ORTHOGONALITY AND PARTIAL POLE ASSIGNMENT FOR THE SYMMETRIC DEFINITE QUADRATIC PENCIL

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Abstract. The eigenvectors of a symmetric matrix can be chosen to form an orthogonal set with respect to the identity and to the matrix itself. Similarly, the eigenvectors of a symmetric definite linear pencil can be chosen to be orthogonal with respect to the pair.This paper presents the three sets of matrix weights, with respect to which the eigenvectors of the symmetric definite quadratic pencil are orthogonal. One of these relations is used to derive an explicit solution of the *partial pole assignment problem* by state feedback control for a control system modeled by a system of second order differential equations. The solution may be of particular interest in the stabilization and control of flexible, large, space structures where only a small part of the spectrum is to be reassigned and the rest of the spectrum is required to remain unchanged.

1. Introduction. The second order matrix differential equation

$$\boldsymbol{M}\frac{d^2}{dt^2}\boldsymbol{v} + \boldsymbol{C}\frac{d}{dt}\boldsymbol{v} + \boldsymbol{K}\boldsymbol{v} = \boldsymbol{o}, \tag{1}$$

leads, with the separation of variables $\boldsymbol{v}(t) = \boldsymbol{x}e^{\lambda t}$, $\boldsymbol{x}$ a constant vector, to the problem of finding the eigenvalues and eigenvectors of the quadratic pencil

$$P(\lambda) = \lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K}. \tag{2}$$

The scalar λ_i is called an eigenvalue and the corresponding vector $\boldsymbol{x}_i \neq \mathbf{o}$ is called an eigenvector if they satisfy

$$P(\lambda_i)\boldsymbol{x}_i = \mathbf{o}. \tag{3}$$

Here, we will assume that $\boldsymbol{M}, \boldsymbol{C}$, and $\boldsymbol{K}$ are symmetric and furthermore, $\boldsymbol{M}$ is positive definite. Such pencils are said to be symmetric definite and usually arise in vibration analysis (see [6, 8]).

If $\boldsymbol{C}$ is a zero matrix, the problem reduces to the well known generalized eigenvalue problem, in which P is a linear pencil

$$P(\mu) = \boldsymbol{K} - \mu\boldsymbol{M} \tag{4}$$

by the substitution $\mu = -\lambda^2$. Since $\boldsymbol{M}$ is positive definite, there are n real eigenvalues μ_i, $i = 1, 2, \ldots, n$ and n linearly independent eigenvectors. Furthermore, the eigenvectors can be chosen so that the following *orthogonality* relations hold ([6], p507):

$$\left.\begin{aligned} \boldsymbol{x}_i^T\boldsymbol{K}\boldsymbol{x}_j &= 0 \\ \boldsymbol{x}_i^T\boldsymbol{M}\boldsymbol{x}_j &= 0 \end{aligned}\right\} i \neq j. \tag{5}$$

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The eigenvalues are given by the *Rayleigh quotients*,

$$\frac{\boldsymbol{x}_i^T \boldsymbol{K} \boldsymbol{x}_i}{\boldsymbol{x}_i^T \boldsymbol{M} \boldsymbol{x}_i} = \mu_i. \tag{6}$$

Besides its theoretical importance, the orthogonality (5) of the eigenvectors has practical application in science and engineering (see [6] p512-531, [8]).

For the general quadratic pencil, the relations (5) no longer apply unless (and only if) $\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{C}$ is symmetric, see [4]. In that case we have, in addition to (5), the following relation

$$\boldsymbol{x}_i^T \boldsymbol{C} \boldsymbol{x}_j = 0, \quad i \neq j$$

and $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are *simultaneously diagonalizable*. In Section 2 we present the orthogonality relations which apply in the case of the quadratic pencil for general $\boldsymbol{C}$. This orthogonality plays a role in our solution to an important control problem, which we now describe.

The system modelled by (1) can be controlled with the application of a forcing function $\boldsymbol{b}u(t)$, $\boldsymbol{b} \in \mathcal{R}^n$ a constant, and $u(t)$ a scalar, in which case (1) is replaced by

$$\boldsymbol{M}\frac{d^2}{dt^2}\boldsymbol{v} + \boldsymbol{C}\frac{d}{dt}\boldsymbol{v} + \boldsymbol{K}\boldsymbol{v} = \boldsymbol{b}u(t). \tag{7}$$

The choice of the *control vector*

$$\boldsymbol{u}(t) = \boldsymbol{f}^T \frac{d}{dt}\boldsymbol{v}(t) + \boldsymbol{g}^T \boldsymbol{v}(t), \tag{8}$$

$\boldsymbol{f}, \boldsymbol{g} \in \mathcal{R}^n$, constants, leads to the *closed loop system*

$$\boldsymbol{M}\frac{d^2}{dt^2}\boldsymbol{v} + (\boldsymbol{C} - \boldsymbol{b}\boldsymbol{f}^T)\frac{d}{dt}\boldsymbol{v} + (\boldsymbol{K} - \boldsymbol{b}\boldsymbol{g}^T)\boldsymbol{v} = \boldsymbol{o}, \tag{9}$$

the dynamics of which are characterised by the eigenvalues of the *closed loop pencil*

$$P_c(\lambda) = \boldsymbol{M}\lambda^2 + (\boldsymbol{C} - \boldsymbol{b}\boldsymbol{f}^T)\lambda + (\boldsymbol{K} - \boldsymbol{b}\boldsymbol{g}^T). \tag{10}$$

The *partial pole assignment problem* for the quadratic pencil requires us to find the feedback vectors $\boldsymbol{f}, \boldsymbol{g}$ which are such that the spectrum of the closed loop pencil (10) has certain of its eigenvalues prescribed and the others are in the spectrum of the *open loop pencil* (2). The most important application of this problem is in the relocation of those eigenvalues associated with instability or which lead to large vibration phenomena of structures modeled by the second order differential equation (1).

