Numerically Robust Pole Assignment for Second-Order Systems

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December 8, 1995

Abstract

We propose two new methods for solution of the eigenvalue assignment problem associated with the second-order control system:

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t) \ .$$

Specifically, the methods construct feedback matrices F_1 and F_2 such that the closed loop quadratic pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D + BF_2) + (K + BF_1)$$

has a desired set of eigenvalues and the associated eigenvectors are well-conditioned. The first method is a modification of the SVD-based method proposed by Juang and Maghami [14, 15] which is a second-order adaptation of the well-known robust eigenvalue assignment method by Kautsky, Nichols and Van Dooren [16] for first order systems. The second one is an extension of the recent nonmodal approach of Datta and Rincón [8] for feedback stabilisation of second order systems. Robustness to numerical round-off errors is achieved by minimising the condition numbers of the eigenvectors of the closed-loop second order pencil. Control robustness to large plant uncertainty will not be explicitly considered in this paper.

Numerical results for both the two methods are favourable. A comparative study of the methods is included in the paper.

1 Introduction

Consider the control of the second-order system

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = B\mathbf{u}(t) \ , \tag{1}$$

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by state-feedback

$$\mathbf{u}(t) = -F_1\mathbf{x}(t) - F_2\dot{\mathbf{x}}(t) \ . \tag{2}$$

Models in the form of (1) arise frequently in a wide variety of applications in vibration and structural analysis (e.g., see [1, 18, 13]). The $n \times n$ real matrices M, D and K are respectively the mass, damping and stiffness matrices. In most applications, these matrices are symmetric and (semi)definite, and the symmetry and definiteness assumptions have important implications on the efficiency of the numerical algorithms involved. Theoretically though, the algorithms suggested in this paper are applicable to general unsymmetric indefinite second-order systems which are *controllable*, in the sense that

$$\text{rank } \left[\lambda^2 M + \lambda D + K, \ B\right] = n \quad \forall \lambda \in \mathcal{C} \ . \tag{3}$$

The definition of controllability in (3) can easily be deduced from the controllability of any first-order linearisation [5, p. 532] of (1), e.g., with $\mathbf{z}^T = \left[\mathbf{x}(t)^T, \ \dot{\mathbf{x}}(t)^T\right]$,

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix} \mathbf{u}(t) \ , \tag{4}$$

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{\mathbf{z}} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ B \end{bmatrix} \mathbf{u}(t) \ , \tag{5}$$

or

$$\begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \dot{\mathbf{z}} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \mathbf{z} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{u}(t) \ . \tag{6}$$

Let us assume that the mass matrix M is nonsingular and, for convenience, that the $n \times m$ input matrix B has full rank ($= m$).

The descriptor systems in (5) and (6) should be preferred over the equivalent nondescriptor system (4), because M does not have to be inverted explicitly. With careful book-keeping, techniques in nondescriptor system control can be applied to (5) or (6) after minor modifications. The system in (6) has the additional advantage that the sparseness, symmetry and definiteness in M, D, K and B are preserved in the augmented matrices in (6), though only for the open-loop system. Many theoretical results for second-order systems have been developed via the corresponding first-order linearisations. Numerically, some efficiency can be gained by designing algorithms which utilise the original data M, D, K and B, so that matrices involved are of dimension n, rather than of $2n$ as will be if the problems are solved via (4)–(6). In addition, the matrices on the left-hand-sides in (5) and (6) can be very ill-conditioned [8] and should be best avoided. We believe that a well-designed algorithm, which deals directly with second-order systems, can out-perform the equivalent first-order algorithm in efficiency.

In this paper, we are interested in the *robust* pole assignment problem:

> Find the feedback matrices F_1 and F_2 such that the eigenvalues of the closed-loop quadratic pencil $P_c(\lambda) \equiv \lambda^2 M + \lambda(D+BF_2)+(K+BF_1)$ equal some prescribed self-conjugate set $\Omega \subset \mathcal{C}$ (i.e. $\lambda \in \Omega$ if and only if $\overline{\lambda} \in \Omega$), and the corresponding eigenvectors are *robust*.

We shall consider the robustness issue in more details in Section 5.

The important special case of stabilisation via feedback has been considered in [8, 12] and some references therein. Also, a method for the partial pole placement that deals exclusively with data matrices of the second-order model has just been proposed in [7].

The pole assignment problem has been considered before using the *independent modal space control* technique in [2, 13, 19]. The technique requires simultaneous diagonalisation of M, D and K, or the complete solution of the quadratic eigenvalue problem for $P(\lambda) \equiv \lambda^2 M + \lambda D + K$, which is difficult and inefficient. Juang and Maghami [14, 15] adapted the first-order approach by Kautsky, Nichols and Van Dooren [16] to the second-order problem. However, the method assign only n eigenvalues (instead of $2n$) and will not work if the matrix $\bar{\Psi}$ is singular (which is required to be inverted in the calculation of the feedback matrices). In Section 3, the Juang-Maghami method will be quoted, re-interpreted and modified (to Method 1), using some results in matrix polynomials (see Section 2). Method 1 can be shown to be robust and generally applicable. One of the methods suggested in [20] implicitly assumes the solvability of certain linear systems and therefore may not always work. This method will be further developed in this paper and the resulting method will be presented in Section 4 as Method 2. It will then be shown that Methods 1 and 2 are closely related.

The paper will be concluded with two illustrative numerical examples in Section 6 and some final remarks will be included in in Section 7.

The proposed algorithms can be, in some sense, expensive. The initiation steps require a QR decomposition [11] to construct a basis of the invariant subspace corresponding to each of the $2n$ prescribed poles in Ω. Furthermore, sweeps of $2n$ steps are required if the *most robust* closed-loop eigenvectors are sought. This is the price we have to pay for optimal robustness. However, as suggested later in Section 5 (and in the numerical examples in Section 6), this full optimisation process of robustness can be replaced by a cheaper random search, which will produce adequate but sub-optimal feedback matrices.

Before proceeding further, it is important to realize that Methods 1 and 2 may be the stepping stones to more efficient approaches in the future, for other related problems, such as approximate pole assignment [3] or partial pole assignment [9, 21].

