EIGENVALUE EMBEDDING IN A QUADRATIC PENCIL USING SYMMETRIC LOW RANK UPDATES *

JOAO CARVALHO[†] , BISWA N. DATTA[†], WEN-WEI LIN[‡], AND CHERN-SHUH WANG[§]

Abstract. In this paper we present theories for a low rank transformation and apply this type of transformation to quadratic eigenvalue problems that arise in model tuning applications. The approach is flexibility, efficiency, and preserves the symmetric structures of the matrices in the problems. Furthermore, it maintains many physical characteristics that are very important in real world applications. A numerical experiment on a simulation data set from the Boeing Company is described.

Key words. non-equivalence transformation, model tuning, eigenvalue embedding, structure preserving

AMS subject classifications. 15A18, 15A24, 65F30.

1. INTRODUCTION. Vibrating structures such as bridges, highways, buildings, and automobiles are most often modeled using finite-element methods. These methods generate structured matrix second-order differential equations

$$M\ddot{x} + C\dot{x} + Kx = 0, \tag{1.1}$$

where the coefficient matrices M, C and K are all symmetric and stand for mass, damping and stiffness matrices, respectively. Model (1.1) will be referred to as the symmetric model. We further note that matrices M and K are positive definite and denoted by $M > 0$ and $K > 0$. The dynamic behavior of a vibrating structure modeled by (1.1) is determined by its response. The response is measured by the natural frequencies of the model that are denoted by the $2n$ eigenvalues $\{\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{2n}\}$ of the associated quadratic matrix pencil

$$F(\lambda) = \lambda^2 M + \lambda C + K. \tag{1.2}$$

Since the eigenvalues measured from a real structure in practice do not match well with system (1.1), the model (1.1) must therefore be corrected. The corrected model is expected to match the experimental and theoretical results and to preserve other meaningful physical properties of the original model.

This consideration gives rise to the following problem, known as the model tuning problem: Given symmetric matrices M, K and C with $M > 0$ and $K > 0$, a part of the spectrum $\{\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{2n}\}$ of $F(\lambda)$, and a set of r complex numbers $\{\mu_1, \ldots, \mu_r\}$, closed under complex conjugation, find the symmetric matrices M_{new}, K_{new} and C_{new} such that the spectrum of the pencil $F_{\text{new}}(\lambda) = \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}$ is $\{\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_{2n}\}$.

---

*This work was supported in part by the National Center for Theoretical Sciences, Hsinchu, Taiwan. The first author was also partially supported by Brazilian CAPES and USA-NSF grant ECS-0074411 and the second author was partially supported by USA-NSF grant ECS-0074411.

[†]Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115-2888, U.S.A., carvalho@math.niu.edu, dattab@math.niu.edu.

[‡]Department of Mathematics, National Tsing Hua University, Hsinchu 30043, Taiwan, wwlin@am.nthu.edu.tw.

[§]Department of Mathematics, National Cheng Kung University, Tainan 701, Taiwan, cswang@math.ncku.edu.tw

A closely related problem to this is the partial eigenvalue assignment problem [1, 2] in control theory. The partial eigenvalue assignment is solved by using feedback control and no symmetry restriction on the updated model is imposed.

There also exists a class of optimization-based methods in the vibration literature [9, 10, 13] which uses optimization techniques to bring symmetry to the model after the eigenvalues have been assigned; unfortunately, however, none of these methods can guarantee the invariance of the remaining eigenvalues of the updated model. This results in a serious loss of many physical properties of the original model; therefore, the updated model cannot be used with confidence in the future.

In this paper, we propose a new method to solve the problem, using the techniques of non-equivalence transformation of the quadratic eigenvalue problem. The non-equivalence transformation of the rational λ-matrix functions has been considered before in the literature [3, 4, 7, 6]. However, the results of this paper cannot be obtained by a straightforward generalization of these papers.

Our approach is purely matrix theoretical. The complex conjugate pairs of the eigenvalues are assigned one pair at a time, and the matrices are updated every time a pair is recovered. If the measured eigenvalues are all real, they can either be assigned simultaneously, or one at a time.

The results of a numerical experiment on mimic data obtained from the aerospace industry [5] are shown to demonstrate the accuracy of the method.

2. Assignment of a Real Eigenvalue. In this section, we construct the updated matrices M_{new}, K_{new} and C_{new}, such that an isolated real eigenpair (λ_1, y_1) of the pencil $F(\lambda) = \lambda^2 M + \lambda C + K$ is replaced by the given eigenvalue μ_1, while the other eigenvalues and eigenvectors remain invariant.

To achieve this goal, we consider a low rank transformation, named the non-equivalence transformation, for the quadratic matrix pencil $F(\lambda)$. A non-equivalence transformation for the rational λ-matrix functions has been previously considered in several papers [3, 4, 7, 6]. However, the non-equivalence transformation reported in this paper cannot be derived by using a straightforward generalization of [3, 4, 7, 6]. Let (λ_1, y_1) be a real isolated eigenpair of $F(\lambda)$, i.e.,

$$F(\lambda_1)y_1 \equiv (\lambda_1^2 M + \lambda_1 C + K)y_1 = 0. \tag{2.1}$$

Since K is positive definite, the eigenvector y_1 can be normalized such that $y_1^\top K y_1 = 1$. Suppose that $\lambda_1 \in I\!R$ is an unwanted eigenvalue that needs to be replaced by a prescribed real number μ_1. The following theorem provides a non-equivalence transformation of $F(\lambda)$ such that the updated matrix pencil, $F_{\text{new}}(\lambda)$, keeps the eigenstructure of $F(\lambda)$ except that μ_1 replaces λ_1 to become an eigenvalue of $F_{\text{new}}(\lambda)$.

Theorem 2.1 (Assignment of a Real Eigenvalue and Spectrum Invariance). *Let (λ_1, y_1) be a real, isolated eigenpair of $F(\lambda)$ with $y_1^\top K y_1 = 1$. Define $\theta_1 = y_1^\top M y_1$ and $\varepsilon_1 = \frac{\lambda_1 - \mu_1}{1 - \lambda_1 \mu_1 \theta_1}$. Suppose $1 - \lambda_1 \mu_1 \theta_1 \neq 0$, and $1 - \lambda_1^2 \theta_1 \neq 0$. Then the updated matrix pencil*

$$F_{new}(\lambda) = \lambda^2 M_{new} + \lambda C_{new} + K_{new}, \tag{2.2}$$

where

$$\begin{aligned} M_{new} &= M - \varepsilon_1 \lambda_1 M y_1 y_1^\top M \\ C_{new} &= C + \varepsilon_1 (M y_1 y_1^\top K + K y_1 y_1^\top M) \\ K_{new} &= K - \frac{\varepsilon_1}{\lambda_1} K y_1 y_1^\top K \end{aligned} \tag{2.3}$$

is symmetric, and has the following spectral properties:

(a) *the number μ_1 is in the spectrum of $F_{new}(\lambda)$ and the remaining eigenvalues of $F_{new}(\lambda)$ are the same as those of $F(\lambda)$.*

(b) (i) *y_1 is also an eigenvector of $F_{new}(\lambda)$ corresponding to the eigenvalue μ_1.*

(ii) *The remaining eigenvectors of $F_{new}(\lambda)$ are the same as those of $F(\lambda)$; that is, if $\lambda_2 \neq \lambda_1$ and (λ_2, y_2) is an eigenpair of $F(\lambda)$, then it is also an eigenpair of $F_{new}(\lambda)$.*

Proof. (a): Substituting the result of (2.1) into $F(\lambda)$, we have

$$\begin{aligned} F(\lambda)y_1 &= \lambda^2 M y_1 + \lambda C y_1 + K y_1 \\ &= \lambda^2 M y_1 + \lambda C y_1 - \lambda_1^2 M y_1 - \lambda_1 C y_1 \\ &= (\lambda - \lambda_1)((\lambda + \lambda_1)M + C)y_1. \end{aligned} \tag{2.4}$$

