Feedback Control in Distributed Parameter Gyroscopic Systems: A Solution of the Partial Eigenvalue Assignment Problem

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Abstract

This paper presents a novel solution to the partial eigenvalue assignment problem of a an undamped gyroscopic distributed parameters system. The partial eigenvalue assignment problem is the problem of reassigning by feedback a few bad eigenvalues of the open-loop operator pencil while leaving the remaining infinite number of eigenvalues unchanged.

The distinctive practical features of our solution are (i) it requires solution of only a small finite dimensional linear algebraic system and knowledge of only a small finite number of eigenvalues and eigenvectors of the infinite dimensional open-loop operator pencil, (ii) no spill-over occurs; that is, the remaining infinite number of eigenvalues and eigenvectors that are required to remain invariant will remain in their places and, (iii) it is obtained completely in distributed parameter setting and no discretization to second-order system of differential equations is invoked so that vital inherent properties of the original system are fully preserved.

Because of the above mentioned practical features, the proposed solution is readily applicable to stabilize or to combat the effects of dangerous vibrations in a large structure.

1 Introduction

The natural model for the vibrating systems, arising in a wide range of applications, especially in the design and analysis of vibrating structures, such as bridges, highways, buildings, airplanes, etc., is a homogenous distributed-parameter system of the form

$$\mathbf{M}(x)\frac{\partial^2 \nu(t,x)}{\partial t^2} + \mathbf{C}(x)\frac{\partial \nu(t,x)}{\partial t} + \mathbf{K}(x)\nu(t,x) \quad = \quad 0\,, \tag{1}$$

where $\mathbf{M}, \mathbf{C} = \mathbf{D} + \mathbf{G}$ and $\mathbf{K}$ are differential operators in the x-domain (spatial domain) of the displacement function $\nu(t,x)$, where for all t the $\nu(t,x)$ belong

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to some Hilbert space $\mathbb{H}$, that accounts for the boundary conditions of (1). The operators $\mathbf{M}$, $\mathbf{K}$, $\mathbf{D}$ and $\mathbf{G}$ are, respectively, called *mass*, *stiffness*, *damping* and *gyroscopic* operators. In many practical applications, $\mathbf{M}$ is self-adjoint and positive definite, $\mathbf{D}$ is self-adjoint and $\mathbf{G}$ is skew-symmetric. That is, for any nonzero functions $\phi(x), \psi(x) \in \mathbb{H}$ and with the scalar product $(\cdot, \cdot)$, associated with the space $\mathbb{H}$, we have

$$(\mathbf{M}\phi, \phi) > 0, (\mathbf{M}\phi, \psi) = (\phi, \mathbf{M}\psi), (\mathbf{D}\phi, \psi) = (\phi, \mathbf{D}\psi) \text{ and } (\mathbf{G}\phi, \psi) = -(\phi, \mathbf{G}\psi).$$

While it is desirable to obtain a solution of a vibration problem in its own natural setting of distributed parameter systems; due to the lack of appropriate computational methods, in practice, very often a distributed parameter system is first discretized to a matrix second-order model using techniques of finite elements or finite differences ([1, 3, 13, 14], etc.), and then the problem is solved for this discretized reduced-order model. A matrix second-order model of the free motion of a vibrating system is a system of differential equations of the form

$$M\ddot{x}(t) + (D + G)\dot{x}(t) + Kx(t) \quad = \quad 0, \tag{2}$$

where

$M = M^T$ is mass or inertia matrix

$D = D^T$ is damping matrix

$G = -G^T$ is skew-symmetric (gyroscopic) matrix

$K = K^T$ is stiffness matrix

The system represented by (2) is called damped gyroscopic system. The eigenvalues of the system (2) are the eigenvalues of the quadratic pencil: $P(\lambda) = \lambda^2 M + \lambda(D + G) + K$.

It is well-known (see [13]) that a dangerous situation called resonance occurs when one or more natural frequencies of the system, which are the eigenvalues of $P(\lambda)$, become equal or close to a frequency of the external force.

To combat undesirable effects of vibrations, such as resonance, caused by a few "bad" eigenvalues of the system, one needs to reassign those few "bad" eigenvalues, leaving the rest unchanged, by using a suitable control force. This problem is known as the *partial pole placement problem* in control theory. Let a control force of the form

$$f = Bh(t),$$

where B is the input (control) matrix, be applied to the vibrating structure and the control vector $h(t)$ be chosen as

$$h(t) = F^T\dot{x}(t) + G^T x(t)\,.$$

Then the closed-loop system corresponding to (2) is

$$M\ddot{x} + (D + G - BF^T)\dot{x}(t) + (K - BG^T)x(t) = 0.$$

So, mathematically, the partial pole placement problem for a matrix second-order system is the problem of finding the matrices F and G such that a few "bad" eigenvalues of the closed-loop quadratic pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D + G - BF^T) + (K - BG^T)$$

are replaced by "suitably" chosen ones leaving the remaining "good" ones unchanged.

Basically, these are two approaches for solving the partial eigenvalue assignment problem in a matrix second-order system: first, via reduction of the problem to a first-order problem and second, using Independent Modal Space Control (IMSC). There are computational and engineering difficulties with both these approaches.

The reduction to a first-order system destroys the structure such as symmetry, positive definiteness, sparsity, bandedness, etc. and if the reduction to the standard first-order system is used, the possibility (DS: necessity) of inversion of a (DS: possibly) ill-conditioned mass matrix exists (see [6]).