It is well known (see [9]) that the system (7) is *completely controllable* if and only if

$$\operatorname{rank}\{\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{C} + \boldsymbol{K}, \boldsymbol{b}\} = n,$$

for every eigenvalue λ of the pencil (2). Complete controllability is a necessary and sufficient condition for the existence of $\boldsymbol{f}$ and $\boldsymbol{g}$ such that the closed-loop pencil has a spectrum that can be assigned arbitrarily. However, if the system is only *partially controllable*, i.e., if

$$\operatorname{rank}\{\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{C} + \boldsymbol{K}, \boldsymbol{b}\} = n,$$

only for m of the eigenvalues $\lambda = \lambda_{i_k}$, $k = 1, 2, \ldots, m$, $m < n$, of the pencil, then only those eigenvalues can be arbitrarily assigned by an appropriate choice of $\boldsymbol{f}$ and $\boldsymbol{g}$. Furthermore, partial controllability can be shown to be equivalent to the following: *the vector* $\boldsymbol{b}$ *must be not orthogonal to* $\{\boldsymbol{x}_{i_k}\}_{k=1}^{m}$*, the eigenvectors corresponding to the assignable eigenvalues* $\{\lambda_{i_k}\}_{k=1}^{m}$.

The partial pole assignment problem can be solved by finding $\hat{\boldsymbol{f}}$ such that the $2n \times 2n$ matrix

$$\boldsymbol{A} - \hat{\boldsymbol{b}}\hat{\boldsymbol{f}}^T, \tag{11}$$

where

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{pmatrix}, \; \hat{\boldsymbol{b}} = \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{M}^{-1}\boldsymbol{b} \end{pmatrix}, \; \hat{\boldsymbol{f}} = \begin{pmatrix} -\boldsymbol{g} \\ -\boldsymbol{f} \end{pmatrix}, \tag{12}$$

has the desired spectrum. Existing methods include the projection and deflation technique of [11] and the matrix equation method proposed in [7].

It is clear that the solution of the partial eigenvalue assignment problem through the first order realization above cannot be recommended in practice because the explicit computation of the inverse of $\boldsymbol{M}$ is not advisable, due to possible ill-conditioning of $\boldsymbol{M}$, and furthermore, that approach does not respect properties such as sparsity or positive definiteness, that are sometimes assets for this problem.

The problem can also be solved by using a nonstandard first order realization that does not require the inverse of $\boldsymbol{M}$, but that approach will give rise to a descriptor control system, and the methods for descriptor systems are not yet well developed [5].

The approach used in this paper works directly with the data matrices $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$, of the second order system, rather than the $2n \times 2n$ non-symmetric, *first order realization* (12) of (10). This allows the exploitation of matrix structural properties, such as symmetry, sparsity and bandedness.

An important practical requirement of any solution method for the partial eigenvalue assignment problem is that the method should not suffer from *spill-over*, the phenomenon in which eigenvalues not intended to be changed are modified by the process.

In Section 3 we use one of the orthogonality relations, mentioned above, to derive the explicit solution to the partial pole assignment problem for the quadratic pencil. Our method ensures that spill-over will not occur. The explicit solution is also a powerful aid in the analysis of such systems. We emphasize that our results are applicable in the general case and do not assume that $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are simultaneously diagonalizable.

The method will be most advantageous in practical applications where n is large and it is required to assign only the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$, where m is much smaller than n. This situation is typical in the vibration control and stabilization of flexible, large, space structures [1, 2]. The method uses the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ and their associated eigenvectors. These can be found by computation using a Krylov subspace method such as Lanczos [10], or by modal analysis measurements when the physical structure is available [8].

In Section 4, a shift of origin technique is derived which allows partial pole assignment in the case where some of the eigenvalues vanish. This is common in vibration, and corresponds to the stabilization of an unsupported system which undergoes rigid body modes of motion [8].

Two illustrative examples are given in Section 5 and a summary is given in Section 6.

2. Orthogonality relations for the symmetric definite quadratic pencil. Throughout this paper, the superscript T in a quantity like $\boldsymbol{x}^T$ denotes the transpose, *not conjugate transpose*, of a vector, even though that vector may be complex.

Suppose $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K} \in \mathcal{R}^{n \times n}$ are symmetric and $\boldsymbol{M}$ is positive definite. Let the $n \times 2n$ system of equations

$$\boldsymbol{M}\boldsymbol{X}\boldsymbol{\Lambda}^2 + \boldsymbol{C}\boldsymbol{X}\boldsymbol{\Lambda} + \boldsymbol{K}\boldsymbol{X} = \boldsymbol{O}, \tag{13}$$

where $\boldsymbol{X} \in \mathcal{C}^{n \times 2n}$ and $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_{2n}\} \in \mathcal{C}^{2n \times 2n}$, λ_i distinct, be an eigendecomposition of the quadratic *open loop* pencil

$$P(\lambda) = \boldsymbol{M}\lambda^2 + \boldsymbol{C}\lambda + \boldsymbol{K}. \tag{14}$$

Isolating the term in $\boldsymbol{C}$ of (13), we have

$$-\boldsymbol{C}\boldsymbol{X}\boldsymbol{\Lambda} = \boldsymbol{M}\boldsymbol{X}\boldsymbol{\Lambda}^2 + \boldsymbol{K}\boldsymbol{X}. \tag{15}$$

Multiplying this on the left by $\boldsymbol{\Lambda}\boldsymbol{X}^T$ gives

$$-\boldsymbol{\Lambda}\boldsymbol{X}^T\boldsymbol{C}\boldsymbol{X}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{X}^T\boldsymbol{M}\boldsymbol{X}\boldsymbol{\Lambda}^2 + \boldsymbol{\Lambda}\boldsymbol{X}^T\boldsymbol{K}\boldsymbol{X}. \tag{16}$$

Transposing (15) and multiplying it on the right by $\boldsymbol{X\Lambda}$ gives

$$-\boldsymbol{\Lambda X}^T\boldsymbol{CX\Lambda} = \boldsymbol{\Lambda}^2\boldsymbol{X}^T\boldsymbol{MX\Lambda} + \boldsymbol{X}^T\boldsymbol{KX\Lambda} \tag{17}$$

Now, subtracting (17) from (16) gives, on rearrangement,

$$\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda}^2 - \boldsymbol{X}^T\boldsymbol{KX\Lambda} = \boldsymbol{\Lambda}^2\boldsymbol{X}^T\boldsymbol{MX\Lambda} - \boldsymbol{\Lambda X}^T\boldsymbol{KX} \tag{18}$$

or

$$(\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda} - \boldsymbol{X}^T\boldsymbol{KX})\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda} - \boldsymbol{X}^T\boldsymbol{KX}). \tag{19}$$

Thus, the matrix $\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda} - \boldsymbol{X}^T\boldsymbol{KX}$ which we denote by $\boldsymbol{D}_1$, must be diagonal since it commutes with a diagonal, the elements of which are distinct. We thus have the *first orthogonality relation*

$$\boxed{\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda} - \boldsymbol{X}^T\boldsymbol{KX} = \boldsymbol{D}_1.} \tag{20}$$