By using the identity

$$\det(I_n + RS) = \det(I_m + SR), \text{ as } R \in \mathbb{C}^{n\times m} \text{ and } S \in \mathbb{C}^{m\times n},$$

and from (2.4), we have

$$\begin{aligned} \det\left(F_{\text{new}}(\lambda)\right) &= \det\left(\lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}\right) \\ &= \det(\lambda^2 M + \lambda C + K - \lambda^2 \varepsilon_1 \lambda_1 M y_1 y_1^\top M \\ &\quad + \lambda\varepsilon_1 (M y_1 y_1^\top K + K y_1 y_1^\top M) - \frac{\varepsilon_1}{\lambda_1} K y_1 y_1^\top K) \\ &= \det\left(F(\lambda) + \varepsilon_1((\lambda + \lambda_1)M + C)y_1 y_1^\top (K - \lambda\lambda_1 M)\right) \\ &= \det\left(F(\lambda) + \frac{\varepsilon_1}{\lambda - \lambda_1} F(\lambda) y_1 y_1^\top (K - \lambda\lambda_1 M)\right) \\ &= \det(F(\lambda)) \cdot \left(1 + \frac{\varepsilon_1}{\lambda - \lambda_1}(1 - \lambda\lambda_1\theta_1)\right) \\ &= \frac{\det(F(\lambda))}{\lambda - \lambda_1} \cdot (\lambda - \lambda_1 + \varepsilon_1(1 - \lambda\lambda_1\theta_1)). \end{aligned}$$

Since $\varepsilon_1 = \frac{\lambda_1 - \mu_1}{1 - \lambda_1\mu_1\theta_1}$, and $1 - \lambda_1^2\theta_1 \neq 0$, we have

$$\lambda - \lambda_1 + \varepsilon_1(1 - \lambda\lambda_1\theta_1) = (\lambda - \mu_1)\frac{(1 - \lambda_1^2\theta_1)}{1 - \lambda_1\mu_1\theta_1}.$$

Therefore, $\det(F_{\text{new}}(\lambda))$ has the same roots as $\det(F(\lambda))$, except λ_1, which is replaced by μ_1.

<u>Proof of part (b)</u>: We first prove part (b)(i). From (2.3), we have

$$\begin{aligned} F_{\text{new}}(\mu_1)y_1 &= \mu_1^2(M - \epsilon_1\lambda_1 M y_1 y_1^\top M)y_1 + \mu_1(C + \epsilon_1(M y_1 y_1^\top K + K y_1 y_1^\top M))y_1 \\ &\quad + (K - \frac{\epsilon_1}{\lambda_1} K y_1 y_1^\top K)y_1 \\ &= (\mu_1^2 - \mu_1^2\epsilon_1\lambda_1\theta_1 + \mu_1\epsilon_1)M y_1 + \mu_1 C y_1 + (\mu_1\epsilon_1\theta_1 + 1 - \frac{\epsilon_1}{\lambda_1})K y_1. \end{aligned} \tag{2.5}$$

Using $\epsilon_1 = \frac{\lambda_1 - \mu_1}{1 - \lambda_1\mu_1\theta_1}$, we have

$$\mu_1\epsilon_1\theta_1 + 1 - \frac{\epsilon_1}{\lambda_1} = \epsilon_1\left(\frac{\lambda_1\mu_1\theta_1 - 1}{\lambda_1}\right) + 1 = \frac{\mu_1}{\lambda_1}. \tag{2.6}$$

Since $F(\lambda_1)y_1 = 0$, we have

$$Ky_1 = -\lambda_1^2 My_1 - \lambda_1 Cy_1. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), we then obtain

$$F_{\text{new}}(\mu_1)y_1 = (\mu_1^2 - \mu_1^2\epsilon_1\lambda_1\theta_1 + \mu_1\epsilon_1 - \lambda_1\mu_1)My_1.$$

Again, substituting the value of $\epsilon_1 = \frac{\lambda_1-\mu_1}{1-\lambda_1\mu_1\theta_1}$, we see that

$$\begin{aligned}\mu_1^2 - \mu_1^2\epsilon_1\lambda_1\theta_1 + \mu_1\epsilon_1 - \lambda_1\mu_1 &= \mu_1(\mu_1-\lambda_1) + \mu_1\epsilon_1(1-\mu_1\lambda_1\theta_1)\\ &= \mu_1(\mu_1-\lambda_1) + \mu_1(\lambda_1-\mu_1)\\ &= 0.\end{aligned}$$

This implies that $F_{\text{new}}(\mu_1)y_1 = 0$, and so (b)(i) is proven.

To prove part (b)(ii), we observe that

$$F(\lambda_2)y_2 = (\lambda_2^2 M + \lambda_2 C + K)y_2 = 0$$

derives $Ky_2 = -\lambda_2^2 My_2 - \lambda_2 Cy_2$. This implies

$$F(\lambda_1)y_2 = (\lambda_1-\lambda_2)((\lambda_1+\lambda_2)M + C)y_2. \tag{2.8}$$

Using the same arguments as in the part (a) proof and (2.8), we obtain

$$\begin{aligned}F_{\text{new}}(\lambda_2)y_2 &= (\lambda_2^2 M_{\text{new}} + \lambda_2 C_{\text{new}} + K_{\text{new}})y_2\\ &= F(\lambda_2)y_2 + \frac{\varepsilon_1}{\lambda_2-\lambda_1}\left(F(\lambda_2)y_1y_1^\top(K-\lambda_2\lambda_1 M)y_2\right)\\ &= \frac{\varepsilon_1}{\lambda_2-\lambda_1}\left(F(\lambda_2)y_1y_1^\top(-\lambda_2((\lambda_1+\lambda_2)M+C))y_2\right)\\ &= \frac{-\lambda_2\varepsilon_1}{(\lambda_2-\lambda_1)^2}\left(F(\lambda_2)y_1y_1^\top F(\lambda_1)y_2\right)\\ &= 0.\end{aligned}$$

Hence, (λ_2, y_2) is also an eigenpair of $F_{\text{new}}(\lambda)$. □

3. Assignment of a Complex Conjugate Pair of Eigenvalues. We now develop the results in this section analogously to those of Theorem 2.1, to show how to compute the updated symmetric matrices M_{new}, K_{new} and C_{new}, such that a complex conjugate pair of eigenvalues, μ_1 and $\bar{\mu}_1$ is assigned to the spectrum of $F_{\text{new}}(\lambda)$, while the other eigenvalues of $F_{\text{new}}(\lambda)$ remain the same as those of $F(\lambda)$. For simplicity, a matrix pair (Λ, Y) satisfying

$$MY\Lambda^2 + CY\Lambda + KY = 0$$

is also called an eigenpair of $F(\lambda)$. The notation spec$(F(\lambda))$, (or spec(Λ)), stands for the spectrum of $F(\lambda)$, (or Λ, respectively).