The IMSC is a sort of text-book approach. Some very stringent requirements on the location of the actuators and sensors are needed for implementation of this approach, which is almost impossible to satisfy in practice (see [13]).

In several recent papers ([5, 7, 9, 8, 24, 22]), the partial pole placement for the damped nongyroscopic systems ($G = 0$) has been considered and a *novel* approach, called the *partial-modal* approach has been developed. This approach has several distinct features.

- The solution requires only those few eigenvalues that need to be reassigned and the corresponding eigenvectors.

- The problem is solved completely in the second-order setting; that is, no transformation to a first-order system is invoked; thus making it possible to preserve the exploitable structures such as the sparsity, definiteness, bandedness, etc., very often offered by many practical problems.

Furthermore, mathematical results are proven that guarantee that there will be no *spill-over* during the process; that is, the eigenvalues and eigenvectors that are not to be changed will remain invariant.

Three new orthogonality relations between the eigenvectors of a symmetric definite matrix quadratic pencil form a basis of these results. These relations generalize those of a symmetric matrix and of a symmetric definite linear matrix pencil (see [6]).

In this paper, we extend our study to an undamped gyroscopic systems. Our study this time goes beyond the matrix second-order system and extends to the operator system of which the former is just a discretized approximation.

Specifically, we solve the *multi-input partial pole placement problem for a gyroscopic operator system.*

Such problem for a gyroscopic operator system arises, for example, in regulating the vibratory effects of small oscillations of a taut string, rotating about its x-axis with constant angular velocity; of small oscillations of a uniform string traveling with constant velocity γ over the fixed supports, etc.

In the special case, when M, G and K are matrices, we obtain a solution of the multi-input partial pole placement problem for an undamped gyroscopic matrix second-order system of the form (2).

The paper is organized as follows.

Section 2 is on modelling. Here and example is given of how a practical life problem can be modelled in (D.S. as) distributed parameter system and then it is shown how this distributed parameter model may be discretized using (D.S. to obtain) a matrix second-order system.

The quadratic operator eigenvalue problem is briefly described in Section 3. This section contains a new result on the orthogonality of eigenfunctions of a quadratic operator pencil, which forms (D.S. plays) a key role in the development of our proposed method of solution.

A formal mathematical statement of the *partial eigenvalue assignment problem* for the gyroscopic distributed parameter system and a theorem on the existence of solution of this problem is stated in Section 4. It is then shown how the recent result [10] on the solution of the single-input partial eigenvalue assignment problem for a gyroscopic distributed parameter system follows from this theorem.

An algorithm, based on the (D.S. constructive) proof of the existence of solution (D.S. of the partial eigenvalue assignment problem), is described in Section 5.

Results of numerical experiments are given in Section 6.

2 Modeling

Consider the small oscillations of a uniform string travelling with constant velocity γ over two fixed supports at $x = 0$ and $x = 1$. This example is both simple enough to be used to illustrate the partial eigenvalue assignment method proposed in this paper and general enough to capture the essence of such problems as stabilization of towed sonar arrays or dampening the waves created by the high speed trains.

The motion of the moving string, shown in Figure 1, is governed by the partial differential equation

$$\nu_{tt} + 2\gamma\nu_{xt} + (\gamma^2 - c^2)\nu_{xx} \quad = \quad 0, \tag{3}$$

where $0 < x < 1$, $t > 0$, $\gamma^2 < c^2$ with boundary conditions given by

$$\nu(0,t) = \nu(1,t) = 0, \tag{4}$$

see *e.g.* [23].

Let the control forces $h_1(t)$ and $h_2(t)$ be applied using the point actuators $b_1(x) = \delta(x - 1/3)$ and $b_2(x) = \delta(x - 2/3)$, respectively, where $\delta(x)$ is the Dirac delta function. Then the closed-loop system motion is governed by

$$\nu_{tt} + 2\gamma\nu_{xt} + (\gamma^2 - c^2)\nu_{xx} \quad = \quad b_1h_1 + b_2h_2, \tag{5}$$

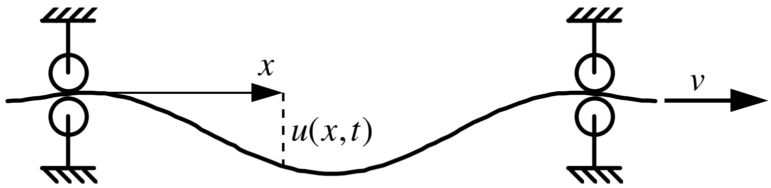

Figure 1: Small oscillations of travelling string

Let us now define the operators $\mathbf{M}$, $\mathbf{G}$ and $\mathbf{K}$ by

$$\mathbf{M}v = v\,,\ \mathbf{G}v = 2\gamma\frac{\partial v}{\partial x}\,,\ \mathbf{K}v = (\gamma^2 - c^2)\frac{\partial^2 v}{\partial x^2}. \tag{6}$$

Then, with respect to the *scalar product* (also called *inner product* or *sesquilinear form*)

$$(v, w) \quad = \quad \int_0^1 \overline{v(x)}w(x)\ dx\,,$$

we have

$$\begin{aligned}(\mathbf{M}v, w) \quad &= \quad \overline{\int_0^1 \overline{w(x)}v(x)\ dx} = \overline{(\mathbf{M}w, v)} = (v, \mathbf{M}w)\,,\\ (\mathbf{M}v, v) \quad &= \quad \int_0^1 |v(x)|^2\ dx \geq 0\,.\end{aligned}$$