2 Some Required Results From Matrix Polynomials

In this section, some useful but not widely known results for matrix polynomials are quoted from [10, Chapters 1, 2, 7 and 8] (or the quotations in Section 2 in [4]). We shall require, and only present, the results for the quadratic pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K \ .$$

$P(\lambda)$ is a *regular* matrix pencil if $\det(P(\lambda))$ does not vanish identically. For regular matrix polynomials, let $(X, T_1 \oplus T_2)$ denote corresponding *decomposable pair*. Here we have $X \equiv [X_1, X_2]$, $X_1 \in \mathcal{C}^{n\times m}$, $X_2 \in \mathcal{C}^{n\times(2n-m)}$, $T_1 \in \mathcal{C}^{m\times m}$, $T_2 \in \mathcal{C}^{(2n-m)\times(2n-m)}$. The spectrum of $P(\lambda)$ is then split up into two clusters in T_1 and T_2, with the latter containing the reciprocals of some eigenvalues, including possibly the infinite ones. Also, note that the matrix

$$S_1 \equiv \left[\begin{array}{cc} X_1 & X_2T_2 \\ X_1T_1 & X_2 \end{array} \right] \tag{7}$$

is nonsingular, and

$$MX_1T_1^2 + DX_1T_1 + KX_1 = 0 = MX_2 + DX_2T_2 + KX_2T_2^2 \ . \tag{8}$$

In the applications we are interested only in nonsingular matrices M and there will be no infinite eigenvalues. So (7) and (8) reduce to, respectively,

$$S_1 \equiv \left[\begin{array}{c} X_1 \\ X_1T_1 \end{array} \right] \tag{9}$$

(which is nonsingular), and

$$MX_1T_1^2 + DX_1T_1 + KX_1 = 0 \ . \tag{10}$$

Here, $X_1 \in \mathcal{C}^{n\times 2n}$ and $T_1 \in \mathcal{C}^{2n\times 2n}$. A useful decomposable pair is the *Jordan pair* with T_1 in Jordan canonical form. Note that the columns of S_1 are the right-eigenvectors of the matrix-pencil associated with the *companion* linearisation in (5).

For the regular quadratic pencil $P(\lambda)$, we have the following *resolvent form* [10, Theorem 7.7, pp. 195–197], when λ is not an eigenvalue of $P(\lambda)$,

$$P^{-1}(\lambda) = X_1(\lambda I - T_1)^{-1}Y_2 \ , \quad Y_2 \equiv S_0^{-1} \left[\begin{array}{c} 0 \\ I_n \end{array} \right] \ , \tag{11}$$

with $S_0 = (I \oplus M)S_1 = \left[\begin{array}{c} X_1 \\ MX_1T_1 \end{array} \right]$ being obviously nonsingular (because S_1 in (9) is).

¿From [4] (see also Section 5), $\|X_1\| \cdot \|Y_2\|$ or $\|\mathbf{x}_{1j}\| \cdot \|\mathbf{y}_{2j}\|$ (where $\mathbf{x}_{1j}$ and $\mathbf{y}_{2j}^T$ are respectively individual or groups of columns and rows of X_1 and Y_2) are condition numbers for the spectrum of $P(\lambda)$.

Finally, apart from using T_1 and T_2 in a decomposable pair to represent finite and infinite subspectra, they can be used to represent subspectra inside and outside the unit circle. This latter representation is important in deriving the condition numbers in [4], or the robustness measures in Section 5.

3 Method 1

We first state the Juang-Maghami method [14, 15].

An eigenvalue λ_k and its corresponding eigenvector $\mathbf{x}_k$ of the closed-loop system for (1), with feedback as shown in (2), satisfy

$$\left[\lambda_k^2 M + \lambda_k (D + BF_2) + (K + BF_1)\right] \mathbf{x}_k = \mathbf{0} \ , \tag{12}$$

with $k = 1, \cdots, n$.

However, the theory of matrix polynomials and the application of control theory on any linearisation of (1) imply that $2n$ poles are assignable.

The eigenvalue equation (12) can be rearranged to $\Gamma_k \phi_k = \mathbf{0}$ with

$$\Gamma_k \equiv \left[(\lambda_k^2 M + \lambda_k D + K), \ B\right] \ , \quad \phi_k \equiv \left[\begin{array}{c} \mathbf{x}_k \\ (\lambda_k F_2 + F_1)\mathbf{x}_k \end{array} \right] \ .$$

Select $\phi_k^T = (\bar{\phi}_k^T, \ \hat{\phi}_k^T)$, with $\bar{\phi}_k \in \mathcal{C}^n$ and $\hat{\phi}_k \in \mathcal{C}^m$. From the null space of Γ_k, the feedback matrices F_1 and F_2 then satisfy

$$[F_1, \ F_2]\tilde{\Psi} = \hat{\Phi} \ , \quad \tilde{\Psi} \equiv \left[\begin{array}{c} \bar{\Phi} \\ \bar{\Phi}\Lambda \end{array} \right] \ ,$$

with $\bar{\Phi} \equiv \left[\bar{\phi}_1, \cdots, \bar{\phi}_n\right]$, $\hat{\Phi} \equiv \left[\hat{\phi}_1, \cdots, \hat{\phi}_n\right]$ and $\Lambda \equiv \mathrm{diag}\left\{\lambda_1, \cdots, \lambda_n\right\}$. Consider the real and complex parts, the above equation can be written in real form

$$[F_1, \ F_2]\bar{\Psi} = \left[\hat{\Phi}_{\Im}, \ \hat{\Phi}_{\Re}\right] \ , \quad \bar{\Psi} \equiv \left[\begin{array}{cc} \bar{\Phi}_{\Im} & \bar{\Phi}_{\Re} \\ \bar{\Phi}_{\Re}\Lambda_{\Im} + \bar{\Phi}_{\Im}\Lambda_{\Re} & \bar{\Phi}_{\Re}\Lambda_{\Re} - \bar{\Phi}_{\Im}\Lambda_{\Im} \end{array} \right] \ ,$$

where the subscripts $(\cdot)_{\Re}$ and $(\cdot)_{\Im}$ indicate respectively the real and imaginary parts.