Let (λ_1, y_1) be an isolated complex eigenpair of $F(\lambda)$, with $\lambda_1 = \alpha_1 + i\beta_1$, α_1, $\beta_1 \in \mathbb{R}$, $\beta_1 \neq 0$, and $y_1 = y_{1r} + iy_{1i}$, y_{1r}, $y_{1i} \in \mathbb{R}^n$. Suppose that y_{1r} and y_{1i} are linearly independent, then y_1 and $\bar{y}_1$ are linearly independent, and $(\bar{\lambda}_1, \bar{y}_1)$ is also an eigenpair of $F(\lambda)$. Since (λ_1, y_1) is an eigenpair of $F(\lambda)$, we have

$$MZ_1\underline{\Lambda}_1^2 + CZ_1\underline{\Lambda}_1 + KZ_1 = 0, \tag{3.1}$$

where

$$\underline{\Lambda}_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \quad \text{and} \quad Z_1 = [y_{1r} \quad y_{1i}].$$

Thus, $(\underline{\Lambda}_1, Z_1)$ is an eigenpair of $F(\lambda)$. Since K is positive definite, $\Sigma_1 = Z_1^\top K Z_1$ is also positive definite. Then there exists an orthogonal matrix $S_1 \in I\!R^{2\times 2}$, and a positive diagonal matrix, $D_1 = \left[\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right]$, such that

$$\Sigma_1 = S_1 D_1 D_1 S_1^\top.$$

Let

$$Y_1 = Z_1 S_1 D_1^{-1}, \tag{3.2}$$

then $Y_1^\top K Y_1 = I_2$. Let

$$\begin{aligned} \Lambda_1 &= D_1 S_1^\top \underline{\Lambda}_1 S_1 D_1^{-1} \\ &= \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 & \beta_1/d \\ -d\beta_1 & \alpha_1 \end{bmatrix}, \end{aligned} \tag{3.3}$$

where $d \equiv d_1/d_2$.

We now present a lemma which will be needed for use later on.

LEMMA 3.1. *Given a complex number, $\mu_1 = \varphi_1 + i\psi_1$, $\psi_1 \neq 0$, there is a real diagonal matrix, E_M, such that μ_1 is an eigenvalue of the matrix pair, $(\Lambda_1\Lambda_1^\top - E_M,\ \Lambda_1^\top - E_M\Theta_1\Lambda_1^\top)$, where $\Theta_1 = Y_1^\top M Y_1$ and Y_1, Λ_1 are given by (3.2), (3.3), respectively.*

Proof. Let $\Theta_1 = Y_1^\top M Y_1 = \left[\begin{smallmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{smallmatrix}\right]$, and $E_M = \left[\begin{smallmatrix} \xi & 0 \\ 0 & \eta \end{smallmatrix}\right] \in I\!R^{2\times 2}$, where ξ, η are two unknowns. From expanding and realizing the following conjugated equations:

$$\begin{aligned} &\det\left[\mu_1(\Lambda_1^\top - E_M\Theta_1\Lambda_1^\top) - (\Lambda_1\Lambda_1^\top - E_M)\right] = 0, \\ &\det\left[\bar{\mu}_1(\Lambda_1^\top - E_M\Theta_1\Lambda_1^\top) - (\Lambda_1\Lambda_1^\top - E_M)\right] = 0. \end{aligned} \tag{3.4}$$

We conclude that ξ, η satisfy a system of two real two degree polynomials

$$\begin{aligned} p_1 + p_2\xi + p_3\eta + p_4\xi\eta = 0, \\ q_1 + q_2\xi + q_3\eta + q_4\xi\eta = 0, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} p_1 &= 2(\varphi_1\rho_1 - \alpha_1\sigma_1), \\ p_2 &= \sigma_1(\alpha_1\theta_{11} + \frac{\alpha_1}{\rho_1} + d\beta_1\theta_{12}) - \frac{2\varphi_1}{\rho_1}(\alpha_1^2 + d^2\beta_1^2), \\ p_3 &= \sigma_1(\frac{\alpha_1}{\rho_1} + \alpha_1\theta_{22} - \frac{\beta_1\theta_{12}}{d}) - \frac{2\varphi_1}{\rho_1}(\alpha_1^2 + \frac{\beta_1^2}{d^2}), \\ p_4 &= \frac{\sigma_1}{\rho_1}(d\beta_1\theta_{12} - \frac{\beta_1\theta_{12}}{d} - \alpha_1\theta_{11} - \alpha_1\theta_{22}) + \frac{2\varphi_1}{\rho_1}, \\ q_1 &= \sigma_1 - \rho_1, \end{aligned}$$

$$
\begin{aligned}
q_2 &= \frac{1}{\rho_1}(\alpha_1^2 + d^2\beta_1^2) - \sigma_1\theta_{11},\\
q_3 &= \frac{1}{\rho_1}(\alpha_1^2 + \frac{\beta_1^2}{d^2}) - \sigma_1\theta_{22},\\
q_4 &= \sigma_1(\theta_{11}\theta_{22} - \theta_{12}^2) - \frac{1}{\rho_1}.
\end{aligned}
$$

Here, $\theta_{j,k}$ is the (j,k)-th entry of Θ_1, $j,\ k = 1,2$, $\rho_1 = \alpha_1^2 + \beta_1^2$ and $\sigma_1 = \varphi_1^2 + \psi_1^2$. Hence, from (3.5), we can find E_M by letting diagonals

$$
\xi = -\frac{q_1 + q_3\eta}{q_2 + q_4\eta} \quad \text{and} \quad \eta = \frac{-\ell_2 \pm \sqrt{\ell_2^2 - 4\ell_1\ell_3}}{2\ell_1}, \tag{3.6}
$$

where $\ell_1 = p_3q_4 - p_4q_3$, $\ell_2 = p_1q_4 - p_2q_3 + p_3q_2 - p_4q_1$ and $\ell_3 = p_1q_2 - p_2q_1$. ☐

REMARK 3.1. *It is easily seen that ξ and η are real provided that $\ell_2^2 - 4\ell_1\ell_3 \geq 0$. This is always true, whenever the assumptions of Lemma 3.1 hold. In particular, formula (3.6) derives $\xi = 0 = \eta$ when the embedded complex number, μ_1, is identical to the original eigenvalue, λ_1.*

The next theorem provides a low rank transformation of matrix pencil $F(\lambda)$, such that the eigenvalues of the updated symmetric pencil $F_{\text{new}}(\lambda)$ are the same as those of $F(\lambda)$, except for the complex pair of eigenvalues $(\lambda_1, \bar{\lambda}_1)$ of $F(\lambda)$ that are replaced by a prescribed complex pair of numbers, $(\mu_1, \bar{\mu}_1)$.

THEOREM 3.2 (Assignment of a Pair of Complex Conjugate Eigenvalues). *Let Y_1 and Λ_1 be the same as those defined in (3.2) and (3.3). Let E_M be the same as that determined in Lemma 3.1. We define*

$$
\begin{aligned}
M_{new} &= M - MY_1E_MY_1^\top M,\\
C_{new} &= C + MY_1E_CY_1^\top K + KY_1E_C^\top Y_1^\top M,\\
K_{new} &= K - KY_1E_KY_1^\top K,
\end{aligned} \tag{3.7}
$$

where

$$
E_K = \Lambda_1^{-1}E_M\Lambda_1^{-\top}, \quad E_C = E_M\Lambda_1^{-\top}. \tag{3.8}
$$

Then the real symmetric pencil, $F_{new}(\lambda) = \lambda^2M_{new} + \lambda C_{new} + K_{new}$, has the following spectral property: the eigenvalues of the matrix pencil $F_{new}(\lambda)$ are the same as those of $F(\lambda)$ except that the complex conjugate eigenvalues $\lambda_1, \bar{\lambda}_1$ of $F(\lambda)$ are replaced by the complex conjugate numbers $\mu_1, \bar{\mu}_1$.