Now, isolating the term in $\boldsymbol{M}$ of (13) we get

$$-\boldsymbol{MX\Lambda}^2 = \boldsymbol{CX\Lambda} + \boldsymbol{KX}, \tag{21}$$

and multiplying this on the left by $\boldsymbol{\Lambda}^2\boldsymbol{X}^T$ gives

$$-\boldsymbol{\Lambda}^2\boldsymbol{X}^T\boldsymbol{MX\Lambda}^2 = \boldsymbol{\Lambda}^2\boldsymbol{X}^T\boldsymbol{CX\Lambda} + \boldsymbol{\Lambda}^2\boldsymbol{X}^T\boldsymbol{KX}. \tag{22}$$

Transposing (21) and multiplying it on the right by $\boldsymbol{X\Lambda}^2$ gives

$$-\boldsymbol{\Lambda}^2\boldsymbol{X}^T\boldsymbol{MX\Lambda}^2 = \boldsymbol{\Lambda X}^T\boldsymbol{CX\Lambda}^2 + \boldsymbol{X}^T\boldsymbol{KX\Lambda}^2. \tag{23}$$

Subtracting (23) from (22) and adding $\boldsymbol{\Lambda X}^T\boldsymbol{KX\Lambda}$ to both sides gives, after some rearrangement,

$$\boldsymbol{\Lambda}(\boldsymbol{\Lambda X}^T\boldsymbol{CX\Lambda}+\boldsymbol{\Lambda X}^T\boldsymbol{KX}+\boldsymbol{X}^T\boldsymbol{KX\Lambda}) = (\boldsymbol{\Lambda X}^T\boldsymbol{CX\Lambda}+\boldsymbol{\Lambda X}^T\boldsymbol{KX}+\boldsymbol{X}^T\boldsymbol{KX\Lambda})\boldsymbol{\Lambda}. \tag{24}$$

Again, this commutivity property implies, if $\boldsymbol{\Lambda}$ has distinct diagonal elements, that

$$\boxed{\boldsymbol{\Lambda X}^T\boldsymbol{CX\Lambda} + \boldsymbol{\Lambda X}^T\boldsymbol{KX} + \boldsymbol{X}^T\boldsymbol{KX\Lambda} = \boldsymbol{D}_2} \tag{25}$$

is a diagonal matrix. This is the *second orthogonality relation.*

Relations (20) and (25) together easily imply the *third orthogonality relation*

$$\boxed{\boldsymbol{\Lambda X}^T\boldsymbol{MX} + \boldsymbol{X}^T\boldsymbol{MX\Lambda} + \boldsymbol{X}^T\boldsymbol{CX} = \boldsymbol{D}_3,} \tag{26}$$

where $\boldsymbol{D}_3$ is also diagonal.

Now, multiplying (26) on the right by $\boldsymbol{\Lambda}$ gives

$$\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda} + \boldsymbol{X}^T\boldsymbol{MX\Lambda}^2 + \boldsymbol{X}^T\boldsymbol{CX\Lambda} = \boldsymbol{D}_3\boldsymbol{\Lambda},$$

which, using (13), gives

$$\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda} + \boldsymbol{X}^T(-\boldsymbol{KX}) = \boldsymbol{D}_3\boldsymbol{\Lambda}.$$

So, from (20) we see that

$$\boxed{\boldsymbol{D}_1 = \boldsymbol{D}_3\boldsymbol{\Lambda}.} \tag{27}$$

Next, using (13), we may write (25) as

$$\begin{aligned} \boldsymbol{D}_2 &= \boldsymbol{\Lambda X}^T(\boldsymbol{CX\Lambda} + \boldsymbol{KX}) + \boldsymbol{X}^T\boldsymbol{KX\Lambda} \\ &= \boldsymbol{\Lambda X}^T(-\boldsymbol{MX\Lambda}^2) + \boldsymbol{X}^T\boldsymbol{KX\Lambda} \\ &= (-\boldsymbol{\Lambda X}^T\boldsymbol{MX\Lambda}^{+}\boldsymbol{X}^T\boldsymbol{KX})\boldsymbol{\Lambda} \end{aligned}$$

Thus, by (20) we have

$$\boxed{\boldsymbol{D}_2 = -\boldsymbol{D}_1\boldsymbol{\Lambda},} \tag{28}$$

and this immediately leads, by (27), to

$$\boxed{\boldsymbol{D}_2 = -\boldsymbol{D}_3\boldsymbol{\Lambda}^2.} \tag{29}$$

We remind the reader that matrix and vector transposition here does not mean conjugation for complex quantities.

It is instructive to view these orthogonality relations componentwise. Thus, equating the off-diagonal elements of the matrices on both sides of relations (20), (25) and (26), we obtain:

$$\left.\begin{aligned} \boldsymbol{x}_i^T(\lambda_i\lambda_j\boldsymbol{M} - \boldsymbol{K})\boldsymbol{x}_j = 0 \\ \boldsymbol{x}_i^T(\lambda_i\lambda_j\boldsymbol{C} + (\lambda_i + \lambda_j)\boldsymbol{K})\boldsymbol{x}_j = 0 \\ \boldsymbol{x}_i^T((\lambda_i + \lambda_j)\boldsymbol{M} + \boldsymbol{C})\boldsymbol{x}_j = 0 \end{aligned}\right\}, \quad i \neq j.$$

Provided the denominators do not vanish we may rewrite these as

$$\left.\begin{aligned} \lambda_i\lambda_j &= \frac{\boldsymbol{x}_i^T\boldsymbol{K}\boldsymbol{x}_j}{\boldsymbol{x}_i^T\boldsymbol{M}\boldsymbol{x}_j} \\ -\frac{\lambda_i+\lambda_j}{\lambda_i\lambda_j} &= \frac{\boldsymbol{x}_i^T\boldsymbol{C}\boldsymbol{x}_j}{\boldsymbol{x}_i^T\boldsymbol{K}\boldsymbol{x}_j} \\ -(\lambda_i+\lambda_j) &= \frac{\boldsymbol{x}_i^T\boldsymbol{C}\boldsymbol{x}_j}{\boldsymbol{x}_i^T\boldsymbol{M}\boldsymbol{x}_j} \end{aligned}\right\}, \quad i \neq j.$$

