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Multi-Input Partial Pole Placement for Distributed Parameter Gyroscopic Systems *

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1 Introduction

The vibrating structures such as beams, buildings, bridges, highways, large space structures, etc., are distributed parameter systems. While it is desirable to obtain a solution of a vibration problem in its own natural setting of distributed parameter systems; due to the lack of appropriate computational methods, in practice, very often a distributed parameter system is first discretized to a matrix second-order model using techniques of finite elements or finite differences ([1, 2, 9, 10], etc.), and then the problem is solved for this discretized reduced-order model. A matrix second-order model of the free motion of a vibrating system is a system of differential equations of the form

$$M\ddot{x}(t) + (D+G)\dot{x}(t) + Kx(t) \quad = \quad 0, \tag{1}$$

where
$M = M^T$ is mass or inertia matrix
$D = D^T$ is damping matrix
$G = -G^T$ is skew-symmetric (gyroscopic) matrix
$K = K^T$ is stiffness matrix

The system represented by (1) is called damped gyroscopic system. The eigenvalues of the system (1) are the eigenvalues of the quadratic pencil: $P(\lambda) = \lambda^2 M + \lambda(D+G) + K$.

It is well-known (see [9]) that a dangerous situation called resonance occurs when one or more natural frequencies of the system, which are the eigenvalues of $P(\lambda)$, become equal or close to a frequency of the external force.

To combat such undesirable effects of vibrations caused by a few "bad" eigenvalues of the system, one needs to reassign those few "bad" eigenvalues, leaving the rest unchanged, by

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using a suitable control force. This problem is known as the *partial pole placement problem* in control theory. Let a control force of the form

$$f = Bh(t)$$

where B is the input (control) matrix, be applied to the vibrating structure and the control vector $h(t)$ be chosen as

$$h(t) = F^T \dot{x}(t) + G^T x(t)\,.$$

Then the closed-loop system corresponding to (1) is

$$M\ddot{x} + (D + G - BF^T)\dot{x}(t) + (K - BG^T)x(t) = 0.$$

So, mathematically, the partial pole placement problem is the problem of finding the matrices F and G such that a few "bad" eigenvalues of the closed-loop quadratic pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D + G - BF^T) + (K - BG^T)$$

are replaced by "suitably" chosen ones leaving the remaining "good" ones unchanged.

In several recent papers ([3, 5, 7, 6, 14, 12]), the partial pole placement for the damped nongyroscopic systems ($G = 0$) has been considered and a *novel* approach, called the *partial-modal* approach has been developed.

This approach has several distinct features. First, the solution requires only those few eigenvalues that need to be reassigned and the corresponding eigenvectors.

Second, the problem is solved completely in the second-order setting; that is, no transformation to a first-order system is invoked; thus making it possible to preserve the exploitable structures such as the sparsity, definiteness, bandness, etc., very often offered by many practical problems.

Furthermore, mathematical results are proven that guarantee that there will be no *spill-over* during the process; that is, the eigenvalues and eigenvectors that are not to be changed will remain invariant. Three new orthogonality relations between the eigenvectors of a symmetric definite matrix quadratic pencil form a basis of these results. These relations generalize those of a symmetric matrix and of a symmetric definite linear matrix pencil (see [4]).

In this paper, we extend our study to the undamped gyroscopic systems. Our study this time goes beyond the matrix second-order system and extends to the operator system of which the former is just a discretized approximation.

Specifically, we solve the multi-input partial pole placement problem for a gyroscopic operator system.

The partial pole assignment problem for a gyroscopic operator system arises, for example, in regulating the vibratory effects of small oscillations of a taut string, rotating about its x-axis with constant angular velocity; of small oscillations of a uniform string traveling with constant velocity γ over the fixed supports, etc. In the special case, when M, G and K are matrices, we obtain a solution of the multi-input partial pole placement problem for an undamped gyroscopic matrix second-order system of the form (1).