Integrating by parts yields

$$(\mathbf{K}v, w) = -\overline{\int_0^1 \overline{(\gamma^2 - c^2)w''(x)}v(x)\ dx} = (v, \mathbf{K}w)\,,$$

and

$$(\mathbf{K}v, v) \quad = \quad \int_0^1 (\gamma^2 - c^2)|v'(x)|^2\ dx > 0\,,$$

in view of the boundary conditions (4). Since $v'(x)$ does not vanish identically, $\mathbf{K}$ is a self-adjoint positive definite operator. Another integration gives

$$(\mathbf{G}v, w) = -\overline{\int_0^1 \overline{2\gamma w'(x)}v(x)\ dx} = -(v, \mathbf{G}w)\,.$$

Thus the system defined by (3) and (4) is an operator system with self-adjoint operators $\mathbf{M}$ and $\mathbf{K}$ and the skew-symmetric operator $\mathbf{G}$. Such a system is called an *undamped gyroscopic operator system.*

In Section 6, the algorithm for the partial eigenvalue assignment problem, proposed in this paper, is illustrated by reassigning the poles, corresponding to the lowest frequency of the vibrating system (3) and (4), to the stable locations.

3 Quadratic Operator Eigenvalue Problem

In this section we briefly describe the quadratic eigenvalue problem for operator pencil. Then we state a new result on the orthogonality of eigenfunctions of an undamped gyroscopic quadratic operator pencil, which plays a key role in the development of our proposed method of solution.

Let us start with following preliminary definitions.

Definition 1 *Let $\mathbb{H}$ be a Hilbert space with an appropriate scalar product*

$$(\alpha\phi(x), \beta\psi(x)) \quad = \quad \overline{\alpha}\beta(\phi(x), \psi(x)) = \overline{(\beta\psi(x), \alpha\phi(x))}\,. \tag{7}$$

An operator function $\mathbf{P} : \mathbb{C} \to \mathbb{H}$ is called a quadratic operator pencil *if*

$$(\mathbf{P}(\lambda)\phi)(x) \quad = \quad \lambda^2\mathbf{M}(x)\phi(x) + \lambda\mathbf{C}(x)\phi(x) + \mathbf{K}(x)\phi(x)\,, \tag{8}$$

where $\phi \in \mathbb{H}$, and $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are differential operators from $\mathbb{H}$ to $\mathbb{H}$.

The quadratic operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ is very often referred to as the *operator polynomial* or *operator bundle* of degree 2 in the mathematical literature [11, 21].

Definition 2 *A scalar $\lambda \in \mathbb{C}$ such that the operator $\mathbf{P}(\lambda)$ is not invertible is called an* eigenvalue *of the quadratic operator pencil $\mathbf{P}(\lambda)$. The set of all eigenvalues is called the* spectrum *of $\mathbf{P}(\lambda)$.*

Definition 3 *The non-zero function $v(x) \in \mathbb{H}$ is called the* eigenfunction*, corresponding to the eigenvalue λ from the* discrete spectrum *of the quadratic operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ if for all functions $\phi(x) \in \mathbb{H}$*

$$\lambda^2(\phi, \mathbf{M}v) + \lambda(\phi, \mathbf{C}v) + (\phi, \mathbf{K}v) \quad = \quad 0. \tag{9}$$

or, equivalently,

$$\mathbf{P}(\lambda)v = (\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})v = 0 \tag{10}$$

Definition 4 *The pair (λ, v) is called the eigenpair of the operator pencil $\mathbf{P}(\lambda)$.*

The *eigenvalue problem for the quadratic operator pencil* is the problem of determining all the eigenvalues and the corresponding eigenfunctions of the given quadratic operator pencil.

Definition 5 *The operator pencil $\mathbf{P}(\lambda)$ is called* singular *if for all $\lambda \in \mathbb{C}$ the operator $\mathbf{P}(\lambda)$ is not invertible. Otherwise the operator pencil is called* regular*.*

In this paper we restrict ourselves to regular quadratic operator pencils.

Definition 6 *The associated functions $v_1, \ldots, v_n \in \mathbb{H}$ of the operator pencil $\mathbf{P}(\lambda)$ at the eigenpair (λ, v_0) are defined by the relations*

$$\begin{aligned}
&\mathbf{P}(\lambda)v_1 + \frac{1}{1!}\frac{\partial \mathbf{P}(\lambda)}{\partial \lambda}v_0 = 0\,, \\
&\vdots \\
&\mathbf{P}(\lambda)v_n + \frac{1}{1!}\frac{\partial \mathbf{P}(\lambda)}{\partial \lambda}v_{n-1} + \cdots + \frac{1}{n!}\frac{\partial^n \mathbf{P}(\lambda)}{\partial \lambda^n}v_0 = 0
\end{aligned}$$

Note, that such a sequence of functions $v_0, v_1, \ldots, v_n$ is often called the *Jordan chain.*

The important, but somewhat technically complicated concept of multiple completeness in the spectral theory of pencils has been described in article [15]. To simplify the exposition we use Lemma 13.3 of [21] to define *two-fold completeness* for the quadratic operator pencil $\mathbf{P}(\lambda)$ as follows.