Similarly, from (27), (28) and (29), and assuming again that the denominators do not vanish, we have the Rayleigh-Quotient like expressions for the quadratic pencil $P(\lambda)$:

$$\left.\begin{aligned} \lambda_i &= \frac{\boldsymbol{x}_i^T(\lambda_i^2\boldsymbol{M}-\boldsymbol{K})\boldsymbol{x}_i}{\boldsymbol{x}_i^T(2\lambda_i\boldsymbol{M}+\boldsymbol{C})\boldsymbol{x}_i} \\ -\lambda_i &= \frac{\boldsymbol{x}_i^T(\lambda_i^2\boldsymbol{C}+2\lambda_i\boldsymbol{K})\boldsymbol{x}_i}{\boldsymbol{x}_i^T(\lambda_i^2\boldsymbol{M}-\boldsymbol{K})\boldsymbol{x}_i} \\ -\lambda_i^2 &= \frac{\boldsymbol{x}_i^T(\lambda_i^2\boldsymbol{C}+2\lambda_i\boldsymbol{K})\boldsymbol{x}_i}{\boldsymbol{x}_i^T(2\lambda_i\boldsymbol{M}+\boldsymbol{C})\boldsymbol{x}_i} \end{aligned}\right\}, \quad i = 1, 2, \ldots, 2n. \tag{30}$$

Note that when $\boldsymbol{C} = \boldsymbol{O}$, the last relation in (30) simplifies to the Rayleigh quotient (6).

3. Partial pole assignment. Using one of the orthogonality relations just proved, we now present a solution to the partial eigenvalue assignment problem for the pencil $P(\lambda) = \lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K}$. For convenience, we restate the problem.

Given m complex numbers $\mu_1, \mu_2, \cdots, \mu_m$, $m \leq n$, and a vector $\boldsymbol{b} \in \mathcal{R}^n$, we are required to find $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{C}^n$ which are such that the *closed loop* pencil

$$P_c(\lambda) = \boldsymbol{M}\lambda^2 + (\boldsymbol{C} - \boldsymbol{b}\boldsymbol{f}^T)\lambda + (\boldsymbol{K} - \boldsymbol{b}\boldsymbol{g}^T), \tag{31}$$

has spectrum

$$\{\mu_1, \mu_2, \ldots, \mu_m, \lambda_{m+1}, \ldots, \lambda_{2n}\}. \tag{32}$$

This is the *partial pole assignment problem* in which we use the vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ to replace the eigenvalues $\{\lambda_j\}_{j=1}^m$ of the pencil $P(\lambda)$ by $\{\mu_j\}_{j=1}^m$, while leaving the other eigenvalues unchanged. To this end we first prove Theorem 3.1.

Let us partition the $n \times 2n$ eigenvector matrix and $2n \times 2n$ eigenvalue matrix as follows:

$$\boldsymbol{X} = \begin{array}{cc} (\boldsymbol{X}_1 & \boldsymbol{X}_2) \\ {\scriptstyle m} & {\scriptstyle 2n-m} \end{array}, \qquad \boldsymbol{\Lambda} = \begin{array}{cc|c} \multicolumn{2}{c}{\begin{pmatrix} \boldsymbol{\Lambda}_1 & \\ & \boldsymbol{\Lambda}_2 \end{pmatrix}} & \begin{array}{c} {\scriptstyle m} \\ {\scriptstyle 2n-m} \end{array} \\ {\scriptstyle m} & {\scriptstyle 2n-m} & \end{array}.$$

THEOREM 3.1. *Let*

$$\boldsymbol{f} = \boldsymbol{M}\boldsymbol{X}_1\boldsymbol{\Lambda}_1\boldsymbol{\beta}, \quad \boldsymbol{g} = -\boldsymbol{K}\boldsymbol{X}_1\boldsymbol{\beta}, \quad \boldsymbol{\beta} \in \mathcal{C}^m. \tag{33}$$

Then, for any choice of $\boldsymbol{\beta}$ *we have*

$$\boldsymbol{M}\boldsymbol{X}_2\boldsymbol{\Lambda}_2^2 + (\boldsymbol{C} - \boldsymbol{b}\boldsymbol{f}^T)\boldsymbol{X}_2\boldsymbol{\Lambda}_2 + (\boldsymbol{K} - \boldsymbol{b}\boldsymbol{g}^T)\boldsymbol{X}_2 = \boldsymbol{O}. \tag{34}$$

In words, this theorem assures us that any choice of $\boldsymbol{\beta}$ with $\boldsymbol{f}, \boldsymbol{g}$ as in (33) guarantees that the last $2n - m$ eigenpairs $(\boldsymbol{\Lambda}_2, \boldsymbol{X}_2)$ of (13) are also eigenpairs of the closed loop pencil (31).

Proof. Expanding the left hand side of (34) gives

$$\begin{aligned} &\boldsymbol{M}\boldsymbol{X}_2\boldsymbol{\Lambda}_2^2 + \boldsymbol{C}\boldsymbol{X}_2\boldsymbol{\Lambda}_2 + \boldsymbol{K}\boldsymbol{X}_2 + \boldsymbol{b}(-\boldsymbol{\beta}^T\boldsymbol{\Lambda}_1\boldsymbol{X}_1^T\boldsymbol{M}\boldsymbol{X}_2\boldsymbol{\Lambda}_2 + \boldsymbol{\beta}^T\boldsymbol{X}_1^T\boldsymbol{K}\boldsymbol{X}_2) \\ &\quad = -\boldsymbol{b}\boldsymbol{\beta}^T(\boldsymbol{\Lambda}_1\boldsymbol{X}_1^T\boldsymbol{M}\boldsymbol{X}_2\boldsymbol{\Lambda}_2 - \boldsymbol{X}_1^T\boldsymbol{K}\boldsymbol{X}_2) \end{aligned}$$

because $\boldsymbol{\Lambda}_2, \boldsymbol{X}_2$ are eigenmatrix pairs of (13). Furthermore,

$$\boldsymbol{\Lambda}_1\boldsymbol{X}_1^T\boldsymbol{M}\boldsymbol{X}_2\boldsymbol{\Lambda}_2 - \boldsymbol{X}_1^T\boldsymbol{K}\boldsymbol{X}_2 = \boldsymbol{O}$$

because the left hand side is the $m \times 2n - m$ top right (and therefore zero) block of a diagonal matrix by virtue of the orthogonality relation (20). ■