Problem 1 (Multi-input Partial Pole Placement Problem for an Undamped Gyroscopic Operator System)

Given self-adjoint positive definite operators M and K, the skew-symmetric operator G, control functions $b_1(x), \ldots, b_p(x)$, and a set of m self-conjugate numbers $\{\mu_i\}_{i=1}^m$; find feedback functions $f_k(x)$ and $g_k(x)$ for $k = 1, \ldots, p$ such that the spectrum of the closed-loop operator system

$$M\nu_{tt} + G\nu_t + K\nu \;\; = \;\; \sum_{k=1}^{p} \left((f_k, \nu_t) + (g_k, \nu) \right) b_k \tag{2}$$

is $\{\{\mu_i\}_{i=1}^m, \{\lambda_j\}_{j \geq m+1}\}$, where λ_j, $j = m+1, \ldots$ are the eigenvalue of the associated open-loop operator system and $(\cdot, \cdot)$ is an appropriate scalar product.

An explicit solution of the above problem is given in the following theorem (Theorem 1). The proof of the Theorem will appear elsewhere.

An orthogonality relation between the eigenfunctions associated with two distinct eigenvalues of the open-loop gyroscopic operator pencil; analogous to the one proved in [5] for the eigenvectors of a symmetric definite quadratic matrix pencil, plays a key role in the proof of Theorem 1.

Besides the above orthogonality relation, several other useful properties of the eigenvalues and eigenvectors of a gyroscopic operator pencil needed in the proof of Theorem 1 will be established in the full version of the paper. The results of this paper generalize those for the singe-input case obtained by the authors earlier ([8]).

Theorem 1 (An Explicit Solution of the Gyroscopic Operator Partial Pole Placement Problem)

Suppose that the open-loop system is partially controllable with respect to its eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_j \notin \{\lambda_k\}_{k \geq 1}$ for $j = 1, \ldots, m$. Then

(i) *Let v_j be the eigenfunctions of the open-loop system corresponding to the eigenvectors λ_j. Then, for any arbitrary matrix $\Phi = (\phi_{jk})_{j,k=1}^{m,p}$ the feedback functions*

$$f_k(x) = \sum_{j=1}^{m} \phi_{jk} \lambda_j M v_j \;, \quad g_k(x) = \sum_{j=1}^{m} \phi_{jk} K v_j \tag{3}$$

are such that $\lambda_{m+1}, \lambda_{m+2}, \ldots$ are the eigenvalues of the closed-loop operator system (2).

(ii) *Let $\Gamma = (\gamma_{kj})_{k,j=1}^{p,m}$ be an arbitrary matrix chosen in such a way that $\mu_j = \overline{\mu_s}$ implies $\gamma_{sj} = \overline{\gamma_{ks}}$ for all $k = 1, \ldots, p$ and $1 \leq j \neq s \leq m$ and that the matrix*

$$Z_1 = \left((\lambda_s M v_s, \mu_j w_j) + (K v_s, w_j) \right)_{s,j=1}^{m} \tag{4}$$

is nonsingular, where $w_1(x), \ldots, w_m(x)$ satisfy

$$\left(\mu_j^2 M + \mu_j G + K \right) w_j \;\; = \;\; \sum_{k=1}^{p} b_k \gamma_{kj} \,. \tag{5}$$

Then, the solution to the Problem 1 is given by (3), where

$$\Phi^H Z_1 = \Gamma \,. \tag{6}$$

Note: Part (i) of Theorem 1 says that an inifinite number of the open-loop eigenvalues starting from λ_{m+1} remain invariant by the feedback defined by (2). In other words, *no spill-over occurs.*

Theorem 1 leads to a computational algorithm for Problem 1.

Algorithm 1 (An Algorithm for the Multi-input Partial Pole Placement Problem for the Quadratic Operator Pencil).

Inputs:

(a) The self-adjoint positive definite operators M and K, and skew-symmetric operator G.

(b) The control functions $b_1, \ldots, b_p$.

(c) The self-conjugate set of scalars $\{\mu_1, \ldots, \mu_m\}$.

(d) The self-conjugate subset $\{\lambda_1, \ldots, \lambda_m\}$ of the open-loop spectrum $\{\lambda_1, \lambda_2, \ldots\}$ and the associated eigenfunction set $\{v_1, \ldots, v_m\}$.