Proof. From (3.1), and the definitions of Y_1 and Λ_1, we see that (Λ_1, Y_1) is an eigenpair of $F(\lambda)$ and satisfies

$$
MY_1\Lambda_1^2 + CY_1\Lambda_1 + KY_1 = 0.
$$

Then, letting $\Lambda = \lambda I_2$, we have

$$
\begin{aligned}
F(\lambda)Y_1 &= (\lambda^2M + \lambda C + K)Y_1\\
&= (MY_1(\Lambda + \Lambda_1) + CY_1)(\Lambda - \Lambda_1).
\end{aligned} \tag{3.9}
$$

From (3.7)–(3.9), we have

$$
\begin{aligned}
F_{\text{new}}(\lambda) &= \lambda^2M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}\\
&= F(\lambda) + \lambda MY_1E_CY_1^\top K - KY_1E_KY_1^\top K + \lambda KY_1E_C^\top Y_1^\top M - \lambda^2MY_1E_MY_1^\top M\\
&= F(\lambda) + (CY_1 + MY_1(\Lambda + \Lambda_1))\Lambda_1E_K(Y_1^\top K - \lambda\Lambda_1^\top Y_1^\top M)\\
&= F(\lambda) + F(\lambda)Y_1(\Lambda - \Lambda_1)^{-1}\Lambda_1E_K(Y_1^\top K - \lambda\Lambda_1^\top Y_1^\top M).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\det[F_{\text{new}}(\lambda)] &= \det[F(\lambda) + F(\lambda)Y_1(\Lambda - \Lambda_1)^{-1}\Lambda_1 E_K(Y_1^\top K - \lambda\Lambda_1^\top Y_1^\top M)] \\
&= \det[F(\lambda)]\det[I_n + Y_1(\Lambda - \Lambda_1)^{-1}\Lambda_1 E_K(Y_1^\top K - \lambda\Lambda_1^\top Y_1^\top M)] \\
&= \det[F(\lambda)]\det[I_2 + (\Lambda - \Lambda_1)^{-1}\Lambda_1 E_K(I_2 - \lambda\Lambda_1^\top \Theta_1)] \\
&= \frac{\det[F(\lambda)]}{(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)}\det[(\lambda I_2 - \Lambda_1) + \Lambda_1 E_K(I_2 - \lambda\Lambda_1^\top\Theta_1)] \\
&= \frac{\det[F(\lambda)]}{(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)}\det[\lambda(I_2 - \Lambda_1 E_K \Lambda_1^\top \Theta_1) - \Lambda_1(I_2 - E_K)].
\end{aligned}
$$

Since E_M is determined in Lemma 3.1, μ_1 and $\bar{\mu}_1$ are a complex conjugate pair of eigenvalues of the matrix pair, $(\Lambda_1\Lambda_1^\top - E_M, \Lambda_1^\top - E_M\Theta_1\Lambda_1^\top)$. This implies

$$
\det[\lambda(I_2 - \Lambda_1 E_K \Lambda_1^\top \Theta_1) - \Lambda_1(I_2 - E_K)] = \frac{(\lambda - \mu_1)(\lambda - \bar{\mu}_1)}{\lambda_1\bar{\lambda}_1}.
$$

Therefore, $F_{\text{new}}(\lambda)$ has the same eigenvalues as $F(\lambda)$, except that λ_1, $\bar{\lambda}_1$ are replaced by μ_1, $\bar{\mu}_1$. ☐

According to Theorem 3.2, we can replace a complex conjugate pair of unwanted eigenvalues with real arithmetic. That means we can obtain the updated symmetric matrix pencil, $F_{\text{new}}(\lambda)$, without adopting any complex arithmetic, when an unwanted complex eigenvalue is replaced by a prescribed complex number.

Next, we show that the remaining eigenvalues and eigenvectors of $F(\lambda)$ remain the same on updating. Let $\lambda_2 = \alpha_2 + i\beta_2$ and $y_2 = y_{2r} + iy_{2i}$. We define Y_2 and Λ_2 in the same way as given in (3.1)–(3.3). Then (Λ_1, Y_1) and (Λ_2, Y_2) are eigenpairs of $F(\lambda)$, with $Y_1^\top KY_1 = I_2$ and $Y_2^\top KY_2 = I_2$. These give the following equations

$$
Y_2^\top KY_1 + Y_2^\top CY_1\Lambda_1 + Y_2^\top MY_1\Lambda_1^2 = 0, \tag{3.10}
$$

$$
Y_2^\top KY_1 + \Lambda_2^\top Y_2^\top CY_1 + \left(\Lambda_1^\top\right)^2 Y_2^\top MY_1 = 0. \tag{3.11}
$$

Eliminating terms involving "$Y_2^\top CY_1$" in (3.10) and (3.11), we have

$$
\Lambda_2^\top\left(Y_2^\top KY_1\right) - \left(Y_2^\top KY_1\right)\Lambda_1 + \Lambda_2^\top\left(Y_2^\top MY_1\right)\Lambda_1^2 - \left(\Lambda_2^\top\right)^2\left(Y_2^\top MY_1\right)\Lambda_1 = 0.
$$

We define $K_Y = Y_2^\top KY_1$, $M_Y = Y_2^\top MY_1$. Let $\otimes$ and $\text{vec}(\cdot)$ denote the Kronecker product and vectorizing operator, respectively. Then vectorizing the previous equation, we have

$$
\begin{aligned}
\left(I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I\right)\text{vec}(K_Y) &= \text{vec}\left(\Lambda_2^\top\left(\Lambda_2^\top M_Y - M_Y\Lambda_1\right)\Lambda_1\right) \\
&= \left(\Lambda_1^\top \otimes \Lambda_2^\top\right)\text{vec}\left(\Lambda_2^\top M_Y - M_Y\Lambda_1\right) \\
&= \left(\Lambda_1^\top \otimes \Lambda_2^\top\right)\left(I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I\right)\text{vec}(M_Y) \\
&= \left(I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I\right)\left(\Lambda_1^\top \otimes \Lambda_2^\top\right)\text{vec}(M_Y).
\end{aligned}
$$

Suppose $\lambda_1 \neq \lambda_2$, then $\text{spec}(\Lambda_1) \cap \text{spec}(\Lambda_2) = \emptyset$. This implies that the matrix $\left(I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I\right)$ is nonsingular and hence, $(\Lambda_1^\top \otimes \Lambda_2^\top)(\text{vec}(M_Y)) = \text{vec}(K_Y)$. Then,

$$
Y_2^\top KY_1 = \Lambda_2^\top\left(Y_2^\top MY_1\right)\Lambda_1. \tag{3.12}
$$

THEOREM 3.3 (Invariance of the Spectrum in Complex Eigenvalue Assignment). *Let (Λ_2, Y_2) be an eigenpair of $F(\lambda)$ with $Y_2^\top K Y_2 = I_2$. Suppose that* $\mathrm{spec}(\Lambda_1) \cap \mathrm{spec}(\Lambda_2) = \emptyset$. *Then, (Λ_2, Y_2) is also an eigenpair of $F_{new}(\lambda)$ given by (3.7).*

Proof. Since (Λ_2, Y_2) is an eigenpair of $F(\lambda)$, we have

$$MY_2\Lambda_2^2 + CY_2\Lambda_2 + KY_2 = 0.$$

From (3.7) it follows that

$$\begin{aligned}
&M_{\rm new}Y_2\Lambda_2^2 + C_{\rm new}Y_2\Lambda_2 + K_{\rm new}Y_2 \\
&= \left(M - MY_1E_MY_1^\top M\right)Y_2\Lambda_2^2 + \left(C + MY_1E_CY_1^\top K + KY_1E_C^\top Y_1^\top M\right)Y_2\Lambda_2 \\
&\quad + \left(K - KY_1E_KY_1^\top K\right)Y_2 \\
&= MY_2\Lambda_2^2 - MY_1E_MY_1^\top MY_2\Lambda_2^2 + CY_2\Lambda_2 + MY_1E_CY_1^TKY_2\Lambda_2 + KY_1E_C^\top Y_1^\top MY_2\Lambda_2 \\
&\quad + KY_2 - KY_1E_KY_1^TKY_2 \\
&= -MY_1\Lambda_1E_K\Lambda_1^\top Y_1^\top MY_2\Lambda_2^2 + MY_1\Lambda_1E_KY_1^\top KY_2\Lambda_2 + KY_1E_k\Lambda_1^\top Y_1^\top MY_2\Lambda_2 \\
&\quad - KY_1E_KY_1^\top KY_2 \\
&= MY_1\Lambda_1E_K\left(Y_1^\top KY_2\Lambda_2 - \Lambda_1^\top Y_1^\top MY_2\Lambda_2^2\right) + KY_1E_K\left(\Lambda_1^\top Y_1^\top MY_2\Lambda_2 - Y_1^\top KY_2\right).
\end{aligned}$$

From (3.12), the assertion holds. □

In this final section, we explore the relationship between the eigenvector associated with the complex eigenvalue λ_1 of the pencil $F(\lambda)$, and that of the complex eigenvalue μ_1 of the updated pencil $F_{new}(\lambda)$.