Unfortunately, $\tilde{\Psi}$ is of dimension $2n \times n$ and consequently $\bar{\Psi}$, has at most rank n and is thus singular. Note that the complex equation involving $\tilde{\Psi}$ and the real equation involving $\bar{\Psi}$ are equivalent. (The difference in the number of equations is a mirage, as some of the real equations are redundant, due to the relationship between the real and complex parts of the eigenvalue equation for a real matrix.) A trivial but revealing observation is that $\bar{\Psi}$ has n zero columns when the poles to be assigned are all real. Note also that although the equation involving $\tilde{\Psi}$ is complex, a real solution is guaranteed because complex conjugate poles are assigned.

Note that $\tilde{\Psi}$ forms n columns of S_1, which is always nonsingular with appropriately chosen eigenvectors. The immediate implication is that a method similar to the Juang-Maghami method will work if $2n$ poles are assigned.

The null-spaces defining the eigenvectors (Step 2 below) can be obtained by pre-multiplying (12) by Q_2^T (from Step 1 below). The calculations involved are simpler and slightly more efficient than those in [14], because slightly smaller matrices are involved. The calculations in Step 3 below then have to be modified accordingly, and (14) can be obtained from pre-multiplying (12) by $B^+ = R_B^{-1}Q_1^T$. With the above refinements (see [16] for details), we arrive at the following algorithm for Method 1:

Algorithm 1

Step 1. (Initiation) Let the prescribed closed-loop poles be $\{\lambda_{1k}\}$ and $\{\lambda_{2k}\}$, $k = 1, \cdots, n$, and $\Lambda_i \equiv \text{diag}\{\lambda_{i1}, \cdots, \lambda_{in}\}$, $i = 1, 2$.

Form the QR decomposition of B, so that

$$B = [Q_1,\ Q_2] \left[\begin{array}{c} R_B \\ 0 \end{array} \right] = Q_1 R_B \ , \quad Q = [Q_1,\ Q_2] \ ,$$

with R_B being upper triangular and nonsingular, and Q being orthogonal.

Step 2. (Eigenvector Selection) For $i = 1, 2$ and $k = 1, \cdots, n$, select $\mathbf{x}_{ik}$ from the null spaces (realized using QR decompositions) defined by

$$Q_2^T (\lambda_{ik}^2 M + \lambda_{ik} D + K) \mathbf{x}_{ik} = \mathbf{0} \ , \tag{13}$$

so that with $X_{1i} = [\mathbf{x}_{i1}, \cdots, \mathbf{x}_{in}]$ $(i = 1, 2)$, $X_1 = [X_{11},\ X_{12}]$ and $T_1 = \Lambda_1 \oplus \Lambda_2$, the condition number of

$$S_1 = \left[\begin{array}{c} X_1 \\ X_1 T_1 \end{array} \right] = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{11}\Lambda_1 & X_{12}\Lambda_2 \end{array} \right] .$$

is minimised.

Step 3. (Calculation of Feedback Matrices) Solve the well-conditioned system

$$R_B [F_2,\ F_1] S_1 = -Q_1^T [R_1,\ R_2] \tag{14}$$

for the feedback matrices F_1 and F_2, where $R_i \equiv M X_{1i} \Lambda_i^2 + D X_{1i} \Lambda_i + K X_{1i}$ $(i = 1, 2)$.

End of Algorithm 1

The eigenvector selection in Step 2 can be done in a similar fashion as in [16]. Note that from the controllability property (3) of (1), we can deduce easily that the dimensions of the null spaces defined by (13) in Step 2 are exactly m. As a result, we can only assign eigenvalues of multiplicity less than or equal m, otherwise a more involved procedure involving defective (thus ill-conditioned) eigenvalues and principle vectors will be required. On the other hand, open-loop eigenvalues can be re-assigned with different eigenvectors. As indicated in Step 2, a possibility is to select $\mathbf{x}_{ik}$ so that the condition number of S_1, the matrix operator in (14), is optimised. Other possibilities will be considered in Section 5.

Finally on the number of assignable poles, we have, from [15], the quotation "However, if the number of eigenvalues to be assigned is less than the number assignable, n, Eq. (11) becomes under-determined, which leads naturally to a minimum gain solution". It is hard to imagine how a system can be *controlled* if part of the closed-loop spectrum is allowed to roam free. These free poles may well affect adversely the performance (such as stability) of the system, though yielding minimum gain feedback matrices. An extreme example will be to assign *no* poles at all in order to obtain a solution with zero gain!

4 Method 2

We next derive another eigenvalue assignment method, which is a generalisation of a feedback-stabilisation method recently proposed by Datta and Rincón [8]. (See also the methods in [20].)

Let X_1 be the same as in Algorithm 1. Assume that X_{11}, the first n columns in X_1, is nonsingular. With $C_0 \equiv X_{11}\Lambda_1 X_{11}^{-1}$, $D_0 \equiv \tilde{Y}\Lambda_2\tilde{Y}^{-1}$ and $\tilde{Y} \equiv [\tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n]$, define

$$L_0 \equiv \begin{bmatrix} I & 0 \\ C_0 & -I \end{bmatrix}, \quad \tilde{T}_1 = \begin{bmatrix} C_0 & -I \\ 0 & D_0 \end{bmatrix}.$$

Obviously, $\tilde{T}_1$ has the same spectrum as $T_1 = \Lambda_1 \oplus \Lambda_2$, and $L_0^2 = I$. Using the linearisation in (4), with $A = \begin{bmatrix} 0 & I \\ M^{-1}K & M^{-1}D \end{bmatrix}$, we have

$$L_0 A - \tilde{T}_1 L_0 = \begin{bmatrix} 0 & 0 \\ M^{-1}K - D_0C_0 & M^{-1}D + (C_0 + D_0) \end{bmatrix}$$

and

$$A - L_0^{-1}\tilde{T}_1 L_0 = \begin{bmatrix} 0 & 0 \\ -M^{-1}K + D_0C_0 & -M^{-1}D - (C_0 + D_0) \end{bmatrix}.$$