Outputs:

The feedback functions $f_1, \ldots, f_p$ and $g_1, \ldots, g_p$ such that the spectrum of the closed-loop operator system, given by (2) and (3), is exactly $\{\mu_1, \ldots, \mu_m; \lambda_{m+1}, \lambda_{m+2}, \ldots\}$.

Assumptions:

(a) The open-loop operator system is partially controllable with respect to its eigenvalues $\lambda_1, \ldots, \lambda_m$.

(b) $\mu_j \notin \{\lambda_1, \lambda_2, \ldots\}$ for each $j = 1, \ldots, m$.

Step 1. *Choose arbitrary scalars γ_{kj}, $k = 1, \ldots, p$, $j = 1, \ldots, m$ in such a way that $\mu_j = \overline{\mu_i}$ implies $\gamma_{kj} = \overline{\gamma_{ki}}$ and solve for $w_1, \ldots, w_m$:*

$$\left(\mu_j^2 M + \mu_j G + K\right) w_j = \sum_{k=1}^{p} b_k \gamma_{kj}\,,\ j = 1, \ldots, m.$$

Step 2. *Form $m \times m$ matrix Z_1:*

$$Z_1 = \left((\lambda_s M v_s, \mu_j w_j) + (K v_s, w_j)\right)_{s,j=1}^{m}$$

If Z_1 is ill-conditioned, then returm to Step 1 and select different scalars γ_{kj}.

Step 3. *Form $\Gamma = (\gamma_{kj})_{k,j=1}^{p,m}$ and solve for Φ:*

$$\Phi^H Z_1 = \Gamma\,.$$

Step 4. *Form $f_1, \ldots, f_p$ and $g_1, \ldots, g_p$:*

$$f_k(x) = \sum_{j=1}^{m} \phi_{jk} \lambda_j M v_j\,, \quad g_k(x) = \sum_{j=1}^{m} \phi_{jk} K v_j$$

Remark 1 *The most distinctive feature of the algorithm is that it computes the solution of an infinite dimensional operator problem by solving a small finite dimensional linear algebraic system and by using only the few eigenvalues of the infinite dimensional operator that need to be reassigned and the associated eigenvectors. The algorithm is thus readily applicable to control dangerous vibration in a structure, where only a small part of the spectrum (which is readily computed) needs to be reassigned and the rest is to remain unchanged.*

Numerical Example. Consider the small oscillations of a uniform string traveling with constant velocity γ over two fixed supports at $x = 0$ and $x = 1$. The motion of the moving string is governed by the partial differential equation

$$u_{tt} + 2\gamma u_{xt} + (\gamma^2 - c^2)v_{xx} \;=\; 0, \tag{7}$$

where $0 < x < 1$, $t > 0$, $\gamma^2 < c^2$ and boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{8}$$

see *e.g.* [13].

To show that the system (7) and (8) is gyroscopic we select the scalar product (v, w) as

$$(v, w) \;=\; \int_0^L \overline{v(x)} w(x)\; dx$$

and define operators M, G and K as

$$Mv = v\,,\; Gv = 2\gamma\frac{\partial v}{\partial x}\,,\; Kv = (\gamma^2 - c^2)\frac{\partial^2 v}{\partial x^2}\;. \tag{9}$$

Then

$$\begin{gathered}(Mv, w) = \overline{\textstyle\int_0^L \overline{w(x)} v(x)\; dx} = \overline{(Mw, v)} = (v, Mw)\,,\\ (Mv, v) = \textstyle\int_0^L |v(x)|^2\; dx \geq 0\,.\end{gathered}$$

Integrating by parts yields

$$(Kv, w) = -\overline{\int_0^L \overline{(\gamma^2 - c^2)w''(x)} v(x)\; dx} = (v, Kw)\,,$$

and

$$(Kv, v) \;=\; \int_0^L (\gamma^2 - c^2)|v'(x)|^2\; dx > 0\,,$$

in view of the boundary conditions (8). Since $v'(x)$ does not vanish identically, K is a self–adjoint positive definite operator. Another integration gives

$$(Gv, w) = -\overline{\int_0^L \overline{2\gamma w'(x)} v(x)\; dx} = -(v, Gw)\,,$$

and hence the traveling string (7) and (8) is a gyroscopic system.