Let $\underline{\Omega}_1 = \left[\begin{smallmatrix} \varphi_1 & \psi_1 \\ -\psi_1 & \varphi_1 \end{smallmatrix}\right]$, where $\mu_1 = \varphi_1 + i\psi_1$ is a complex eigenvalue of $F_{\rm new}(\lambda)$, with $\psi_1 \neq 0$. From (3.4), there exists a nonsingular matrix, $V_1 \in \mathbb{R}^{2\times 2}$, such that

$$(I_2 - \Lambda_1E_K\Lambda_1^\top\Theta_1)V_1\underline{\Omega}_1 - \Lambda_1(I_2 - E_K)V_1 = 0.$$

By setting $\Omega_1 = V_1\underline{\Omega}_1V_1^{-1}$,

$$(I_2 - \Lambda_1E_K\Lambda_1^\top\Theta_1)\Omega_1 - \Lambda_1(I_2 - E_K) = 0. \tag{3.13}$$

The next theorem derives the relationship between the eigenvectors of $F_{\rm new}(\lambda)$ corresponding to μ_1, $\bar{\mu}_1$, and the eigenvectors of $F(\lambda)$ corresponding to λ_1, $\bar{\lambda}_1$ when the low rank transformation (3.7) is applied to replace the unwanted eigenvalues λ_1, $\bar{\lambda}_1$ by μ_1, $\bar{\mu}_1$.

THEOREM 3.4. *Let (Λ_1, Y_1) be an eigenpair of $F(\lambda)$ with $Y_1^\top KY_1 = I_2$, where Λ_1 derives from (3.3), and is similar to $\left[\begin{smallmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{smallmatrix}\right]$, with $\lambda_1 = \alpha_1 + i\beta_1$ being an isolated complex eigenvalue of $F(\lambda)$. Suppose that the non-equivalence transformation (3.7) is applied to replace the unwanted eigenvalues λ_1, $\bar{\lambda}_1$ by μ_1, $\bar{\mu}_1$. Then,*

$$M_{new}Y_1\Omega_1^2 + C_{new}Y_1\Omega_1 + K_{new}Y_1 = 0,$$

where Ω_1 satisfies (3.13), and is similar to $\left[\begin{smallmatrix} \varphi_1 & \psi_1 \\ -\psi_1 & \varphi_1 \end{smallmatrix}\right]$. Also, $\mu_1 = \varphi_1 + i\psi_1$ is an embedded eigenvalue of $F_{new}(\lambda)$.

Consequently, there exists a nonsingular matrix $T_1 \in \mathbb{C}^{2\times 2}$ such that

$$T_1\Omega_1T_1^{-1} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \bar{\mu}_1 \end{bmatrix},$$

and $\tilde{y}_1 = Y_1T_1e_1$ is the eigenvector of $F_{new}(\lambda)$, corresponding to the eigenvalue μ_1. Here, e_1 denotes the first column of the identity matrix.

Proof. From (3.7),

$$M_{\text{new}}Y_1\Omega_1^2 + C_{\text{new}}Y_1\Omega_1 + K_{\text{new}}Y_1 \tag{3.14}$$
$$= MY_1(\Omega_1^2 - E_M\Theta_1\Omega_1^2 + E_C\Omega_1) + CY_1\Omega_1 + KY_1(E_C^\top\Theta_1\Omega_1 + I_2 - E_K),$$

where $\Theta_1 = Y_1^\top MY_1$ and E_M, E_C, E_K are defined as in (3.8). Since (Λ_1, Y_1) is an eigenpair of $F(\lambda)$, and Ω_1 satisfies (3.13), we conclude that

$$\begin{aligned}
&CY_1\Omega_1 + KY_1(E_C^\top\Theta_1\Omega_1 + I_2 - E_K)\\
&= CY_1\Omega_1 + (-MY_1\Lambda_1^2 - CY_1\Lambda_1)(E_K\Lambda_1^\top\Theta_1\Omega_1 + I_2 - E_K)\\
&= -MY_1\Lambda_1^2(E_K\Lambda_1^\top\Theta_1\Omega_1 + I_2 - E_K) + CY_1[(I_2 - \Lambda_1E_K\Lambda_1^\top\Theta_1)\Omega_1 - \Lambda_1(I_2 - E_K)]\\
&= -MY_1\Lambda_1^2(E_K\Lambda_1^\top\Theta_1\Omega_1 + I_2 - E_K).
\end{aligned}$$

Therefore, (3.14) becomes

$$\begin{aligned}
&MY_1[\Omega_1^2 - E_M\Theta_1\Omega_1^2 + E_C\Omega_1 - \Lambda_1^2(E_K\Lambda_1^\top\Theta_1\Omega_1) - \Lambda_1^2(I_2 - E_K)]\\
&= MY_1[\Omega_1^2 - E_M\Theta_1\Omega_1^2 - \Lambda_1\Omega_1 + \Lambda_1E_K\Omega_1 + \Lambda_1\Omega_1 - \Lambda_1^2(E_K\Lambda_1^\top\Theta_1\Omega_1) - \Lambda_1^2(I_2 - E_K)]\\
&= MY_1[((I_2 - E_M\Theta_1)\Omega_1 - \Lambda_1(I_2 - E_K))\Omega_1 + \Lambda_1((I_2 - E_M\Theta_1)\Omega_1 - \Lambda_1(I_2 - E_K))]\\
&= 0.
\end{aligned}$$

Hence the theorem follows. □

Next, we provide an algorithm for assigning a pair of complex conjugate numbers to be eigenvalues of the updated symmetric matrix pencil.

ALGORITHM 3.1. (Assignment of a Pair of Complex Conjugate Eigenvalues)

Input:

- *An unwanted complex eigenvalue,* $\lambda_1 = \alpha_1 + i\beta_1$, α_1, $\beta_1 \in \mathbb{R}$, $\beta_1 \neq 0$, *and the corresponding eigenvector,* $y_1 = y_{1r} + iy_{1i}$, y_{1r}, $y_{1i} \in \mathbb{R}^n$, *with* y_{1r}, y_{1i} *being linearly independent;*
- *A pair of complex conjugate numbers,* μ_1 *and* $\bar{\mu}_1$, *which are assigned to be eigenvalues of the update symmetric matrix pencil;*
- *Symmetric matrices,* M, C *and* K, *such that* M *and* K *are positive definite.*

Output: *Symmetric matrices* M_{new}, C_{new} *and* K_{new}.

Step 1. *Use (3.2) and (3.3) to find the eigenpair* (Λ_1, Y_1) *of the original matrix pencil* $F(\lambda) = \lambda^2M + \lambda C + K$ *such that* $\text{spec}(\Lambda_1) = \{\lambda_1, \bar{\lambda}_1\}$ *and* $Y_1^\top KY_1 = I_2$.

Step 2. *Determine* ξ *and* η *by using formula (3.6).*

Step 3. *Let* $E_M = \left[\begin{smallmatrix} \xi & 0 \\ 0 & \eta \end{smallmatrix}\right]$, $E_K = \Lambda_1^{-1}E_M\Lambda_1^{-\top}$ *and* $E_C = E_M\Lambda_1^{-\top}$.