Re-arranging and comparing with the closed-loop linearisation

$$A - \begin{bmatrix} 0 & 0 \\ M^{-1}BF_1 & M^{-1}BF_2 \end{bmatrix},$$

we have the sufficient (but not necessary) conditions for F_1 and F_2 assigning the poles in T_1:

$$BF_1 = -K + MD_0C_0 \tag{15}$$

and

$$BF_2 = -D - M(C_0 + D_0) \tag{16}$$

The feedback matrices F_1 and F_2 can now be obtained via solutions of (15) and (16), possibly in the least squares sense, for given C_0 and D_0. Unfortunately, (15) and (16) are consistent only for suitably chosen D_0 and C_0 (see the modified approach later), and the least squares solution does not guarantee the assignment, approximate or otherwise, of the prescribed closed-loop poles.

We now show how the above approach can be made to work (as Method 2), and we shall establish a relationship between Methods 1 and 2. From (16), we have the expression

$$MD_0 = -D - BF_2 - MC_0 .$$

Substituting into (15), we have

$$B(F_1 + F_2C_0) = -(K + DC_0 + MC_0^2) . \tag{17}$$

With $C_0 \equiv X_{11}\Lambda_1 X_{11}^{-1}$, pre-multiplying (17) by Q_2^T and then post-multiplying by $\mathbf{x}_{1k}$ yield

$$Q_2^T(\lambda_{1k}^2 M + \lambda_{1k} D + K)\mathbf{x}_{1k} = \mathbf{0} \quad (k = 1, \cdots, n) \ , \tag{18}$$

exactly the same equation as (13), when $i = 1$. With $X_{11} = [\mathbf{x}_{11}, \cdots, \mathbf{x}_{1n}]$ selected from the null spaces defined by (18), C_0 can then be calculated. Pre-multiplying (17) by $B^+ = R_B^{-1} Q_1^T$ gives rise to

$$F_1 = -F_2 C_0 - R_B^{-1} Q_1^T (K + DC_0 + MC_0^2) \ ,$$

from which F_1 can be calculated in terms of F_2. However, a similar but simpler formula for F_1 can be obtained by pre-multiplying (15) by B^+, so that

$$F_1 = R_B^{-1} Q_1^T (-K + MD_0 C_0) \ , \tag{19}$$

which is applicable after both C_0 and D_0 are obtained. A similar formula for F_2 can be obtained from (16), using the same technique, so that

$$F_2 = -R_B^{-1} Q_1^T [D + M(C_0 + D_0)] \ . \tag{20}$$

To obtain $D_0 \equiv \tilde{Y}\Lambda_2\tilde{Y}$, pre-multiply (16) by Q_2^T and post-multiply by $\tilde{\mathbf{y}}_k$ $(k = 1, \cdots, n)$ to produce

$$Q_2^T(-D - MC_0 - \lambda_{2k} M)\tilde{\mathbf{y}}_k = \mathbf{0} \quad (k = 1, \cdots, n) \ . \tag{21}$$

Columns in $\tilde{Y}$ can thus be selected from the null spaces defined by (21), after C_0 has been calculated.

It is now evident that Method 2 is quite similar to Method 1. This can be seen by replacing the definition of the null spaces in (13) for λ_{2k} $(k = 1, \cdots, n)$ and making corresponding modifications in the calculations of the feedback matrices in (19) and (20).

Algorithm 2

Step 1. (Initialisation) Let the prescribed closed-loop poles be $\{\lambda_{1k}\}$ and $\{\lambda_{2k}\}$, $k = 1, \cdots, n$, and $\Lambda_i \equiv \mathrm{diag}\{\lambda_{i1}, \cdots, \lambda_{in}\}$, $i = 1, 2$.

Form the QR decomposition of B, so that

$$B = [Q_1, \ Q_2] \begin{bmatrix} R_B \\ 0 \end{bmatrix} = Q_1 R_B \ , \quad Q = [Q_1, \ Q_2] \ ,$$

with R_B being upper triangular and nonsingular, and Q being orthogonal.

Step 2. (Eigenvector Selection for X_{11}) For $k = 1, \cdots, n$, select $\mathbf{x}_{ik}$ from the null spaces (realized using QR decompositions) defined by (18):

$$Q_2^T(\lambda_{1k}^2 M + \lambda_{1k} D + K)\mathbf{x}_{1k} = \mathbf{0} \ ,$$

so that $X_{11} = [\mathbf{x}_{11}, \cdots, \mathbf{x}_{1n}]$ is well-conditioned.

Solve for C_0: $C_0 X_{11} = X_{11}\Lambda_1$.

Step 3. (Eigenvector Selection for $\tilde{Y}$) For $k = 1, \cdots, n$, select $\tilde{\mathbf{y}}_k$ from the null spaces defined by (21):

$$Q_2^T(-D - MC_0 - \lambda_{2k}M)\tilde{\mathbf{y}}_k = \mathbf{0} \ ,$$

so that $\tilde{Y} = [\tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n]$ is well-conditioned.

Solve for D_0: $D_0\tilde{Y} = \tilde{Y}\Lambda_2$.

Step 4. (Calculation of Feedback Matrices) Solve for $F = (F_1, F_2)$:

$$R_B[F_1, \ F_2] = Q_1^T[-K + MD_0C_0, \ -D - M(C_0 + D_0)] \ .$$

End of Algorithm 2

It can be easily shown (using (15) and (16), or the definition of C_0) that C_0 satisfies

$$MC_0^2 + (D + BF_2)C_0 + (K + BF_1) = 0 \ .$$

This relation is analogous to (10). A similar relation also holds for D_0.

¿From (16), we have

$$C_0 = -D_0 - M^{-1}BF_2 - M^{-1}D \ .$$

Substituting this into (15), we have

$$\tilde{D}_0^2M + \tilde{D}_0(D + BF_2) + (K + BF_1) = 0 \ , \quad \tilde{D}_0 = MD_0M^{-1} \ . \tag{22}$$

We see that from (22), $\tilde{D}_0$, and thus D_0, and $\tilde{\mathbf{y}}_k$, are related to the left-eigenvectors of (1) (more on this later).

