7.46×10^4	1602	8838	1.87×10^4
40	42058	109139	–	66806	327638	8.20×10^5	11044	67504	1.38×10^5

5.2. Comparison of the proposed algorithms with Brahma–Datta's algorithm

Both these algorithms have the same efficiency in the sense that operation count are of the same order of magnitudes. Below we compare Brahma–Datta's algorithm [4] with the proposed algorithm for actual performances, taking $\alpha = 1$ and $\beta = 1$. The notations used there are:

- IV = values of the concerned quantities with the initial parameter Γ .
- FV = values of the concerned quantities with the final parameter Γ (obtained by the algorithm).
- RE—relative change (in percent): $= 100 \times (IV - FV)/IV$.

The results in Table 3 show that the proposed algorithm performed better in almost all cases.

5.3. Comparison of the proposed algorithms with Ram–Elhay's algorithm

In Table 4, we display the results of comparison of the proposed algorithm with both single and multi-input versions of Ram–Elhay's algorithm [34], on a test problem in [34], see Problem 5.5 below. For convenience, we set $\alpha = 1$ and $\beta = 1$. We note that the eigenvector matrix Y_1 can also be computed from the Ram–Elhay's single-step method. Thus the associated condition number $\kappa_2(Y)$ can be estimated.

Problem 5.5 (See [34, Example 1]). In this example, M, D, K , and B are taken as follows:

$$M = I_n, \quad D = 0, \quad K = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} I_m \\ 0 \end{bmatrix}.$$

We take $m = 3$, $p = 4$, and $n = 10, 20, 40$. The four eigenvalues with smallest absolute values are replaced by $\lambda_{2k-1} = -k + \sqrt{-10k}$, $\lambda_{2k} = \bar{\lambda}_{2k-1}$, $k = 1, 2$. The other eigenvalues were preserved.

The results in Table 4 again demonstrate the superiority of the proposed algorithm on both aspects of feedback norms and the closed-loop eigenvalue condition number over the **Ram**-Elhay algorithm. Similar results are expected if compared with the **Datta**-Sarkissian multi-input algorithm [20] as well. It is to be noted in this context, that neither the **Ram**-Elhay algorithm nor the **Datta**-Sarkissian algorithm considers the robustness issue and feedback norm minimization. Thus, such performance of these algorithms compared to the proposed algorithm is not surprising.

5.4. Comparison of system responses

Problem 5.6 below is used to demonstrate the robustness of the proposed algorithm under small perturbations of the stiffness matrix K . Comparisons are made of systems responses both for open-loop and closed-loop systems as (Fig. 1) well as for closed-loop systems under different perturbations (Fig. 2).

Problem 5.6. In this example, M, D, K , and B are taken as follows:

$$M = I_3, \quad D = \begin{bmatrix} 12.5 & 10.0 & 0 \\ 10.0 & 8.5 & 2.0 \\ 0 & 2.0 & 12.5 \end{bmatrix}, \quad K = \begin{bmatrix} 16 & 12 & 0 \\ 12 & 13 & 4 \\ 0 & 4 & 29 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ has **six** open-loop eigenvalues $\{-19.4889, -9.4839, -0.0586 \pm 1.4429i, -1.3620, -3.0479\}$. We replace the two open-loop complex eigenvalues $\{-0.0586 \pm 1.4429i\}$ by $\{-0.5 \pm 1.4429i\}$. The other eigenvalues are kept unchanged.