For the purpose of making a simple illustrative numerical example we set $c = 1$, $\gamma = 1/2$ and take the control functions $b_1(x) = 1$ and $b_2(x) = x$. Then the spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$ and eigenfunctions $\{v_k\}_{k\in\mathbb{Z}}$ of traveling string system is given by

$$\lambda_k = \frac{3}{4}\pi ik\,,\ v_k(x) = e^{-\frac{1}{2}\pi ik} - e^{\frac{3}{2}\pi ik} \text{ for all } k \in \mathbb{Z}\,. \tag{10}$$

We will solve Problem 1 reassigning only the eigenvalues $\pm 3\pi i/4$ to the locations $\mu_{1,2} = -1 \pm i$. That is we want the closed-loop system (2) to have the spectrum

$$\{-1 \pm i\} \cup \{\frac{3}{4}\pi ik : k \in \mathbb{Z}, k \neq \pm 1\}. \tag{11}$$

Arbitrary choice of Γ in Step 1 of Algorithm 1 in the form $\Gamma = \begin{pmatrix} \gamma_{11} & \overline{\gamma_{11}} \\ \gamma_{21} & \overline{\gamma_{21}} \end{pmatrix}$ with $(\gamma_{11}, \gamma_{21}) \neq (0,0)$ results in real feedback. To simplify the symbolic expressions for w_1 and w_2 we set $\gamma_{11} = (1+2i)\left(e^{2i/3} - ie^{2/3}\right)$, $\gamma_{21} = (1+i)\left(e^{2/3} - ie^{2i/3}\right)$ and obtain

$$w_1(x) = \left(\frac{1}{2} - \frac{i}{2}\right)\left(e^{\frac{2}{3}(i+x)} + (x-1)e^{\frac{2}{3}i(1+x)} - e^{\frac{2}{3}(1+ix)}\right)\,,\ w_2(x) = \overline{w_1(x)}.$$

Such Γ yields the matrix $Z_1 = \begin{pmatrix} -0.509 + 0.892i & 0.614 + 0.034i \\ 0.614 - 0.034i & -0.509 - 0.892i \end{pmatrix}$ with condition number $\mathrm{Cond_2(Z_1)} = ||Z_1||_2||Z_1^{-1}||_2 = 3.978$ in Step 2. (All numerical results are rounded to 3 decimal digits.)

At the Step 3 the matrix $\Phi = \begin{pmatrix} 1.078 - 0.608i & -2.207 + 2.171i \\ 1.078 + 0.608i & -2.207 - 2.171i \end{pmatrix}$ is computed.

Finally, in Step 4 the real feedback functions

$$\begin{aligned}
f_1(x) &= 2.867\cos(\frac{\pi}{2}x) - 2.867\cos(\frac{3\pi}{2}x) + 5.079\sin(\frac{\pi}{2}x) + \sin(\frac{3\pi}{2}x) \\
f_2(x) &= -10.231\cos(\frac{\pi}{2}x) + 10.231\cos[\frac{3\pi}{2}x) - 10.399\sin(\frac{\pi}{2}x) + \sin(\frac{3\pi}{2}x) \\
g_1(x) &= 3.989\cos(\frac{\pi}{2}x) - 35.903\cos(\frac{3\pi}{2}x) - 2.251\sin(\frac{\pi}{2}x) - 20.262\sin(\frac{3\pi}{2}x) \\
g_2(x) &= -8.167\cos(\frac{\pi}{2}x) + 73.506\cos(\frac{3\pi}{2}x) + 8.035\sin(\frac{\pi}{2}x) + 72.316\sin(\frac{3\pi}{2}x)
\end{aligned}$$

are produced such that the spectrum of the closed-loop system

$$u_{tt} + 2\gamma u_{xt} + (\gamma^2 - c^2)v_{xx} = 1\int_0^1 f_1(x)u_t + g_1(x)u\ dx + x\int_0^1 f_2(x)u_t + g_2(x)u\ dx$$

with the boundary conditions (8) is precisely (11).

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