Definition 7 *The system of eigenfunctions and associated functions of the quadratic operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ is* two-fold complete *in $\mathbb{H}$ if the system of eigenfunctions and associated functions of the generalized operator eigenvalue problem of the linear pencil*

$$\begin{pmatrix} K & C \\ 0 & I \end{pmatrix} - \lambda \begin{pmatrix} 0 & -M \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & \mathbf{C}+\lambda\mathbf{M} \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathbf{P}(\lambda) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\lambda I & I \end{pmatrix}$$

is complete in $\mathbb{H}^2$, where $\mathbb{H}^2$ is the orthogonal sum of two copies of $\mathbb{H}$.

Definition 8 *The eigenvalue λ of the quadratic operator pencil $\mathbf{P}(\lambda)$ is called* semi-simple *if λ is not a finite accumulation point of the spectrum and λ has no associated functions.*

The detailed spectral theory of the quadratic operator pencil is beyond the scope of this paper, therefore we make the following assumption for the rest of this paper.

Assumption 1 *The open-loop quadratic operator pencil $\mathbf{P}(\lambda)$ has discrete spectrum without finite accumulation points, every eigenvalue of $\mathbf{P}(\lambda)$ is semi-simple, and the system of eigenfunctions of $\mathbf{P}(\lambda)$ is two-fold complete.*

There is a large body of research that deals with the spectral theory of operators, in particular we mention [2, 4, 11, 12, 16, 17, 18, 19, 20, 21, 25].

3.1 Orthogonality Relations

In this section, we quote the **new orthogonality relations** between the eigenfunctions of an undamped gyroscopic quadratic operator pencil established in [26], which generalize the recent result established in [10]. This result (the relation (12)) plays a key role in obtaining the constructive solution to the partial eigenvalue assignment problem.

Theorem 1 (Orthogonality of the Eigenfunctions of an Undamped Gyroscopic Quadratic Operator Pencil).
Let $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}$ *be the undamped gyroscopic quadratic operator pencil, that is let* $\mathbf{M}$ *be positive definite self-adjoint operator,* $\mathbf{G}$ *be skew-symmetric operator and* $\mathbf{K}$ *be self-adjoint operator. Let* $(\lambda_1, \mathbf{w}_1)$, $(\lambda_2, \mathbf{w}_2)$, $\ldots$ *be the eigenpairs of* $\mathbf{P}(\lambda)$.

If $\lambda_j \neq \lambda_k$ *then*

$$(\lambda_j \mathbf{v}_j, \lambda_k \mathbf{M}\mathbf{w}_k) - (\mathbf{v}_j, \mathbf{K}\mathbf{w}_k) \quad = \quad 0 \tag{11}$$

and

$$(\mathbf{v}_j, \lambda_k \mathbf{M}\mathbf{w}_k) - (\lambda_j \mathbf{v}_j, \mathbf{M}\mathbf{w}_k) + (\mathbf{v}_j, \mathbf{C}\mathbf{w}_k) \quad = \quad 0\,. \tag{12}$$

Proof. See Theorem 3.10 and Corollary 3.3 of [26]. ■

4 Problem Statement and Theorem on Existence of Solution

In this section a formal mathematical statement of the *partial eigenvalue assignment problem* (Problem 1) for the gyroscopic distributed parameter system is given and Theorem 2 which gives a family of parametric feedback matrices that solve Problem 1 is developed. It is then shown how the recent result [10] on the solution of the single-input partial eigenvalue assignment problem for an undamped gyroscopic distributed parameter system follows from this theorem.

Problem 1 (Multi-input Partial Eigenvalue Assignment Problem for an Undamped Gyroscopic Quadratic Operator Pencil).
Given

1. *Positive definite self-adjoint operator* $\mathbf{M}$*, skew-symmetric operator* $\mathbf{G}$ *and self-adjoint operator* $\mathbf{K}$*, such that the quadratic operator pencil*

$$\mathbf{P}(\lambda) \quad = \quad \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} \tag{13}$$

has a discrete spectrum without finite accumulation points, every eigenvalue of (13) is semi-simple and the system of eigenfunctions of (13) is two-fold complete.

2. m *real control functions* $b_1, \ldots, b_m$.

3. *The self-conjugate subset* $\{\lambda_1, \ldots, \lambda_p\}$ *of the set of eigenvalues* $\{\lambda_1, \lambda_2, \ldots\}$ *and the corresponding left eigenfunction set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *of (13)*

4. *The self-conjugate set* $\{\mu_1, \ldots, \mu_p\}$ *of scalars.*

Find

Real feedback functions $\mathbf{f}_{11},\ldots,\mathbf{f}_{1m}$ *and* $\mathbf{f}_{21},\ldots,\mathbf{f}_{2m}$ *such that the spectrum of the closed-loop pencil*

$$\mathbf{P}_c(\lambda)\phi = \lambda^2\mathbf{M}\phi + \lambda\left(\mathbf{C}\phi - \sum_{k=1}^{m}(\mathbf{f}_{1k},\phi)b_k\right) + \left(\mathbf{K}\phi - \sum_{k=1}^{m}(\mathbf{f}_{2k},\phi)b_k\right) \tag{14}$$

is the set $S = \{\mu_1,\ldots,\mu_p;\lambda_{p+1},\lambda_{p+2},\ldots\}$.