Step 4. *Update*

$$\begin{aligned}
M_{new} &= M - MY_1E_MY_1^\top M,\\
C_{new} &= C + MY_1E_CY_1^\top K + KY_1E_C^\top Y_1^\top M,\\
K_{new} &= K - KY_1E_KY_1^\top K.
\end{aligned}$$

Return

Suppose there are r pairs of complex conjugate eigenvalues which must be replaced by r pairs of complex conjugate numbers. We can repeatedly apply Algorithm 3.1 until the positive definite of updated matrix M_{new} or K_{new} is violated .

REMARK 3.2. *Above, we have discussed how to replace unwanted complex conjugate eigenvalues,* λ_1 *and* $\bar{\lambda}_1$, *by prescribed conjugate numbers,* μ_1 *and* $\bar{\mu}_1$, *whenever* *the real and the imaginary parts of the associated eigenvector, y_{1r} and y_{1i}, are linearly independent.*

We now consider the degenerate case where the real and the imaginary parts of the eigenvector, y_{1r} and y_{1i} are linearly dependent. In this case, the eigenvectors corresponding to λ_1 and $\bar{\lambda}_1$, are also linearly dependent. Hence, the eigenvector y_1 can be a real vector, i.e., $y_1 \in \mathbb{R}^n$. Since both (λ_1, y_1) and $(\bar{\lambda}_1, y_1)$ are eigenpairs of $F(\lambda)$, we have

$$
\begin{aligned}
\lambda_1^2 M y_1 + \lambda_1 C y_1 + K y_1 &= 0, \\
\bar{\lambda}_1^2 M y_1 + \bar{\lambda}_1 C y_1 + K y_1 &= 0.
\end{aligned}
$$

Then, we obtain $(\lambda_1 + \bar{\lambda}_1) M y_1 + C y_1 = 0$. This implies that $Cy_1 \,/\!/\, My_1$, and thus, $Ky_1 \,/\!/\, My_1$. Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix such that $Q^\top y_1 = e_1$. Here, e_1 stands for the first column vector of the identity matrix. When we let

$$
\widetilde{M} = Q^\top M Q, \quad \widetilde{C} = Q^\top C Q, \quad \widetilde{K} = Q^\top K Q,
$$

we can see that the first columns of $\widetilde{M}$, $\widetilde{C}$, $\widetilde{K}$ are mutually parallel. Furthermore, since $\widetilde{M}$, $\widetilde{C}$, $\widetilde{K}$ are symmetric, the first row vectors of $\widetilde{M}$, $\widetilde{C}$, $\widetilde{K}$ are also mutually parallel. Hence, if we apply a Gaussian elimination matrix, L, to eliminate those elements from the second component to the n-th component of the first column of $\widetilde{M}$, we know that the first columns and rows of the matrices $L\widetilde{M}L^\top$, $L\widetilde{C}L^\top$, $L\widetilde{K}L^\top$ are parallel to e_1. Hence the dimension of the quadratic problem can be reduced to $n-1$, whenever we deflate the first rows and columns of matrices $L\widetilde{M}L^\top$, $L\widetilde{C}L^\top$, $L\widetilde{K}L^\top$ simultaneously. In the above case, the unwanted eigenvalues λ_1 and $\bar{\lambda}_1$ can be deflated simultaneously, and the dimension of the considered system can be reduced to $n-1$.

REMARK 3.3. *Suppose the unwanted eigenvalue, λ_1, of $F(\lambda)$ is a root of a nonlinear divisor of $\det(F(\lambda))$. We cannot only exploit the associated eigenvectors to develop a low rank transformation for replacing the unwanted eigenvalue λ_1 by the prescribed number μ_1, because in this case, the dimension of the null space of $F(\lambda_1)$ is less than the multiplicity of λ_1. Hence, a low rank transformation for replacing unwanted eigenvalue λ_1 by μ_1 does not only consist of the eigenvectors corresponding to λ_1, but also of the generalized eigenvectors corresponding to λ_1. For simplicity, we illustrate here by letting $(\lambda - \lambda_1)^2$, $\lambda_1 = \alpha_1 + i\beta_1 \in \mathbb{C}$, be a nonlinear divisor of* $\det(F(\lambda))$ *and* $\operatorname{rank}(F(\lambda_1)) = n-1$. *To find the eigenvector y_1 corresponding to λ_1 and the associated generalized eigenvector z_1, we can apply [8, 11] to an enlarged linear eigenvalue problem derived from the quadratic problem, $F(\lambda)y = 0$. The formula (3.7)*

$$
\begin{aligned}
M_{new} &= M - MY_1 E_M Y_1^\top M, \\
C_{new} &= C + MY_1 E_C Y_1^\top K + KY_1 E_C^\top Y_1^\top M, \\
K_{new} &= K - KY_1 E_K Y_1^\top K
\end{aligned}
$$

can still be applied to replace λ_1 by μ_1, where Y_1 is derived from normalizing $[y_{1r}, y_{1i}, z_{1r}, z_{1i}]$ with $Y_1^\top K Y_1 = I_4$, and y_{1r}, z_{1r} (y_{1i}, z_{1i}) are respectively, the real (and imaginary) parts of y_1, z_1, and E_M is diagonal and is yet to be determined, $E_C = E_M \widetilde{\Lambda}_1^{-\top}$, $E_K = \widetilde{\Lambda}_1^{-1} E_M \widetilde{\Lambda}_1^{-\top}$. In addition, $\widetilde{\Lambda}_1 \in \mathbb{R}^{4\times 4}$ is similar to $\begin{bmatrix} \Lambda_1 & I \\ 0 & \Lambda_1 \end{bmatrix}$, and

$\Lambda_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}$. *To find E_M, we must solve the following system of equations:*

$$\begin{aligned} &\det(F_{new}(\mu_1)) = 0, \\ &\det(F_{new}(\bar{\mu}_1)) = 0, \\ &\frac{d}{d\lambda}\left[\det(F_{new}(\mu_1))\right] = 0, \\ &\frac{d}{d\lambda}\left[\det(F_{new}(\bar{\mu}_1))\right] = 0. \end{aligned}$$

4. Error Analysis for the Assignment of a Complex Conjugate Pair of Eigenvalues . We now give an error analysis for the non-equivalence transformation (3.7). Throughout this section, $\|A\|$ denotes a 2-norm of the matrix A. A matrix or a vector with a hat, say $\widehat{A}$ or $\widehat{v}$, means the resulting matrix or vector deriving from the numerical implementation of the matrix, A, or vector, v. We will let ε be the round-off unit. The higher order terms of ε are defined as "H.O.T.".

First, we estimate the error bounds for computing the matrix M_{new}. From (3.7), we have

$$\begin{aligned} \left\|\widehat{M}_{\text{new}} - M_{\text{new}}\right\| &= \|M\widehat{Y}_1\widehat{E}_M\widehat{Y}_1^\top M - MY_1E_MY_1^\top M\| \\ &\le \|M\|^2\|\widehat{Y}_1\widehat{E}_M\widehat{Y}_1^\top - Y_1E_MY_1^\top\|. \end{aligned} \tag{4.1}$$