Let

$$\Gamma = 10^2 \cdot \begin{bmatrix} -8i & 8i \\ 3 + 8i & 3 - 8i \end{bmatrix}. \quad (65)$$

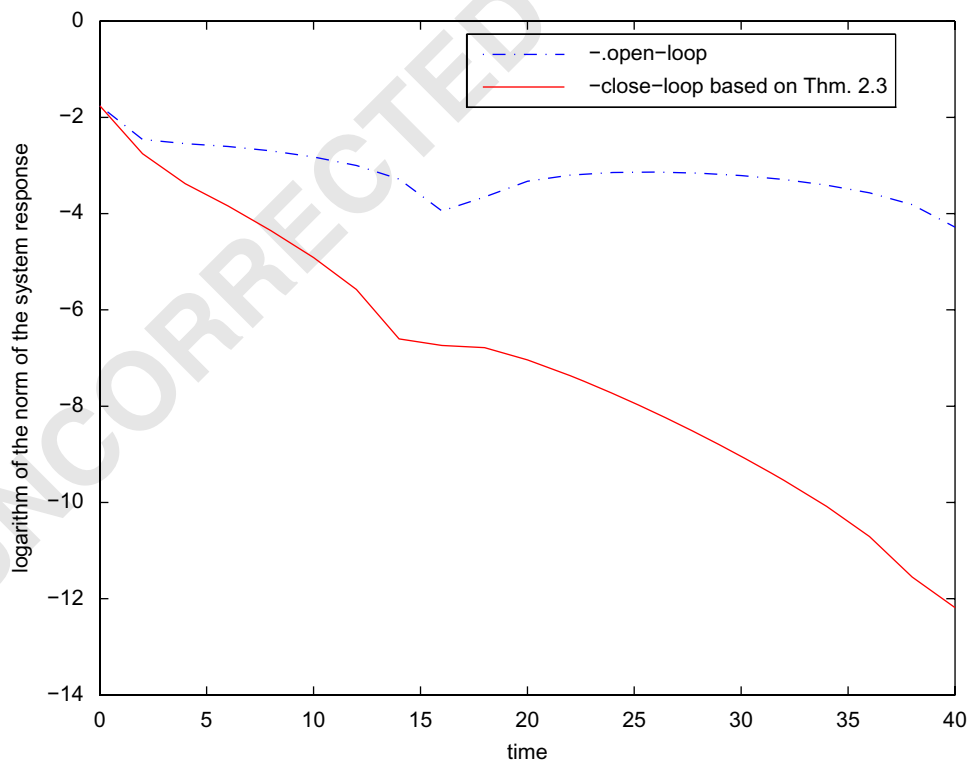


Fig. 1. Comparison of the norms of the system responses for Example 5.6.