Without loss of generality we make the following simplifying assumptions for the rest of the paper.

Assumption 2 *The control functions* $b_1,\ldots,b_m$ *are linearly independent.*

Justification: Indeed, if the control functions are linearly dependent then the modified Gram-Schmidt method can be used to obtain $m_1 < m$ functions $\hat{b}_1,\ldots,\hat{b}_{m_1}$ that form a basis in the subspace spanned by $b_1,\ldots,b_m$. Applying the algorithm proposed in this paper with the new control functions $\hat{b}_1,\ldots,\hat{b}_{m_1}$, one obtain the feedback force that can also be implemented using the original control functions.

Assumption 3 *The sets* $\{\lambda_1,\lambda_2,\ldots\}$ *and* $\{\mu_1,\ldots,\mu_p\}$ *are disjoint.*

Justification: If $\{\mu_1,\ldots,\mu_p\}\cap\{\lambda_1,\ldots,\lambda_p\}\neq\emptyset$, it means that some open-loop eigenvalues that we have selected to reassign in fact would not move. In this case, we should renumber the open-loop eigenvalues in such a way that the eigenvalues that would remain unaltered would go last and the number p of the eigenvalues to be reassigned will be decreased. This way we obtain the partial eigenvalue assignment problems with $\{\mu_1,\ldots,\mu_p\}\cap\{\lambda_1,\ldots,\lambda_p\}=\emptyset$.

Designing a closed-loop system such that $\{\mu_1,\ldots,\mu_p\}\cap\{\lambda_{p+1},\lambda_{p+2},\ldots\}\neq\emptyset$ is generally considered a "bad practice" in engineering. Systems with such artificially created multiple eigenvalues are usually less robust compared to the systems designed with slightly perturbed $\mu_1,\ldots,\mu_p$, because multiple eigenvalues are usually very sensitive to perturbations.

The following theorem investigates the parameterization of the feedback functions $\mathbf{f}_{11},\ldots,\mathbf{f}_{1m}$ and $\mathbf{f}_{21},\ldots,\mathbf{f}_{2m}$ using the parameter matrix Γ.

Theorem 2 (An Explicit Solution of the Multi-input Partial Eigenvalue Assignment Problem for an Undamped Gyroscopic Operator Pencil)
Suppose that the Assumptions 2 and 3 hold and the pair $(\mathbf{P}(\lambda),\{b_1,\ldots,b_m\})$ *is partially controllable with respect to* $\{\lambda_1,\ldots,\lambda_p\}$, *that is, for every* $j=1,\ldots,p$ *at least one of the following scalar products is nonzero*

$$(\mathbf{v}_j,b_1)\neq 0,\ldots,(\mathbf{v}_j,b_m)\neq 0\,.$$

Let $\Gamma=(\gamma_1,\ldots,\gamma_p)\in\mathbb{C}^{m\times p}$ *be a matrix such that*

$$\gamma_j=\overline{\gamma_k}\ \textit{whenever}\ \mu_j=\overline{\mu_k}\,. \tag{15}$$

Let Z_1 be the unique nonsingular solution of the Sylvester equation

$$\Lambda_1 Z_1 + Z_1 \Lambda_{c1} = -\begin{pmatrix} (\mathbf{v}_1, b_1) & \cdots & (\mathbf{v}_1, b_m) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_p, b_1) & \cdots & (\mathbf{v}_p, b_m) \end{pmatrix} \Gamma\,, \tag{16}$$

where $\Lambda_1 = \mathrm{diag}(\overline{\lambda}_1, \ldots, \overline{\lambda}_p)$ *and* $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$.
Let the matrix $\Phi = (\phi_{kj}) \in \mathbb{C}^{m \times p}$ *be the solution of the linear system*

$$\Phi Z_1 = \Gamma\,. \tag{17}$$

Then the real feedback functions $\mathbf{f}_{1k}$ *and* $\mathbf{f}_{2k}$ *for* $k = 1, 2, \ldots, m$ *that solve Problem 1 are given by*

$$\mathbf{f}_{1k} = \sum_{j=1}^{p} \overline{\phi}_{kj}\, \mathbf{M}\mathbf{v}_j \quad \textit{and} \quad \mathbf{f}_{2k} = -\sum_{j=1}^{p} \overline{\phi}_{kj}\, (\lambda_j \mathbf{M}\mathbf{v}_j + \mathbf{G}\mathbf{v}_j)\,. \tag{18}$$

Conversely, if there exist real feedback functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ *and* $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ *of the form (18) that constitute the solution of Problem 1, then the matrix* Φ *in (18) can be constructed to satisfy (15) through (17).*

Proof. See Theorem 6.3 and Corollary 6.1 of [26]. ■

Note, that Theorem 2 guarantees that an infinite number of the open-loop eigenvalues starting from λ_{m+1} remain invariant by the feedback obtained by (14) through (18). In other words, *no spill-over occurs.*

5 Algorithm for Solution of the Partial Eigenvalue Assignment Problem

Based on the Theorem 2 we state the following algorithm for the solution of Problem 1.

Algorithm 1 (Parametric Solution to the Partial Eigenvalue Assignment Problem for an Undamped Gyroscopic Quadratic Operator Pencil).