By using the triangular inequality,

$$\begin{aligned} \|\widehat{Y}_1\widehat{E}_M\widehat{Y}_1^\top - Y_1E_MY_1^\top\| &\le \|\widehat{Y}_1\widehat{E}_M\widehat{Y}_1^\top - \widehat{Y}_1\widehat{E}_MY_1^\top\| + \|\widehat{Y}_1\widehat{E}_MY_1^\top - \widehat{Y}_1E_MY_1^\top\| \\ &\quad + \|\widehat{Y}_1E_MY_1^\top - Y_1E_MY_1^\top\| \\ &\le \|\widehat{Y}_1\widehat{E}_M\|\,\|\widehat{Y}_1 - Y_1\| + \|\widehat{Y}_1\|\,\|Y_1\|\,\|\widehat{E}_M - E_M\| + \|\widehat{Y}_1 - Y_1\|\,\|E_MY_1^\top\| \\ &\le \Big[\Big(\|\widehat{Y}_1 - Y_1\| + \|Y_1\|\Big)\|\widehat{E}_M - E_M\| + \|\widehat{Y}_1 - Y_1\|\,\|E_M\| + \|Y_1E_M\|\Big]\|\widehat{Y}_1 - Y_1\| \\ &\quad + \Big(\|\widehat{Y}_1 - Y_1\| + \|Y_1\|\Big)\|Y_1\|\,\|\widehat{E}_M - E_M\| + \|\widehat{Y}_1 - Y_1\|\,\|E_MY_1^\top\|. \end{aligned} \tag{4.2}$$

From the definition of Y_1 in (3.2), we have

$$\begin{aligned} \|\widehat{Y}_1 - Y_1\| &= \|\widehat{Z}_1\widehat{S}_1\widehat{D}_1^{-1} - Z_1S_1D_1^{-1}\| \\ &\le \Big[\Big(\|\widehat{Z}_1 - Z_1\| + \|Z_1\|\Big)\|\widehat{S}_1 - S_1\| + \|\widehat{Z}_1 - Z_1\|\,\|S_1\| + \|Z_1S_1\|\Big]\|\widehat{D}_1^{-1} - D_1^{-1}\| \\ &\quad + \Big(\|\widehat{Z}_1 - Z_1\| + \|Z_1\|\Big)\|D_1^{-1}\|\,\|\widehat{S}_1 - S_1\| + \|\widehat{Z}_1 - Z_1\|\,\|S_1D_1^{-1}\| \end{aligned} \tag{4.3}$$

From [12], we know the error bound for Z_1

$$\|\widehat{Z}_1 - Z_1\| \le c_1\varepsilon, \tag{4.4}$$

where

$$c_1 = \sum_{k=2}^{2n} \frac{\|Z_k\|}{|\lambda_k - \lambda_1|(1 + |\overline{\lambda}_k\lambda_1|)|z_k^H y_1|},$$

z_k and y_k are, respectively, the left and right eigenvector corresponding to the eigenvalue λ_k of $F(\lambda)$, $Z_k = [y_{kr} \;\; y_{ki}]$. From [12] again, the error bound for S_1 is

$$\|\widehat{S}_1 - S_1\| \le c_2\varepsilon, \tag{4.5}$$

where $c_2 = \chi(2)$ is a constant given by evaluating a low degree polynomial $\chi(\varpi)$. Since $D_1 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \in I\!\!R^{2\times 2}$, $d_1 > d_2$, and $S_1 \in I\!\!R^{2\times 2}$ is orthogonal, we have

$$\|D_1^{-1}\| = \|S_1 D_1^{-1}\| = \frac{1}{d_2} \tag{4.6}$$

$$\begin{aligned} \|\widehat{D}_1^{-1} - D_1^{-1}\| &= \|D_1^{-1}(D_1 - \widehat{D}_1)\widehat{D}_1^{-1}\| \\ &\le \|D_1^{-1}\|^2 \|D_1 - \widehat{D}_1\| + \text{H.O.T.} \\ &= \frac{d_1}{d_2^2}\varepsilon + \text{H.O.T.}. \end{aligned} \tag{4.7}$$

Substituting the upper bounds (4.4)–(4.7) into (4.3), we obtain

$$\|\widehat{Y}_1 - Y_1\| \le c_3\varepsilon + \text{H.O.T.}, \tag{4.8}$$

where

$$c_3 = \left(\frac{d_1}{d_2^2} + \frac{c_2}{d_2}\right)\|Z_1\| + \frac{c_1}{d_2}.$$

As $E_M = \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix}$, we have

$$\|\widehat{E}_M - E_M\| = \max\{|\widehat{\xi} - \xi|,\ |\widehat{\eta} - \eta|\}. \tag{4.9}$$

From (3.6) and relations (3.3)–(3.5), we know that

$$\begin{aligned} \xi &= \xi(\lambda_1, \mu_1) = \xi(\alpha_1, \beta_1, \varphi_1, \psi_1), \\ \eta &= \eta(\lambda_1, \mu_1) = \eta(\alpha_1, \beta_1, \varphi_1, \psi_1), \end{aligned}$$

where $\lambda_1 = \alpha_1 + i\beta_1$ and $\mu_1 = \varphi_1 + i\psi_1$. In addition,

$$\begin{aligned} \widehat{\xi} &= \widehat{\xi}(\alpha_1, \beta_1, \varphi_1, \psi_1) = \xi(\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\varphi}_1, \widehat{\psi}_1) + \text{H.O.T.}, \\ \widehat{\eta} &= \widehat{\eta}(\alpha_1, \beta_1, \varphi_1, \psi_1) = \eta(\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\varphi}_1, \widehat{\psi}_1) + \text{H.O.T.}. \end{aligned}$$

Then,

$$\widehat{\xi} - \xi = \frac{\partial \xi}{\partial \alpha_1}\Delta\alpha + \frac{\partial \xi}{\partial \beta_1}\Delta\beta + \frac{\partial \xi}{\partial \varphi_1}\Delta\varphi + \frac{\partial \xi}{\partial \psi_1}\Delta\psi + \text{H.O.T.} \tag{4.10}$$

$$\widehat{\eta} - \eta = \frac{\partial \eta}{\partial \alpha_1}\Delta\alpha + \frac{\partial \eta}{\partial \beta_1}\Delta\beta + \frac{\partial \eta}{\partial \varphi_1}\Delta\varphi + \frac{\partial \eta}{\partial \psi_1}\Delta\psi + \text{H.O.T.} \tag{4.11}$$

where $\Delta\alpha = \widehat{\alpha}_1 - \alpha_1$, $\Delta\beta = \widehat{\beta}_1 - \beta_1$, $\Delta\varphi = \widehat{\varphi}_1 - \varphi_1$, and $\Delta\psi = \widehat{\psi}_1 - \psi_1$. Since $\mu_1 = \varphi + i\psi$ is a prescribed number, we need not calculate it. Thus, $\Delta\varphi$ and $\Delta\psi$ are usually much smaller than $\Delta\alpha$ or $\Delta\beta$. We can hence ignore terms involving $\Delta\varphi$ or $\Delta\psi$ in equations (4.10) and (4.11). Hence, we are only concerned with those terms related to $\Delta\alpha$ or $\Delta\beta$ in the estimation of the error bounds for ξ and η. Thus, we have

$$\begin{aligned} |\widehat{\xi} - \xi| &\le \left|\frac{\partial \xi}{\partial \alpha_1}\right| |\Delta\alpha| + \left|\frac{\partial \xi}{\partial \beta_1}\right| |\Delta\beta| + \text{H.O.T.} \le \left(\left|\frac{\partial \xi}{\partial \alpha_1}\right| + \left|\frac{\partial \xi}{\partial \beta_1}\right|\right) |\widehat{\lambda}_1 - \lambda_1| + \text{H.O.T.}, \\ |\widehat{\eta} - \eta| &\le \left|\frac{\partial \eta}{\partial \alpha_1}\right| |\Delta\alpha| + \left|\frac{\partial \eta}{\partial \beta_1}\right| |\Delta\beta| + \text{H.O.T.} \le \left(\left|\frac{\partial \eta}{\partial \alpha_1}\right| + \left|\frac{\partial \eta}{\partial \beta_1}\right|\right) |\widehat{\lambda}_1 - \lambda_1| + \text{H.O.T.}. \end{aligned}$$