In order to use Theorem 3.1 to solve the partial pole assignment problem, we need to choose $\boldsymbol{\beta}$ which will move $\{\lambda_j\}_{j=1}^m$ of the pencil $P(\lambda)$ to $\{\mu_j\}_{j=1}^m$ in $P_c(\lambda)$, if that is possible. If there is such a vector $\boldsymbol{\beta}$, then there exist an eigenvector matrix $\boldsymbol{Y} \in \mathcal{C}^{n\times m}$,

$$\boldsymbol{Y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m), \quad \boldsymbol{y}_j \neq 0, \quad j = 1, 2, \ldots, m,$$

and $\boldsymbol{D} = \operatorname{diag}\{\mu_1, \mu_2, \cdots, \mu_m\}$ which are such that

$$\boldsymbol{M}\boldsymbol{Y}\boldsymbol{D}^2 + (\boldsymbol{C} - \boldsymbol{b}\boldsymbol{f}^T)\boldsymbol{Y}\boldsymbol{D} + (\boldsymbol{K} - \boldsymbol{b}\boldsymbol{g}^T)\boldsymbol{Y} = \boldsymbol{O}. \tag{35}$$

Substituting for $\boldsymbol{f}, \boldsymbol{g}$ and rearranging, we have

$$\begin{aligned} \boldsymbol{M}\boldsymbol{Y}\boldsymbol{D}^2 + \boldsymbol{C}\boldsymbol{Y}\boldsymbol{D} + \boldsymbol{K}\boldsymbol{Y} &= \boldsymbol{b}\boldsymbol{\beta}^T(\boldsymbol{\Lambda}_1\boldsymbol{X}_1^T\boldsymbol{M}\boldsymbol{Y}\boldsymbol{D} - \boldsymbol{X}_1^T\boldsymbol{K}\boldsymbol{Y}) \\ &= \boldsymbol{b}\boldsymbol{\beta}^T\boldsymbol{Z}_1^T \\ &= \boldsymbol{b}\boldsymbol{c}^T, \end{aligned}$$

where $\boldsymbol{Z}_1 = \boldsymbol{D}\boldsymbol{Y}^T\boldsymbol{M}\boldsymbol{X}_1\boldsymbol{\Lambda}_1 - \boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{X}_1$ and

$$\boldsymbol{c} = \boldsymbol{Z}_1\boldsymbol{\beta} \tag{36}$$

is a vector that will depend on the scaling chosen for the eigenvectors in $\boldsymbol{Y}$. To obtain $\boldsymbol{Y}$, we can solve for each of the eigenvectors $\boldsymbol{y}_i$ using the equations

$$(\mu_j^2\boldsymbol{M} + \mu_j\boldsymbol{C} + \boldsymbol{K})\boldsymbol{y}_j = \boldsymbol{b}, \quad j = 1, 2, \ldots, m. \tag{37}$$

This corresponds to choosing the vector $\boldsymbol{c} = (1, 1, \ldots, 1)^T$, so, having computed the eigenvectors we could solve the m square system

$$\boldsymbol{Z}_1 \boldsymbol{\beta} = (1, 1, \ldots, 1)^T$$

for $\boldsymbol{\beta}$, and hence determine the vectors $\boldsymbol{f}, \boldsymbol{g}$. However, there exists an explicit solution for this problem which we now give.

Theorem 3.2. *Suppose the open loop quadratic pencil (14) has eigendecomposition (13) and $\boldsymbol{f}, \boldsymbol{g}$ are chosen as in (33) with the components β_j of $\boldsymbol{\beta}$ chosen as*

$$\beta_j = \frac{1}{\boldsymbol{b}^T \boldsymbol{x}_j} \frac{\mu_j - \lambda_j}{\lambda_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}, \quad j = 1, 2, \ldots, m. \tag{38}$$

Then, the closed loop pencil (31) has spectrum (32) and its first m eigenvectors can be scaled to satisfy (37).

Proof. In view of Theorem 3.1, we need only show that

$$\Phi_k(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \left[\mu_k^2 \boldsymbol{M} + \mu_k (\boldsymbol{C} - \boldsymbol{b}\boldsymbol{f}^T) + (\boldsymbol{K} - \boldsymbol{b}\boldsymbol{g}^T)\right] \boldsymbol{y}_k = \mathbf{o}, \quad k = 1, 2, \ldots, m, \tag{39}$$

where

$$(\mu_k^2 \boldsymbol{M} + \mu_k \boldsymbol{C} + \boldsymbol{K}) \boldsymbol{y}_k = \boldsymbol{b}, \tag{40}$$

for the choice of $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{\beta}$ indicated. Now, $\Phi_k(\boldsymbol{\beta})$ with $\boldsymbol{f}, \boldsymbol{g}$ replaced by the expressions in (33), gives

$$\begin{aligned} \Phi_k(\boldsymbol{\beta}) &= \left[\mu_k^2 \boldsymbol{M} + \mu_k (\boldsymbol{C} - \boldsymbol{b} \sum_{j=1}^{m} \beta_j \lambda_j \boldsymbol{x}_j^T \boldsymbol{M}) + (\boldsymbol{K} + \boldsymbol{b} \sum_{j=1}^{m} \beta_j \boldsymbol{x}_j^T \boldsymbol{K})\right] \boldsymbol{y}_k, \\ &= \left[(\mu_k^2 \boldsymbol{M} + \mu_k \boldsymbol{C} + \boldsymbol{K}) - \boldsymbol{b} \left(\mu_k \sum_{j=1}^{m} \beta_j \lambda_j \boldsymbol{x}_j^T \boldsymbol{M} - \sum_{j=1}^{m} \beta_j \boldsymbol{x}_j^T \boldsymbol{K}\right)\right] \boldsymbol{y}_k, \\ &= \boldsymbol{b} - \left[\boldsymbol{b} \sum_{j=1}^{m} \beta_j \boldsymbol{x}_j^T (\mu_k \lambda_j \boldsymbol{M} - \boldsymbol{K})\right] \boldsymbol{y}_k, \end{aligned}$$

by (40). Now, substituting for β_j using (38) gives

$$\begin{aligned} \Phi_k(\boldsymbol{\beta}) &= \boldsymbol{b} - \left[\boldsymbol{b} \sum_{j=1}^{m} \frac{1}{\boldsymbol{b}^T \boldsymbol{x}_j} \frac{\mu_j - \lambda_j}{\lambda_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j} \boldsymbol{x}_j^T (\mu_k \lambda_j \boldsymbol{M} - \boldsymbol{K})\right] \boldsymbol{y}_k, \\ &= \boldsymbol{b} - \left[\boldsymbol{b} \sum_{j=1}^{m} \frac{1}{\boldsymbol{b}^T \boldsymbol{x}_j} \frac{\mu_j - \lambda_j}{\lambda_k - \lambda_j} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j} \boldsymbol{x}_j^T \left(\mu_k (\mu_k - \lambda_j) \boldsymbol{M} - \frac{\mu_k - \lambda_j}{\lambda_j} \boldsymbol{K}\right)\right] \boldsymbol{y}_k. \end{aligned}$$