Inputs:

(a) *The self-adjoint positive definite operator* $\mathbf{M}$, *skew-symmetric operator* $\mathbf{G}$ *and self-adjoint operator* $\mathbf{K}$.

(b) *The* m *control functions* $b_1, \ldots, b_m$.

(c) *The set of scalars* $\{\mu_1, \ldots, \mu_p\}$, *closed under complex conjugation.*

(d) *The self-conjugate subset* $\{\lambda_1, \ldots, \lambda_p\}$ *of the open-loop spectrum* $\{\lambda_1, \lambda_2, \ldots\}$ *and the associated eigenfunction set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Outputs:

The feedback functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ *and* $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ *such that the spectrum of the closed-loop operator pencil*

$$\mathbf{P}_c(\lambda)\phi = \lambda^2 \mathbf{M}\phi + \lambda \left(\mathbf{G}\phi - \sum_{r=1}^{m} (\mathbf{f}_{1r}, \phi)\, b_r \right) + \left(\mathbf{K}\phi - \sum_{r=1}^{m} (\mathbf{f}_{2r}, \phi)\, b_r \right)$$

is $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \lambda_{p+2}, \ldots\}$.

Assumptions:

(a) The control functions $b_1, \ldots, b_m$ *are linearly independent.*

(b) The open-loop undamped gyroscopic quadratic operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}$ *with control functions* $b_1, \ldots, b_m$ *is partially controllable with respect to the eigenvalues* $\lambda_1, \ldots, \lambda_p$.

(c) The sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \lambda_{p+1}, \ldots\}$ *and* $\{\mu_1, \ldots, \mu_p\}$ *are disjoint.*

Step 1. *Form*

$$\Lambda_1 = \mathrm{diag}(\overline{\lambda}_1, \ldots, \overline{\lambda}_p), \mathbf{V}_1 = (\mathbf{v}_1, \ldots, \mathbf{v}_p) \text{ and } \Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)\,.$$

Step 2. *Choose arbitrary* $m \times 1$ *vectors* $\gamma_1, \ldots, \gamma_p$ *in such a way that* $\overline{\mu_j} = \mu_k$ *implies* $\overline{\gamma_j} = \gamma_k$ *and form* $\Gamma = (\gamma_1, \ldots, \gamma_p)$.

Step 3. *Solve the following Sylvester equation for* Z_1*:*

$$\Lambda_1 Z_1 + Z_1 \Lambda_{c1} = -\begin{pmatrix} (\mathbf{v}_1, b_1) & \cdots & (\mathbf{v}_1, b_m) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_p, b_1) & \cdots & (\mathbf{v}_p, b_m) \end{pmatrix} \Gamma\,.$$

If Z_1 *is ill-conditioned then return to Step 2 and select different* $\gamma_1, \ldots, \gamma_p$.

Step 4. *Solve* $\Phi Z_1 = \Gamma$ *for* Φ.

Step 5. *If none of the* $\lambda_1, \ldots, \lambda_p$ *is zero, form for all* $k = 1, \ldots, m$

$$\mathbf{f}_{1k} = \sum_{j=1}^{p} \overline{\phi}_{kj}\, \mathbf{M}\mathbf{v}_j \quad \text{and} \quad \mathbf{f}_{2k} = \sum_{j=1}^{p} \overline{\phi}_{kj}\, \frac{1}{\lambda_j}\, \mathbf{K}\mathbf{v}_j$$

otherwise form for all $k = 1, \ldots, m$*, form*

$$\mathbf{f}_{1k} = \sum_{j=1}^{p} \overline{\phi}_{kj}\, \mathbf{M}\mathbf{v}_j \quad \text{and} \quad \mathbf{f}_{2k} = -\sum_{j=1}^{p} \overline{\phi}_{kj}\, (\lambda_j \mathbf{M}\mathbf{v}_j + \mathbf{G}\mathbf{v}_j)\,.$$

An example for the Algorithm 1 is worked out in Section 6.

Remark 1 *The most distinctive feature of the algorithm is that it computes the solution of an infinite dimensional operator problem by solving a small finite dimensional linear algebraic system and by using only the few eigenvalues of the infinite dimensional operator that need to be reassigned and the associated eigenvectors. The algorithm is thus readily applicable to control dangerous vibration in a structure, where only a small part of the spectrum needs to be reassigned and the rest is to remain unchanged.*

6 Numerical Example

For the purpose of making a simple illustrative numerical example we set $c = 1$, $\gamma = 1/2$ and take the control functions

$$b_1(x) = \delta(x - 1/3) \text{ and } b_2(x) = \delta(x - 2/3)\,.$$

Then the spectrum $\{\lambda_k\}_{k=\pm1,\pm2,\ldots}$ and eigenfunctions $\{\mathbf{v}_k\}_{k=\pm1,\pm2,\ldots}$ of travelling string system (3) and (4) are the nonzero solutions of the following boundary value problem

$$\lambda_k^2 \mathbf{v}_k(x) + \lambda_k \mathbf{v}_k'(x) - \frac{3}{4}\mathbf{v}_k''(x) = 0\,,\ \mathbf{v}(0) = \mathbf{v}(1) = 0\,. \tag{19}$$

Thus, the eigenpairs of (19) are given by

$$\lambda_k = i\frac{3\pi}{4}k\,,\ \mathbf{v}_k(x) = e^{i\frac{3\pi}{2}kx} - e^{-i\frac{\pi}{2}kx} \text{ for } k = \pm1, \pm2, \ldots. \tag{20}$$