There are several differences and similarities between the Methods 1 and 2. Firstly, the assumption that X_{11} and $\tilde{Y}$ are nonsingular is required for Method 2 for, but not for Method 1. From the theory of matrix polynomials (see Section 2), it is easy to see that S_1 is always nonsingular, implying that X_1 always has full rank. As a result, it is always possible to find a partition $T_1 = \Lambda_1 \oplus \Lambda_2$ such that X_{11}, which contains as columns the eigenvectors corresponding to Λ_1, is nonsingular. However, this partition is not known in advance and we may have to repeat our calculations when X_{11} is detected to be ill-conditioned. (The invertibility of $\tilde{Y}$ will be considered in the paragraph after next.)

Secondly in Method 1, the selection of $\{\mathbf{x}_{1j}\}$ and $\{\mathbf{x}_{2k}\}$ can be performed in parallel, in contrast to the fact that X_{11} and C_0 have to be determined before the selection of $\{\tilde{\mathbf{y}}_k\}$ in Method 2.

We shall now elaborate on the left-eigenvectors and the equation (22) further. Re-writing the linearisation (5) for (1), we have

$$\frac{d\tilde{\mathbf{z}}}{dt} = \begin{bmatrix} 0 & I \\ -(K + BF_1)M^{-1} & -(D + BF_2)M^{-1} \end{bmatrix} \tilde{\mathbf{z}} \ .$$

Let the rows of Y be the left-eigenvectors of closed-loop system matrix, for given feedback matrices F_1 and F_2, of the above linearised first-order system with corresponding eigenvalues on the diagonal of $T_1 = \Lambda_1 \oplus \Lambda_2$. Let $Y = S_0^{-1}$ and

$$Y = [Y_1, \ Y_2] = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} .$$

Recall that Y_2 appears in (11) (and later in Section 5 in the condition numbers). From (22), we can show that $M\tilde{Y} = Y_{22}^{-1}$, implying that $\tilde{Y}$ will be nonsingular for a suitable partition of $T_1 = \Lambda_1 \oplus \Lambda_2$ (which yields a nonsingular Y_{22}; c.f. the similar argument for the invertibility of X_{11} above). It remains an open question whether the controllability of (1), or more generally the solvability of the pole assignment problem, guarantee the existence of a partition of T_1 which implies the invertibility of X_{11} and $\tilde{Y}$ *simultaneously*.

Finally in Steps 2 and 3 for Algorithm 2, the eigenvectors can be selected in a similar fashion as in [18]. Similar to Algorithm 1, the controllability property (3) of (1) implies that the dimensions of the null-spaces defined in Step 2 are exactly m. We do not know much about the null-spaces in Step 3, except that they are not empty and generically should be of dimension m as well. Also, alternatives to minimising the condition numbers of X_{11} and $\tilde{Y}$ will be discussed in the next section.

5 Eigenvector Selection

We shall denote the condition number of a matrix Z by $\kappa(Z) \equiv \|Z\| \, \|Z^+\|$, where Z^+ is the pseudoinverse of Z. The norm used in this paper is either 2-norm or Frobenius norm, unless otherwise stated.

In [4], condition numbers were obtained, using the theory of matrix polynomials [12] in general; and the resolvent form (11) in particular. From Bauer-Fike type theorem [4, Theorem 4.2], we know that

$$\kappa \equiv \|X_1\| \, \|Y_2\|$$

is a condition number for the whole spectrum, without any restriction on the sizes of the perturbations to the original data of M, D and K. When the perturbations are asymptotic, we have a condition number for λ_{ik} [4, Section 5], ($i = 1, 2; \ k = 1, \cdots, n$),

$$\kappa_{ik} = \|\mathbf{x}_{ik}\| \, \|\mathbf{y}_{ik}\| ,$$

where $\mathbf{x}_{ik}$ is the kth column of X_{1i} and $\mathbf{y}_{ik}^T$ is the kth row of Y_{2i}.

For clusters of eigenvalues (see details in [4, Section 5]), under some mild restriction on the sizes of the perturbations (to stop the clusters from merging), we have the condition number for a cluster similarly defined as κ_{ik}, with $\mathbf{x}_{ik}$ ($\mathbf{y}_{ik}$) replaced by the corresponding columns (rows) in X_{1i} (Y_{2i}).

It is difficult to select eigenvectors to minimise κ. However, basic matrix inequalities imply

$$\kappa \leq \|X_1\| \, \left\|S_0^{-1}\right\| \leq \kappa(S_0) \leq \kappa(\tilde{M}) \; \kappa(S_1) \; , \quad \tilde{M} = I \oplus M \; .$$

These inequalities give rise to the possibilities of minimising various upper bounds of κ. When the right-eigenvectors are normalised, we may minimise the simpler quantities in

$$\|Y_2\| \leq \left\|S_0^{-1}\right\| \leq \|\tilde{M}^{-1}\| \; \|S_1^{-1}\| \ .$$

For Method 2, the above quantities are not available from the computations directly, and it may be more efficient to minimise the related quantities which have to be inverted, e.g., $\kappa(X_{11})$, $\kappa(\tilde{Y})$, $\|X_{11}^{-1}\|$ or $\|\tilde{Y}^{-1}\|$.

For both Methods 1 and 2, another alternative is to choose a few randomly chosen eigenvectors, evaluate the corresponding condition numbers (or related upper bounds), and then select the best sub-optimal solution. This "random-search" approach will be adequate and yet especially powerful for large control systems with sparseness in M, D, K and B. For such systems, the full iterative approach for minimising condition numbers may be prohibitively expensive, especially when only the most ill-conditioned solutions need to be avoided. We shall consider the full iterative approach in what follows.