The j-th column of (13) can be rewritten as

$$\boldsymbol{C}\boldsymbol{x}_j = -(\lambda_j \boldsymbol{M} + \boldsymbol{K}/\lambda_j)\boldsymbol{x}_j, \quad \lambda_j \neq 0.$$

Hence, for any choice of $1 \leq k, j \leq m$,

$$\begin{aligned}(\mu_k^2 \boldsymbol{M} + \mu_k \boldsymbol{C} + \boldsymbol{K})\boldsymbol{x}_j &= (\mu_k^2 \boldsymbol{M} - \mu_k(\lambda_j \boldsymbol{M} + \boldsymbol{K}/\lambda_j) + \boldsymbol{K})\boldsymbol{x}_j \\ &= (\mu_k(\mu_k - \lambda_j)\boldsymbol{M} - \frac{1}{\lambda_j}(\mu_k - \lambda_j)\boldsymbol{K})\boldsymbol{x}_j.\end{aligned} \tag{41}$$

Substituting (41) into the last expression for $\Phi_k(\boldsymbol{\beta})$ gives

$$\begin{aligned}\Phi_k(\boldsymbol{\beta}) &= \boldsymbol{b} - \left[\boldsymbol{b} \sum_{j=1}^{m} \frac{1}{\boldsymbol{b}^T \boldsymbol{x}_j} \frac{\mu_j - \lambda_j}{\lambda_k - \lambda_j} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j} \boldsymbol{x}_j^T (\mu_k^2 \boldsymbol{M} + \mu_k \boldsymbol{C} + \boldsymbol{K})\right] \boldsymbol{y}_k, \\ &= \boldsymbol{b} - \boldsymbol{b} \sum_{j=1}^{m} \frac{1}{\boldsymbol{b}^T \boldsymbol{x}_j} \frac{\mu_j - \lambda_j}{\lambda_k - \lambda_j} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j} \boldsymbol{x}_j^T \boldsymbol{b},\end{aligned}$$

using (40) again. Canceling the common term, we get

$$\begin{aligned}\Phi_k(\boldsymbol{\beta}) &= \boldsymbol{b} - \boldsymbol{b} \sum_{j=1}^{m} \frac{\mu_j - \lambda_j}{\lambda_k - \lambda_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}, \\ &= \boldsymbol{b} \left(1 - \sum_{j=1}^{m} \frac{\prod_{\substack{i=1 \\ i \neq k}}^{m} \mu_i - \lambda_j}{\prod_{\substack{i=1 \\ i \neq j}}^{m} \lambda_i - \lambda_j}\right).\end{aligned}$$

Now, it so happens that the terms under the summation sign in this last expression are the formulae given in [3] for the squares of the elements of a vector normalized to have length unity! In [3] it is assumed that the λ_i and the μ_i satisfy an interlacing property. We now show that

$$\sum_{j=1}^{m} \frac{\prod_{\substack{i=1 \\ i \neq k}}^{m} \mu_i - \lambda_j}{\prod_{\substack{i=1 \\ i \neq j}}^{m} \lambda_i - \lambda_j} = 1, \quad k = 1, 2, \ldots, m, \tag{42}$$

for any sets of $\{\lambda_i\}_{i=1}^m$ and $\{\mu_i\}_{i=1}^m$, in which the λ_i are distinct, and thus establish that $\Phi_k(\boldsymbol{\beta})$ vanishes, as required.

Define the monic polynomial

$$p_{m-1}(t) = \prod_{\substack{i=1 \\ i \neq k}}^{m} (t - \mu_i), \quad 1 \leq k \leq m. \tag{43}$$

The Lagrange polynomial which interpolates p_{m-1} at the m distinct points $t=\lambda_i$, $i=1,2,\ldots,m$, recovers p_{m-1} itself, so we may write

$$
\begin{aligned}
p_{m-1}(t) &= \sum_{j=1}^{m} p_{m-1}(\lambda_j)\frac{\prod_{\substack{i=1\\ i\neq j}}^{m}(t-\lambda_i)}{\prod_{\substack{i=1\\ i\neq j}}^{m}(\lambda_j-\lambda_i)}\\
&= \sum_{j=1}^{m} a_j \prod_{\substack{i=1\\ i\neq j}}^{m}(t-\lambda_i) \qquad (44)
\end{aligned}
$$

where

$$
a_j = p_{m-1}(\lambda_j)/\prod_{\substack{i=1\\ i\neq j}}^{m}(\lambda_j-\lambda_i).
$$

Moreover,

$$
\sum_{j=1}^{m} a_j = 1 \qquad (45)
$$

because $p_{m-1}(t)$ is monic. Equating the two forms (43) and (44) of $p_{m-1}(t)$ at $t=\lambda_k$, $1\le k\le m$, gives

$$
a_j = \frac{\prod_{\substack{i=1\\ i\neq k}}^{m}(\lambda_j-\mu_i)}{\prod_{\substack{i=1\\ i\neq j}}^{m}(\lambda_j-\lambda_i)},
$$

from which (42) follows by (45). This completes the proof. ■

From the expression (38) it is clear that sufficient conditions for the existence of $\boldsymbol{\beta}$, and consequently for a solution to the partial pole assignment problem to exist, are that

(a) no λ_j, $j=1,2,\ldots,m$ vanishes,
(b) the $\{\lambda_j\}_{j=1}^m$ are distinct, and
(c) $\boldsymbol{b}$ must be not orthogonal to $\boldsymbol{x}_j$, $j=1,2,\ldots,m$.

It is worth emphasizing that these restrictions apply only to the m eigenvalues which will be replaced, and their associated eigenvectors. In the next section we show how restriction (a) can always be overcome by using an appropriate shift of origin. Furthermore, as mentioned in Section 1, condition (c) is equivalent to the condition of partial controllability of system (7)

Clearly, if all the $\{\lambda_j\}_{j=1}^m$ are real, then $\boldsymbol{X}_1$ is real as well. If, in addition, all the $\{\mu_j\}_{j=1}^m$ are real, then $\boldsymbol{\beta}$ and so $\boldsymbol{f},\boldsymbol{g}$ are also real.