We will solve Problem 1 reassigning only the eigenvalues $\pm i3\pi/4$ to the locations $\mu_{1,2} = -1 \pm i3\pi/4$. That is we want the closed-loop system (14) to have the spectrum

$$\left\{-1 \pm i\frac{3\pi}{4}\right\} \cup \left\{i\frac{3\pi}{4}k : k = \pm2, \pm3, \ldots\right\}. \tag{21}$$

The step-wise results of our implementation of Algorithm 1 are given in the following

Step 1. We form

$$\Lambda_1 = \operatorname{diag}\left(-i\frac{3\pi}{4}, i\frac{3\pi}{4}\right)\,,\ \mathbf{V}_1 = \left(e^{i\frac{3\pi}{2}x} - e^{-i\frac{\pi}{2}x}, e^{-i\frac{3\pi}{2}x} - e^{i\frac{\pi}{2}x}\right)$$

and

$$\Lambda_{c1} = \operatorname{diag}(-1 + i\frac{3\pi}{4}, -1 - i\frac{3\pi}{4})\,.$$

Step 2. We choose arbitrary

$$\Gamma = \begin{pmatrix} 0.62303 + 0.1478i & 0.62303 - 0.1478i \\ 0.2818 - 0.71455i & 0.2818 + 0.71455i \end{pmatrix}.$$

Step 3. Solving the Sylvester equation for Z_1

$$\Lambda_1 Z_1 + Z_1 \Lambda_{c1} = -\begin{pmatrix} (\mathbf{v}_1, b_1) & (\mathbf{v}_1, b_2) \\ (\mathbf{v}_2, b_1) & (\mathbf{v}_2, b_2) \end{pmatrix} \Gamma$$

we obtain

$$Z_1 = \begin{pmatrix} 0.10625 - 0.26592i & 0.56514 + 2.12242i \\ 0.56514 - 2.12242i & 0.10625 + 0.26592i \end{pmatrix}.$$

Step 4. Solving $\Phi Z_1 = \Gamma$ for Φ_1, we obtain

$$\Phi = \begin{pmatrix} 0.0024296 - 0.33472i & 0.0024296 + 0.33472i \\ 0.30701 - 0.040761i & 0.30701 + 0.040761i \end{pmatrix}.$$

Step 5. The velocity feedback functions $\mathbf{f}_{11}$ and $\mathbf{f}_{12}$, plotted in Figure 2, are given by

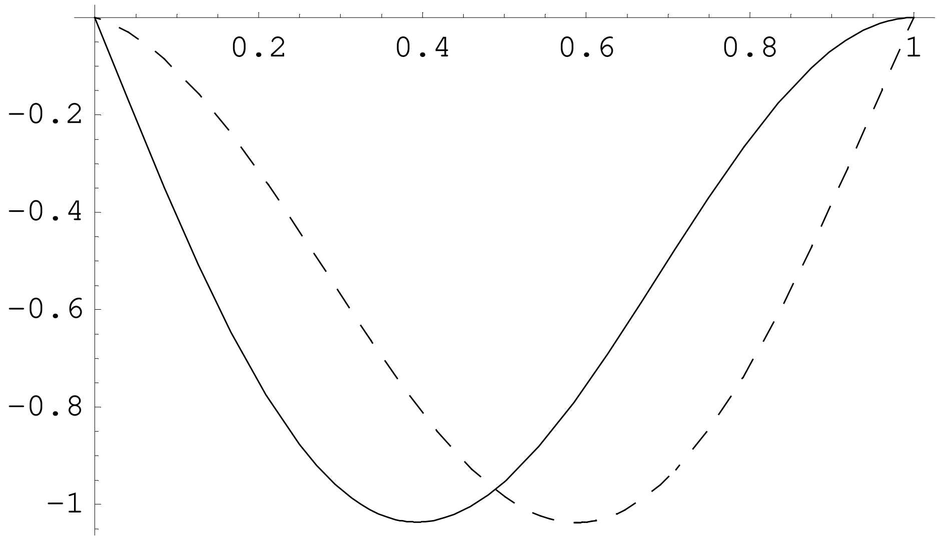

Figure 2: The velocity feedback functions $\mathbf{f}_{11}$ (solid line) and $\mathbf{f}_{12}$ (dashed line).

$$\begin{aligned}
\mathbf{f}_{11} &= -2.67775 \cos^2\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi x}{2}\right) - 0.0194366 \cos\left(\frac{\pi x}{2}\right) \sin^2\left(\frac{\pi x}{2}\right) \\
\mathbf{f}_{12} &= -0.32609 \cos^6\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi x}{2}\right) - 2.4561 \cos^5\left(\frac{\pi x}{2}\right) \sin^2\left(\frac{\pi x}{2}\right) - \\
&\quad 0.65217 \cos^4\left(\frac{\pi x}{2}\right) \sin^3\left(\frac{\pi x}{2}\right) - 4.9122 \cos^3\left(\frac{\pi x}{2}\right) \sin^4\left(\frac{\pi x}{2}\right) - \\
&\quad 0.32609 \cos^2\left(\frac{\pi x}{2}\right) \sin^5\left(\frac{\pi x}{2}\right) - 2.4561 \cos\left(\frac{\pi x}{2}\right) \sin^6\left(\frac{\pi x}{2}\right)
\end{aligned}$$