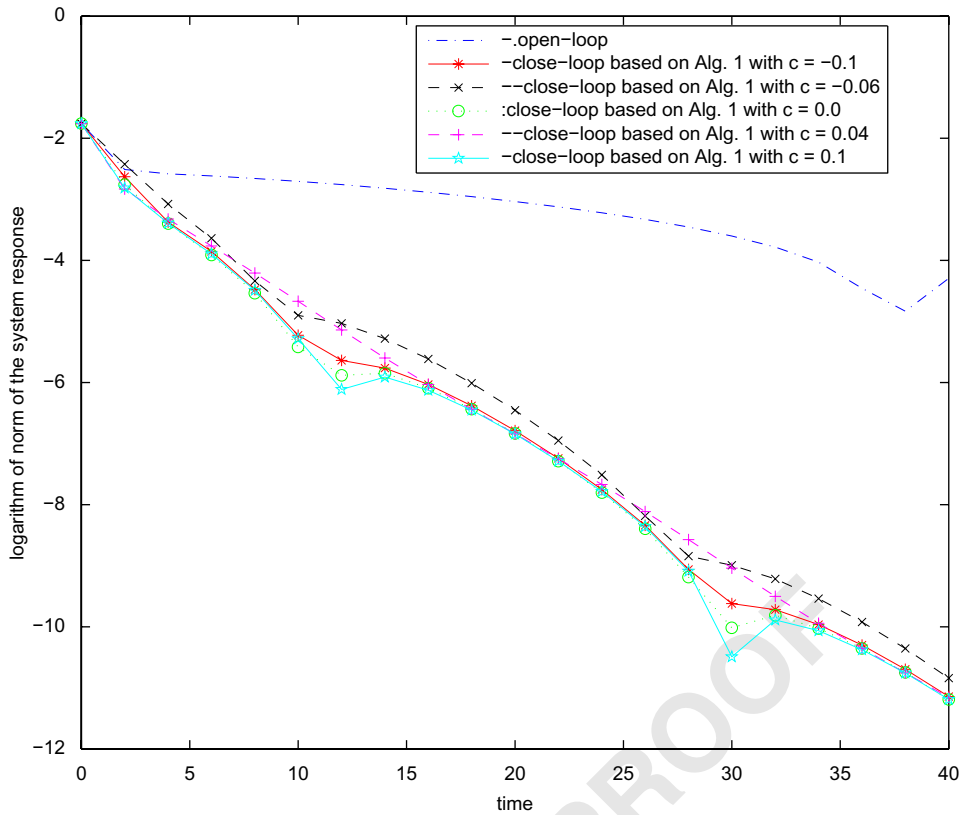


Fig. 2. Comparison of the norm of the system responses for Example 5.6 with perturbed K .

By Theorem 4.3, we obtain the feedback matrices

$$F_1 = \begin{bmatrix} 0.3665 & -0.3917 \\ -0.4440 & 0.4803 \\ 0.0679 & -0.0746 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} 0.1860 & -0.3485 \\ -0.0057 & 0.3017 \\ -0.0377 & -0.0106 \end{bmatrix}.$$

The eigenvalues of the corresponding closed-loop system are $\{-19.4889, -9.4839, -0.5 \pm 1.4429i, -1.3620, -3.0479\}$.

Then we compute the system responses of the open-loop and the closed-loop systems with the feedback matrices F_1 and F_2 obtained by Theorem 4.3. Fig. 1 depicts the base 10 logarithm of the norm of the system responses over the given time period. The initial condition is $\mathbf{z}(0) = 0.01 \cdot \mathbf{1}_{2n}$, where $\mathbf{1}_{2n} = (1, \dots, 1)^T \in \mathbb{R}^{2n}$. We observe from Fig. 1 that the system response behavior of the closed-loop system with the feedback matrices F_1 and F_2 obtained by Theorem 4.3 is better than that of the original open-loop system which was expected.

Next, we perturb the stiffness matrix K to $K + cK$ with the parameter c varying from -0.1 to 0.1 , and keep the matrices M and D , and B fixed.

We then compute the system responses of the open-loop system and the perturbed closed-loop systems with the feedback matrices F_1 and F_2 obtained by Algorithm 4.1 (we set $\alpha = \beta = 1$). Fig. 2 shows the base 10 logarithm of the norm of the system responses over the defined time period for different values of c . The initial condition is $\mathbf{z}(0) = 0.01 \cdot \mathbf{1}_{2n}$. We can see from Fig. 1 that the system responses of the perturbed closed-loop system with the feedback matrices F_1 and F_2 obtained by Algorithm 4.1 are all insensitive to perturbation and successfully tend to the steady state, showing the robustness of the algorithm.

6. Summary and conclusions

A new optimization-based method for robust design of active vibration control in structures modeled by a system of second-order differential equations is proposed. The method can be implemented with the knowledge of only a small number of eigenvalues that need to be reassigned to control vibration, and the associated eigenvectors. The no spill-over of the unassigned eigenvalues and eigenvectors is established by means of a mathematical theory. Furthermore, the method is implemented in the original second-order system itself without requiring transformation into the standard first-order state-space control system, and any reduction of the order of the original model. These attractive practical features make the method suitable for even large-scale practical applications. The natural mathematical models of vibrating structures

are, however, distributed parameter systems. Though there now exist some methods for partial eigenvalue assignment in distributed parameter systems, methods for robust active vibration control design in distributed parameter systems do not exist. A natural extension of the proposed method to such systems is presently being investigated. Also, the possibility of using the present scheme in conjunction with some passive control device in robust active vibration control of real-life models, such as, beams, plates, etc., that have infinite degree of freedom, is also being currently explored.

7. Uncited reference

[35].

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