Let us first reiterate a technique from [18] for the minimisation of $\kappa(X)$ in the Frobenius norm, $X = [X_j, \ \mathbf{x}_j]$ by updating the jth column $\mathbf{x}_j$ in X. Other techniques in [18] can also be adapted but shall not be elaborated here. Notice that we have rearranged the columns in X so that the jth column appears last, without loss of generality and for simplicity in notation. The technique can be applied in the minimisation of $\kappa(X_{11})$, $\kappa(\tilde{Y})$, $\|X_{11}^{-1}\|$ or $\|\tilde{Y}\|$ for both Methods 1 and 2.

Consider the QR decomposition of

$$X_j = [Q_j, \ \mathbf{q}_j] \begin{bmatrix} R_j \\ \mathbf{0}^T \end{bmatrix} = Q_j R_j \ ,$$

where R_j is upper triangular and assumed to be nonsingular, and $\tilde{Q} \equiv [Q_j, \ \mathbf{q}_j]$. We then have

$$X = [X_j, \ \mathbf{x}_j] = \tilde{Q} \begin{bmatrix} R_j & Q_j^H \mathbf{x}_j \\ \mathbf{0}^T & \mathbf{q}_j^H \mathbf{x}_j \end{bmatrix} \ .$$

Here $(\cdot)^H$ denotes the conjugate transpose, or Hermitian, of matrices. Note that

$$X^{-1} = \begin{bmatrix} R_j^{-1} & -\alpha_j^{-1} R_j^{-1} Q_j^H \mathbf{x}_j \\ 0 & \alpha_j^{-1} \end{bmatrix} \tilde{Q}^H \ , \quad \alpha_j \equiv \mathbf{q}_j^H \mathbf{x}_j \ .$$

Assuming that the columns of X are normalised to unity, minimising $\kappa(X)$ is then equivalent to $\min_{\mathbf{x}_j} \|X^{-1}\|$ or

$$\min_{\mathbf{x}_j} \left\| \alpha_j^{-1} \begin{bmatrix} R_j^{-1} Q_j^H \\ I \end{bmatrix} \mathbf{x}_j \right\| \ . \tag{23}$$

Let N_j be a unitary matrix with columns spanning the null-space from which $\mathbf{x}_j$ should be selected. Thus $\mathbf{x}_j = N_j \mathbf{u}_j$ with $\|\mathbf{u}_j\| = 1$. Form the QR decomposition

$$N_j^T \mathbf{q}_j = \hat{Q}_j \begin{bmatrix} \beta_j \\ 0 \end{bmatrix} = \beta_j \hat{Q}_j \mathbf{e}_1 \ ,$$

where $\mathbf{e}_j$ is the jth column of the identity matrix. Consequently, we have $\mathbf{q}_j^H N_j \hat{Q}_j = \overline{\beta}_j \mathbf{e}_1^T$, $\alpha_j = (\mathbf{q}_j^H N_j \hat{Q}_j)(\hat{Q}_j^H \mathbf{u}_j) = \overline{\beta}_j \mathbf{e}_1^T(\hat{Q}_j^H \mathbf{u}_j)$, and the minimisation problem (23) reduces to (after ignoring β_j)

$$\min_{\hat{\mathbf{u}}_j} \left\| \begin{bmatrix} R_j^{-1} Q_j^H N_j \\ I \end{bmatrix} \hat{Q}_j \left(\frac{\hat{Q}_j^H \mathbf{u}_j}{\mathbf{e}_1^T(\hat{Q}_j^H \mathbf{u}_j)} \right) \right\| .$$

Let

$$\frac{\hat{Q}_j^H \mathbf{u}_j}{\mathbf{e}_1^T(\hat{Q}_j^H \mathbf{u}_j)} = \begin{bmatrix} 1 \\ \hat{\mathbf{u}}_j \end{bmatrix} , \quad \hat{\mathbf{u}}_j \equiv \begin{bmatrix} \hat{\mathbf{u}}_{j2} \\ \vdots \\ \hat{\mathbf{u}}_{jn} \end{bmatrix} .$$

We then have to solve the equivalent minimisation problem

$$\min_{\hat{\mathbf{u}}_j} \left\| \begin{bmatrix} R_j^{-1} Q_j^H N_j \\ I \end{bmatrix} \hat{Q}_j \begin{bmatrix} 1 \\ \hat{\mathbf{u}}_j \end{bmatrix} \right\| = \min_{\hat{\mathbf{u}}_j} \| Z_j \hat{\mathbf{u}}_j + \mathbf{z}_j \| \ ,$$

which is a standard least squares problem and can be solved using the QR decomposition of Z_j, with $\hat{\mathbf{u}}_j = -Z_j^+ \mathbf{z}_j$. The eigenvector $\mathbf{x}_j$ can be recovered from $\mathbf{z}_j$.

When the eigenvalues to be assigned are complex, the above rank-1 update technique will no longer be adequate, as two eigenvectors corresponding to a pair of complex conjugate eigenvalues have to be selected simultaneously. However, a rank-2 update technique can easily be developed from the procedure described above. The easy-to-solve least squares problem will then become a minimisation problem involving a quadratic pencil (see [16] for details).

The minimisation of $\kappa(S_1)$ can then be carried out as follows. Recall that

$$S_1 = \begin{bmatrix} X_1 \\ X_1 T_1 \end{bmatrix} .$$

The jth column of S_1, $\mathbf{z}_j$, satisfies

$$\|\mathbf{z}_j\|_2^2 = (1 + |\lambda_{ik}|^2)\|\mathbf{x}_{ik}\|_2^2 \ ,$$

for some values of i and k with $j = (i-1)n + k$. Assume that the columns in X_1 are normalised to unity. As $1 + |\lambda_{ik}|^2$ is constant for the purpose of the minimisation, the process is then equivalent to $\min \|S_1^{-1}\|$. With

$$S_1 = \begin{bmatrix} X_j & \mathbf{x}_j \\ X_j \Lambda_j & \lambda_{ik} \mathbf{x}_j \end{bmatrix} , \quad T_1 = \Lambda_j \oplus \lambda_{ik} \ ,$$

$$\begin{bmatrix} X_j \\ X_j \Lambda_j \end{bmatrix} = \tilde{Q} \begin{bmatrix} R_j \\ \mathbf{0}^T \end{bmatrix} , \quad \tilde{Q} \equiv [Q_j, \ \mathbf{q}_j] \ ,$$

and

$$S_1 = \tilde{Q} \begin{bmatrix} Q_j^H R_j & Q_j^H Z_j \mathbf{x}_j \\ \mathbf{0}^T & \mathbf{q}_j^H Z_j \mathbf{x}_j \end{bmatrix} , \quad Z_j \equiv \begin{bmatrix} I \\ \lambda_{ik} I \end{bmatrix} . \tag{24}$$

The above rank-1 update procedure can then be generalised in a straight forward manner to update columns of S_1. Similar techniques can obviously be applied to $\min \|S_0^{-1}\|$, with Z_j in (24) replaced by $\begin{bmatrix} I \\ \lambda_{ik} M \end{bmatrix}$.