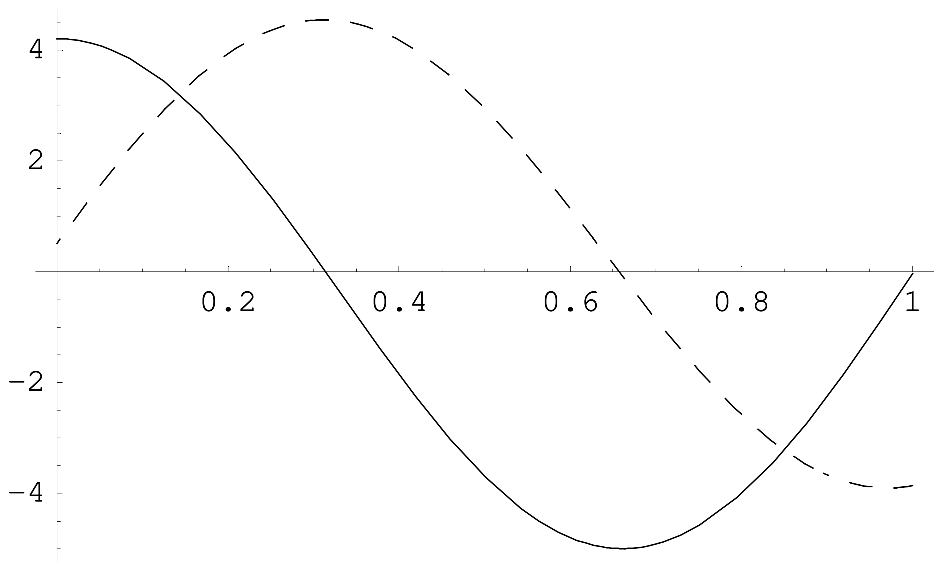

Figure 3: The position feedback functions $\mathbf{f}_{21}$ (solid line) and $\mathbf{f}_{22}$ (dashed line).

and the position feedback functions $\mathbf{f}_{21}$ and $\mathbf{f}_{22}$, plotted in Figure 3, are given by

$$\begin{aligned}
\mathbf{f}_{21} &= 4.2062\cos^5\left(\frac{\pi x}{2}\right) + 0.10686\cos^4\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right) - \\
&\quad 10.5155\cos^3\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right) + 0.076327\cos^2\left(\frac{\pi x}{2}\right)\sin^3\left(\frac{\pi x}{2}\right) - \\
&\quad 14.722\cos\left(\frac{\pi x}{2}\right)\sin^4\left(\frac{\pi x}{2}\right) - 0.030531\sin^5\left(\frac{\pi x}{2}\right) \\
\mathbf{f}_{22} &= 0.51222\cos^7\left(\frac{\pi x}{2}\right) + 13.5031\cos^6\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right) - \\
&\quad 0.76832\cos^5\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right) + 23.1482\cos^4\left(\frac{\pi x}{2}\right)\sin^3\left(\frac{\pi x}{2}\right) - \\
&\quad 3.0733\cos^3\left(\frac{\pi x}{2}\right)\sin^4\left(\frac{\pi x}{2}\right) + 5.78704\cos^2\left(\frac{\pi x}{2}\right)\sin^5\left(\frac{\pi x}{2}\right) - \\
&\quad 1.79276\cos\left(\frac{\pi x}{2}\right)\sin^6\left(\frac{\pi x}{2}\right) - 3.85803\sin^7\left(\frac{\pi x}{2}\right) .
\end{aligned}$$

The closed-loop operator pencil

$$\mathbf{P}_c(\mu)\mathbf{w}_c(x) = \mu^2\mathbf{w}_c(x) + \mu\left(\mathbf{w}_c{}'(x) - \sum_{r=1}^{2}\left(\mathbf{f}_{1r}(x), \mathbf{w}_c(x)\right)\,\delta(x - \frac{r}{3})\right) +$$
$$\left(-\frac{3}{4}\mathbf{w}_c{}''(x) - \sum_{r=1}^{2}\left(\mathbf{f}_{2r}(x), \mathbf{w}_c(x)\right)\,\delta(x - \frac{r}{3})\right)$$

has the eigenvalues $\mu_1 = -1 - i3\pi/4$ and $\overline{\mu}_1$ with eigenfunctions given by

$$\begin{aligned}\mathbf{w}_{c1} \quad = \quad & e^{-(2+i\frac{3\pi}{2})x}\Big((0.13271 + 0.11997i)\Big(1 - e^{(\frac{8}{3}+2\pi i)x}\Big) + \\ & \Big((0.58872 + 0.43312i) - (0.033192 - 0.29864i)e^{(\frac{8}{3}+2\pi i)x}\Big)\, h(x - \frac{1}{3}) + \\ & \Big((1.21755 + 1.19732i) + (0.27814 - 0.077031)e^{(\frac{8}{3}+2\pi i)x}\Big)\, h(x - \frac{2}{3})\Big)\end{aligned}$$

and $\overline{\mathbf{w}_{c1}(x)}$, respectively, where

$$h(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

is the unit step (Heaviside) function. Furthermore, the eigenvalues λ_k, $k = \pm 2, \pm 3, \ldots$, of the open-loop pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}$ remain unchanged.

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