More generally, if the set of eigenvalues which are to be replaced is self conjugate then the set of associated eigenvectors is also self conjugate. From (38), it can then be seen that the set $\{\beta_j\}_{j=1}^m$ is self conjugate and thus, it follows from (33) that $\boldsymbol{f}$ and $\boldsymbol{g}$ are real. An important consequence of this is that the whole calculation can, for such a case, be done in real arithmetic. More important from the control point of view is the fact that real $\boldsymbol{f}$ and $\boldsymbol{g}$ specify a solution which can be physically realized.

4. State control with shift of origin. As mentioned above, we do not need to assume that the $\{\lambda_j\}_{j=1}^m$ do not vanish to use the assignment method of the previous section. In this section we show how to shift the origin of the original pencil (so that no shifted λ_j vanishes), do a shifted assignment and then compute the controls which should be applied to $P(\lambda)$ to achieve the required assignment. This effectively removes the restriction (a) of the previous section.

LEMMA 4.1. *Let the pencil*

$$Q(\lambda) = \lambda^2 \boldsymbol{U} + \lambda \boldsymbol{V} + \boldsymbol{W}, \quad \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W} \in \mathcal{C}^{n\times n} \tag{46}$$

$\boldsymbol{U}$ *invertible, have spectrum* $\{\lambda_j\}_{j=1}^{2n}$. *Then, the pencil*

$$\hat{Q}(\lambda) = \lambda^2 \boldsymbol{U} + \lambda \hat{\boldsymbol{V}} + \hat{\boldsymbol{W}}$$

with

$$\hat{\boldsymbol{V}} = \boldsymbol{V} + 2p\boldsymbol{U}, \quad \hat{\boldsymbol{W}} = \boldsymbol{W} + p\boldsymbol{V} + p^2\boldsymbol{U} \tag{47}$$

has spectrum $\{\lambda_j - p\}_{j=1}^{2n}$.

If, in addition, $Q(\lambda)$ *is symmetric definite, then* $\hat{Q}(\lambda)$ *is also symmetric definite.*

Proof. Define

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ -\boldsymbol{U}^{-1}\boldsymbol{W} & -\boldsymbol{U}^{-1}\boldsymbol{V} \end{pmatrix}.$$

The proof follows from the fact that

$$\boldsymbol{L}^{-1}(\boldsymbol{A} - p\boldsymbol{I})\boldsymbol{L} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ -p^2\boldsymbol{I} - p\boldsymbol{U}^{-1}\boldsymbol{V} - \boldsymbol{U}^{-1}\boldsymbol{W} & -2p\boldsymbol{I} - \boldsymbol{U}^{-1}\boldsymbol{V} \end{pmatrix}$$

with

$$\boldsymbol{L} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ p\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}$$

is in block companion form. Note that the inverse of $\boldsymbol{L}$ has the same form as $\boldsymbol{L}$ with p replaced by $-p$.

If, in addition, $Q(\lambda)$ is symmetric, then $\hat{Q}(\lambda)$ is symmetric because relations (47) are (scalar) linear combinations of symmetric matrices.

Finally, if $Q(\lambda)$ is symmetric and definite, then $\hat{Q}(\lambda)$ is symmetric and definite because $\boldsymbol{U}$ is unchanged by the shift. ■

Thus, if necessary, we can shift the origin of $P(\lambda)$ by some scalar p, and then perform the assignment of the required, shifted, eigenvalues on the (symmetric) pencil $\hat{P}$. Denote the feedback vectors for the shifted closed loop pencil by $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{g}}$ and denote the closed loop shifted pencil by

$$\hat{P}_c(\lambda) = \lambda^2 \boldsymbol{M} + \lambda \tilde{\boldsymbol{C}} + \tilde{\boldsymbol{K}}. \tag{48}$$

Thus, $\tilde{\boldsymbol{C}} = \hat{\boldsymbol{C}} - \boldsymbol{b}\hat{\boldsymbol{f}}^T$ and $\tilde{\boldsymbol{K}} = \hat{\boldsymbol{K}} - \boldsymbol{b}\hat{\boldsymbol{g}}^T$. To restore the shift to this pencil we must add p to each eigenvalue. Doing this transforms the (no longer symmetric) pencil (48), according to (47), to

$$\begin{aligned}
\hat{P}_c(\lambda) &= \lambda^2 \boldsymbol{M} + \lambda(\tilde{\boldsymbol{C}} + 2p\boldsymbol{M}) + (\tilde{\boldsymbol{K}} + p\tilde{\boldsymbol{C}} + p^2 \boldsymbol{M}) \\
&= \lambda^2 \boldsymbol{M} + \lambda(\hat{\boldsymbol{C}} + 2p\boldsymbol{M} - \boldsymbol{b}\hat{\boldsymbol{f}}^T) \\
&\quad\; + (\boldsymbol{K} + p\boldsymbol{C} + p^2\boldsymbol{M} - \boldsymbol{b}\hat{\boldsymbol{g}}^T + p(\hat{\boldsymbol{C}} - \boldsymbol{b}\hat{\boldsymbol{f}}^T) + p^2\boldsymbol{M}) \\
&= \lambda^2 \boldsymbol{M} + \lambda(\boldsymbol{C} - \boldsymbol{b}\hat{\boldsymbol{f}}^T) + \boldsymbol{K} - \boldsymbol{b}(\hat{\boldsymbol{g}}^T + p\hat{\boldsymbol{f}}^T).
\end{aligned}$$

Thus, we see that the feeedback vectors to be applied to the original pencil $P(\lambda)$ are

$$\boldsymbol{f} = \hat{\boldsymbol{f}}, \quad \boldsymbol{g} = \hat{\boldsymbol{g}} + p\hat{\boldsymbol{f}}.$$

5. Examples. In this section we present two illustrative examples to show the operation of the method on the finite difference model of an axially vibrating, non-conservative rod. The results of modest numerical tests (by no means intended to be exhaustive or even extensive) are reported to show that the method can usefully be applied to cases such as this. All calculations were performed in IEEE standard double precision arithmetic (machine $\epsilon \approx 2 \times 10^{-16}$).