6 Numerical Examples

The number 6.7774×10^3 is denoted by 6.7774(3) in this section. All the norms used in tables are the 2-norm. The numerical results were obtained using MATLAB [17] on the Mdw-Work-Station network at Monash University. The machine accuracy $eps \approx 2.22(-16)$ (see [17] for details on eps).

6.1 Example 1

Example 1 is a small multi-input example with $n = 3$ and $m = 2$:

$$M = \text{ diag } \{10,\ 10,\ 10\} \ , \quad D = \text{diag } \{5,\ 2.5,\ 0.5\} \ ,$$

$$K = \begin{bmatrix} 1500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix} \ , \quad B = \begin{bmatrix} 0.1 & -0.2 \\ 0.2 & -0.3 \\ -0.5 & 0.1 \end{bmatrix} \ ,$$

$$\Lambda_1 = \text{ diag } \{-0.1,\ -0.2,\ -0.3\} \ , \quad \Lambda_2 = \text{ diag } \{-0.4,\ -0.5,\ -0.6\} \ .$$

The open-loop eigenvalues are

$$\{-1.2290 \pm 13.1266i,\ -0.1366 \pm 6.3592i,\ -0.0344 \pm 2.6775i\} \ .$$

As $m = 2$, the solution is not unique. A full minimisation in Step 2 of Algorithm 1 or Steps 2 and 3 of Algorithm 2 has not been performed. Instead, three combinations (cases (a), (b) and (c)) of eigenvectors from the null-spaces in (13), (18) and (21) are selected and tested. In cases (a) and (c), all the eigenvectors were selected, respectively, from the first and last vectors in the null-spaces. In case (b), the average of the two vectors in the null-spaces are selected. (A similar selection process is also applied in Example 2 later.) Method 1 produced ill-conditioned solutions in cases (a) and (b), while Method 1 failed in case (c) and Method 2 failed in all cases. A successful strategy — select the first vector in the null-spaces for λ_{ik} when k is odd and select the second vector for λ_{ik} when k is even — produces the same feedback matrices, ignoring round-off errors, from Methods 1 and 2. The results are summarised in Table 1 below. In Table 1, column 2 contains various quantities (as indicated in column 1) from Method 1. Column 2 contains the differences of a particular quantity obtained using Methods 1 and 2, measured in 2-norm.

Note that it will be difficult to compare the two methods in general, as solutions from them may correspond to different closed-loop eigenvectors.

The solution in Table 1, and most likely Example 1, are ill-conditioned, where $\kappa(S_0) = 3.1177(6)$ and $\kappa = 1.5917(6)$. Judging from Table 1, we see that there is very little difference between the solutions obtained by the two methods. Method 1 produced slightly more accurate closed-loop eigenvalues, although the difference is negligible.

Recall that S_1 has to be inverted in Method 1, and X_{11} and $\tilde{Y}$ in Method 2. The associated condition numbers are:

$$\kappa(S_1) = 1.0007(6) \ , \quad \kappa(X_{11}) = 1.4966(4) \ , \quad \kappa(\tilde{Y}) = 1.4878(4) \ .$$

Table 1

	Method 1	$\|\text{Diff.}\|$
F_1	$\begin{matrix} 6.7774(3) & -3.3905(3) & 3.9070(2) \\ 3.3176(4) & -1.5103(4) & 9.3977(2) \end{matrix}$	10^{-1}
F_2	$\begin{matrix} -4.8163(1) & 2.8523(1) & -1.4400(1) \\ -2.4098(2) & 1.4187(2) & -7.2218 \end{matrix}$	10^{-2}
λ	$-9.999999995798320(-2)$	10^{-7}
	$-1.999999999999902(-1)$	10^{-13}
	$-3.000000029776527(-1)$	10^{-7}
	$-3.999999985153957(-1)$	10^{-6}
	$-5.000000000000144(-1)$	10^{-12}
	$-6.000000005189579(-1)$	10^{-7}

Interestingly, we have better accuracy from Method 1 with the larger condition number $\kappa(S_1)$.

Both Methods 1 and 2 were successful in obtaining the feedback matrices accurately.

6.2 Example 2

The example is quoted from [8, Case 2, Example 1, Section 2, p. 150]:

$$M = 10 I_3 \ , \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix} \ , \quad D = 0 \ , \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix} \ ,$$

$$\Lambda_1 = \ \text{diag}\ \{-1,\ -2,\ -3\}\ , \quad \Lambda_2 = \ \text{diag}\ \{-4,\ -5,\ -6\}\ .$$

Λ_i $(i = 1, 2)$ are chosen arbitrarily. The open-loop eigenvalues are

$$\{\pm 3.6039i,\ \pm 2.4940i,\ \pm 0.8901i\}\ .$$

Similar to Example 1, $m = 2$ and the solution is not unique. A full minimisation in Step 2 of Algorithm 1 or Steps 2 and 3 of Algorithm 2 has not been performed and the three combinations of eigenvectors from the null-spaces in (13), (18) and (21), cases (a), (b) and (c) as in Example 1, are selected. The results for cases (a)–(c) from Methods 1 and 2 are summarised respectively in Tables 2a, 2b and 2c. Similar notations as in Table 1 are used, together with $\Re(\cdot)$ and $\Im(\cdot)$ denoting respectively the real and imaginary parts.