Define the shift matrix $\boldsymbol{S} = [\delta_{i+1,j}]$, δ_{ij} the Kroenecker delta, and let $\boldsymbol{F} = \boldsymbol{I} - \boldsymbol{S}$. The matrices $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are all tridiagonal and are defined, for this model, by $\boldsymbol{M} = 2(\boldsymbol{I} + \boldsymbol{S}\boldsymbol{S}^T) + \boldsymbol{S} + \boldsymbol{S}^T$, $\boldsymbol{C} = \boldsymbol{F}\boldsymbol{\Gamma}\boldsymbol{F}^T$, $\boldsymbol{\Gamma} = \mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, $\gamma_i = \frac{1}{100}\sin 2\pi i/n$, and $\boldsymbol{K} = 1000\boldsymbol{F}\boldsymbol{F}^T$. Thus, for the case $n = 4$ these matrices are

$$\boldsymbol{M} = \begin{pmatrix} 4 & 1 & & \\ 1 & 4 & 1 & \\ & 1 & 4 & 1 \\ & & 1 & 2 \end{pmatrix},$$

$$
\boldsymbol{C} = \frac{1}{100}\begin{pmatrix} 1.0898 & -0.7071 & & \\ -0.7071 & 1.6310 & -0.9239 & \\ & -0.9239 & 1.9239 & -1.0000 \\ & & -1.0000 & 1.0000 \end{pmatrix}
$$

$$
\boldsymbol{K} = 1000\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix}.
$$

The pencil

$$
P(\lambda) = \boldsymbol{M}\lambda^2 + \boldsymbol{C}\lambda + \boldsymbol{K},
$$

for this choice of $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ has eigenvalues, ordered according to their real parts, shown in Table 1. We now compute the *feedback*

i	λ_i
1	$-7.6813(-05) - 5.1024(+00)i$
2	$-7.6813(-05) + 5.1024(+00)i$
3	$-9.6435(-04) - 1.6096(+01)i$
4	$-9.6435(-04) + 1.6096(+01)i$
5	$-3.2590(-03) - 2.9239(+01)i$
6	$-3.2590(-03) + 2.9239(+01)i$
7	$-8.1947(-03) - 4.2282(+01)i$
8	$-8.1947(-03) + 4.2282(+01)i$

TABLE 1
Eigenvalues of the open loop pencil for $n = 4$.

vectors $\boldsymbol{f}, \boldsymbol{g}$ for the case in which $\boldsymbol{b} = (1, 1, 1, 1)^T$ and we assign the first $m = 2$ eigenvalues λ_1, λ_2 to the conjugate pair $\mu_{1,2} = -(1 \pm i)$. Using the explicit formula (38) gives

$$
\boldsymbol{\beta} = \begin{pmatrix} -1.2888 + 0.4586i \\ -1.2888 - 0.4586i \end{pmatrix},
$$

from which we get, in view of (33),

$$
\boldsymbol{f} = \begin{pmatrix} -1.4850 \\ -2.7439 \\ -3.5852 \\ -1.9403 \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 17.8477 \\ 32.9769 \\ 43.0857 \\ 23.3177 \end{pmatrix}.
$$

As expected, $\boldsymbol{f}, \boldsymbol{g}$ are real because we have replaced one conjugate pair in the spectrum by another. The relative error of the assigned eigenvalues nowhere exceeds 10^{-14} and the relative error of the fixed eigenvalues nowhere exceeds 10^{-13}, in this example.

For $n = 100$ the absolute values of the real parts of the eigenvalues of the pencil range between about 10^{-8} and 10^{-4}, and the absolute values of their imaginary parts range between about 0.2 and 45. With $\boldsymbol{b} = (1, 1, \ldots, 1)^T$ we took $\mu_{1,2} = -(1 \pm i)$ and $\mu_{3,4} = -(2 \pm i)$. The first 10 eigenvalues of the open and closed loop pencils are shown in Table 2. The relative errors of the closed loop eigenvalues which were required to remain unchanged by the assignment process were nowhere greater than 3×10^{-12} and the assigned eigenvalues had relative errors smaller than 2×10^{-11}. We stress again that, the examples here are intended only to demonstrate the method and not to comprehensively test it. In this context we consider these results to be very satisfactory, especially for the engineering application used.

Open loop eigenvalues	Closed loop eigenvalues
$-8.8639(-08) - 2.0279(-01)i$	$-2.0000(+00) - 1.0000(+00)i$
$-8.8639(-08) + 2.0279(-01)i$	$-2.0000(+00) + 1.0000(+00)i$
$-1.1542(-06) - 6.0842(-01)i$	$-1.0000(+00) - 1.0000(+00)i$
$-1.1542(-06) + 6.0842(-01)i$	$-1.0000(+00) + 1.0000(+00)i$
$-3.2670(-06) - 1.0142(+00)i$	$-3.2670(-06) - 1.0142(+00)i$
$-3.2670(-06) + 1.0142(+00)i$	$-3.2670(-06) + 1.0142(+00)i$
$-6.4382(-06) - 1.4202(+00)i$	$-6.4382(-06) + 1.4202(+00)i$
$-6.4382(-06) + 1.4202(+00)i$	$-6.4382(-06) - 1.4202(+00)i$
$-1.0671(-05) - 1.8266(+00)i$	$-1.0671(-05) - 1.8266(+00)i$
$-1.0671(-05) + 1.8266(+00)i$	$-1.0671(-05) + 1.8266(+00)i$
$-1.5970(-05) - 2.2335(+00)i$	$-1.5970(-05) + 2.2335(+00)i$
$-1.5970(-05) + 2.2335(+00)i$	$-1.5970(-05) - 2.2335(+00)i$

TABLE 2
First 10 open and closed loop eigenvalues for $n = 100$.

6. Summary. The well known orthogonality relations for the eigenvectors of a symmetric matrix or a symmetric definite pair are generalized to the triplet that defines a symmetric definite quadratic pencil. One of the three orthogonality relations for the quadratic pencil is then used to derive an explicit solution to the partial pole assignment problem for a second order system. The explicit solution is a powerful tool in the analysis of eigenvalue assignment problems and it lends itself naturally to the solution of the problem of stabilization and control of flexible, large, space structures where only a small part of the spectrum is to be assigned and the rest of the spectrum is required to remain unchanged. The orthogonality relations may also be useful in investigating eigenvalue sensitivity. We leave these as open problems.

An advantage of the proposed method is that it allows us to work directly with the data matrices $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ of the second order pencil, thus allowing the exploitation of structural properties such as symmetry, sparsity, and bandedness which occur frequently in practical applications.

We have also shown how to modify the matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ in the pencil $Q(\lambda) = \lambda^2\boldsymbol{U} + \lambda\boldsymbol{V} + \boldsymbol{W}$ to shift all $2n$ eigenvalues by a constant p. This overcomes the restriction on the method that every eigenvalue to be reassigned must be different from zero. In vibration, vanishing eigenvalues correspond to an unsupported system which undergoes rigid body modes of motion.

Some modest examples have been provided to illustrate the results.

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