First, let us consider the results for Case (a) in Table 2a. Method 1 gave a more well-conditioned solution, in the sense that the condition numbers κ and $\kappa(S_0)$ are slightly smaller. Interestingly, the closed-loop eigenvalues from Method 2 are slightly more accurate. It is worth reminding ourselves that condition numbers are usually conservative upper bounds, and large condition numbers are necessary but not sufficient for an ill-conditioned problem. Example 2 is obviously ill-conditioned, with large condition numbers, although the

Table 2a

	Method 1		Method 2	
F_1^T	$-3.3291(3)$	$2.6273(3)$	$1.0105(4)$	$-8.6740(3)$
	$-2.9716(4)$	$2.3555(4)$	$1.2150(4)$	$-1.0392(4)$
	$6.8797(4)$	$-5.4501(4)$	$-5.1170(4)$	$4.3855(4)$
F_2^T	$-8.6617(3)$	$6.8633(3)$	$1.9502(2)$	$-1.9141(2)$
	$2.7906(4)$	$-2.2112(4)$	$9.7301(2)$	$-8.5292(2)$
	$-3.3444(4)$	$2.6500(4)$	$-1.4141(3)$	$1.3132(3)$
κ	$2.1544(6)$		$8.1493(6)$	
$\kappa(S_0)$	$7.3384(6)$		$3.2436(7)$	
$\Re(\lambda)$	-6.000000018342428		-5.999999999607684	
	-4.999999948895886		-5.000000001614143	
	-4.000000055236776		-3.999999997502286	
	-2.999999969681889		-3.000000001794087	
	-2.000000008120103		-1.999999999371852	
	$-9.999999992679218(-1)$		-1.000000000095840	
$\Im(\lambda)$	$O(10^{-11})$		$O(10^{-11})$	

closed-loop eigenvalues in both the methods are still accurate enough (up to eight significant figures) for most applications.

We next consider the results for case (b) in Table 2b. Method 1 is definitely superior to Method 2, producing a smaller condition number for the solution and smaller gain for the feedback matrices. Also, the closed-loop eigenvalues are at least a hundred times more accurate. The closed-loop eigenvalues from Method 1 are accurate up to seven significant figures; those from Method 2 are accurate up to only five significant figures. The results for case (c) is summarised in Table 2c and conclusions similar to those for case (b) in Table 2b can be drawn.

¿From Tables 2a–c, Methods 1 and 2 were both successful in obtaining the feedback matrices accurately, provided that the eigenvectors are selected appropriately. Method 1 seems to be producing better conditioned solutions, consistent with the theory in the previous sections. Case (a) produced the best sub-optimal solutions for both Methods 1 and 2, without going through the more expensive full-minimisation process.

7 Final Remarks

We have developed two methods for the robust eigenvalue assignment problem of a control system governed by a system of second-order differential equations.

The first method is a modification of a method by Juang and Maghami [14], and the second method generalises a feedback stabilisation method recently proposed by Datta and Rincón [8]. The development of these methods is based on some important, but not so well-known, results on matrix polynomials.

Table 2b

	Method 1		Method 2	
F_1^T	$-4.6181(4)$	$2.8661(4)$	$-9.7709(5)$	$8.0021(5)$
	$-1.2857(5)$	$7.9758(4)$	$-1.7775(6)$	$1.4555(6)$
	$1.5868(4)$	$-9.8317(3)$	$3.1970(3)$	$-2.5930(3)$
F_2^T	$5.7132(4)$	$-3.5433(4)$	$9.4011(3)$	$-6.6629(3)$
	$1.1536(5)$	$-7.1543(4)$	$3.3083(3)$	$-8.2563(2)$
	$-4.1116(3)$	$2.5500(3)$	$-2.9920(1)$	$2.1227(1)$
κ	$1.1410(7)$		$9.9619(9)$	
$\kappa(S_0)$	$1.9647(7)$		$3.0617(10)$	
$\Re(\lambda)$	-5.999999946546182		-5.999996413586461	
	-5.000000166766821		-5.000014093971046	
	-3.999999805231928		-3.999978091004949	
	-3.000000107589290		-3.000017029943044	
	-1.999999970741010		-1.999993225845674	
	-1.000000003037593		-1.000001146454624	
$\Im(\lambda)$	$O(10^{-13})$		$O(10^{-8})$	

Table 2c

	Method 1		Method 2	
F_1^T	$1.3564(4)$	$2.5644(3)$	$-6.5140(5)$	$4.5089(5)$
	$1.3933(4)$	$2.5203(3)$	$-6.3163(5)$	$4.3729(5)$
	$-1.4171(4)$	$-2.5154(3)$	$6.4590(5)$	$-4.4716(5)$
F_2^T	$-5.8738(3)$	$-1.0347(3)$	$-4.6153(3)$	$3.6042(3)$
	$-7.8963(3)$	$-1.3910(3)$	$6.3244(3)$	$-3.9812(3)$
	$7.5902(3)$	$1.3371(3)$	$-6.3401(3)$	$3.9830(3)$
κ	$7.9887(6)$		$1.8077(9)$	
$\kappa(S_0)$	$1.2204(7)$		$5.5354(9)$	
$\Re(\lambda)$	-5.999999991388784		-5.999999615792164	
	-5.000000018032756		-5.000001557116362	
	-3.999999990186220		-3.999997525600821	
	-2.999999997226952		-3.000001935263001	
	-2.000000003539084		-1.999999235603004	
	$-9.999999992460400(-1)$		-1.000000128258480	
$\Im(\lambda)$	$O(10^{-15})$		$O(10^{-9})$	

Our limited numerical experiments on these methods show that the both methods are accurate, though the first method has a slight edge over the second, as far as the condition of the closed-loop eigenvalues are concerned. This behaviour is consistent with out theoretical understanding of Method 2, in terms of the invertibility of X_{11} and $\tilde{Y}$.

Apart from suggesting the two new methods, our paper have also raised many interesting questions or related problems, e.g., in eigenvector selection, optimisation of condition number, etc.. Further studies, theoretical as well as numerical, are obviously required.

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