After performing a tedious calculation, as well as applying the chain rule to some functions of several variables, we can conclude that $\left|\frac{\partial \xi}{\partial \alpha_1}\right|$, $\left|\frac{\partial \xi}{\partial \beta_1}\right|$, $\left|\frac{\partial \eta}{\partial \alpha_1}\right|$ and $\left|\frac{\partial \eta}{\partial \beta_1}\right|$ are bound by the relative rational functions in α_1, β_1 and $|\lambda_1|$. More precisely, we conclude that

$$|\widehat{\xi} - \xi| \leq \frac{|\zeta_1(\alpha_1, \beta_1)|}{\zeta_2(|\lambda_1|)} |\widehat{\lambda}_1 - \lambda_1| + \text{H.O.T.}, \tag{4.12}$$

$$|\widehat{\eta} - \eta| \leq \frac{|\varsigma_1(\alpha_1, \beta_1)|}{\varsigma_2(|\lambda_1|)} |\widehat{\lambda}_1 - \lambda_1| + \text{H.O.T.}, \tag{4.13}$$

where ζ_1, ς_1 are low degree polynomials in α_1, β_1, and ζ_2, ς_2 are low degree polynomials in $|\lambda_1|$. Since $|\lambda_1| \neq 0$, ζ_2 and ς_2 are nonzero. Hence, both bounds in (4.12) and (4.13) are finite. By using [12], we know

$$|\widehat{\lambda}_1 - \lambda_1| \leq \frac{1}{(1 + |\lambda_1|^2)|z_1^H y_1|} \varepsilon. \tag{4.14}$$

Substituting (4.12)–(4.14) into (4.9), the error estimation of E_M can be derived using the following:

$$\|\widehat{E}_M - E_M\| = \max\{|\widehat{\xi} - \xi|,\ |\widehat{\eta} - \eta|\} \leq c_4 \varepsilon + \text{H.O.T.}, \tag{4.15}$$

where

$$\begin{aligned} c_4 &= \frac{\zeta(\alpha_1, \beta_1)}{\varsigma(|\lambda_1|)} \frac{1}{(1 + |\lambda_1|^2)|z_1^H y_1|}, \\ \zeta(\alpha_1, \beta_1) &= \max\{|\zeta_1(\alpha_1, \beta_1)|,\ |\varsigma_1(\alpha_1, \beta_1)|\}, \\ \varsigma(|\lambda_1|) &= \min\{\zeta_1(|\lambda_1|),\ \varsigma_1(|\lambda_1|)\}. \end{aligned}$$

By using (4.2), (4.8) and (4.15), we obtain an upper bound for

$$\|\widehat{Y}_1 \widehat{E}_M \widehat{Y}_1^\top - Y_1 E_M Y_1^\top\| \leq \left[2\|Y_1\|\,\|E_M\| c_3 + \|Y_1\|^2 c_4\right] \cdot \varepsilon + \text{H.O.T.}.$$

Hence, the error bound for M_{new} is

$$\left\|\widehat{M}_{\text{new}} - M_{\text{new}}\right\| \leq \|M\|^2 \left[2\|Y_1\|\,\|E_M\| c_3 + \|Y_1\|^2 c_4\right] \cdot \varepsilon + \text{H.O.T.}. \tag{4.16}$$

To estimate the error bounds for C_{new} and K_{new}, we first need to find the error bound for Λ_1^{-1}. From (3.3), we have

$$\|\widehat{\Lambda}_1^{-1} - \Lambda_1^{-1}\| = \|\widehat{D}_1 \widehat{\underline{\Lambda}}_1^{-1} \widehat{D}_1^{-1} - D_1 \underline{\Lambda}_1^{-1} D_1^{-1}\| \leq c_5 \varepsilon + \text{H.O.T.},$$

where

$$c_5 = \frac{d_1}{d_2} \left(\frac{d_1}{d_2 |\lambda_1|} + \frac{1}{|\lambda_1|} + \frac{1}{(1 + |\lambda_1|^2)|z_1^H y_1|} \right).$$

Hence, by a similar process as above, we obtain the error bounds for C_{new} and K_{new} as given below:

$$\left\|\widehat{C}_{\text{new}} - C_{\text{new}}\right\| \leq 2\|M\|\,\|K\| \left[\|E_M\| \left(\frac{d_1 c_3}{d_2 |\lambda_1|} + c_5 \right) + \frac{c_4}{|\lambda_1|} \right] \cdot \varepsilon + \text{H.O.T.}, \tag{4.17}$$

$$\left\| \widehat{K}_{\text{new}} - K_{\text{new}} \right\| \leq \|K\|^2 \left[2\|Y_1\| \, \|E_M\| \frac{d_1^2 c_3}{d_2^2 |\lambda_1|^2} + \|Y_1\|^2 \left(\frac{2d_1}{d_2|\lambda_1|} \|E_M\| c_5 + \frac{d_1^2 c_4}{d_2^2 |\lambda_1|^2} \right) \right] \cdot \varepsilon + \text{H.O.T.}. \tag{4.18}$$

Consequently, from (4.16), (4.17) and (4.18), we know that the non-equivalence transformation (3.7) is numerically backward stable.

5. Simultaneous Assignment of Several Real Eigenvalues. So far, we have considered the problem of assigning either one real or a pair of complex conjugate eigenvalues. In this section, we consider the simultaneous assignment of several real eigenvalues.

It is to be noted that it may not always be possible to assign an arbitrary number of real eigenvalues $\{\mu_1, \ldots, \mu_r\}$ to the spectrum of $F_{\text{new}}(\lambda)$. This is because of the requirement that the mass and stiffness matrices remain positive definite, which is crucial to preserve symmetry, and which may be violated at some stage of the assignment process.

Given r real numbers, $\{\mu_1, \ldots, \mu_r\}$, the following method computes a positive integer, $m_r \leq r$, the matrices W and U, and the diagonal matrices D_M, D_C and D_K, such that the symmetric matrix pencil, $F_{\text{new}}(\lambda) = \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}$, has the spectrum

$$\text{spec}(F_{\text{new}}(\lambda)) = \{\mu_1, \ldots, \mu_{m_r}, \lambda_{m_r+1}, \ldots, \lambda_{2n}\},$$

where

$$M_{\text{new}} = M - W D_M W^\top \tag{5.1a}$$

$$C_{\text{new}} = C + U D_C W^\top + W D_C U^\top \tag{5.1b}$$

$$K_{\text{new}} = K - U D_K U^\top. \tag{5.1c}$$

It is easily seen that the non-equivalence transformation (2.3) can be successively exploited to assign a sequence of real eigenvalues, $\{\mu_1, \ldots, \mu_{m_r}\}$, to the updated symmetric matrix pencil, $F_{\text{new}}(\lambda)$. We obtain recursive formulae: For $s = 1, \ldots, m_r$,

$$\begin{aligned} M_s &= M_{s-1} - \varepsilon_s \lambda_s M_{s-1} y_s y_s^\top M_{s-1} \\ C_s &= C_{s-1} + \varepsilon_s (M_{s-1} y_s y_s^\top K_{s-1} + K_{s-1} y_s y_s^\top M_{s-1}) \\ K_s &= K_{s-1} - \frac{\varepsilon_s}{\lambda_s} K_{s-1} y_s y_s^\top K_{s-1} \end{aligned} \tag{5.2}$$

whenever $M_0 = M$, $C_0 = C$ and $K_0 = K$, and θ_s and ε_s are given by

$$\theta_s = y_s^\top M_{s-1} y_s \quad \text{and} \quad \varepsilon_s = \frac{\lambda_s - \mu_s}{1 - \lambda_s \mu_s \theta_s}. \tag{5.3}$$

These formulae can be repeatedly applied to assign a sequence of real eigenvalues, and to replace an unwanted real eigenvalue by a real number at a time.

The method proposed below delays the updating of the coefficient matrices until all the real numbers, $\{\theta_s\}$ and $\{\varepsilon_s\}$, needed for the multi-assignment, have been computed. After all these quantities have been computed, the coefficient matrices are updated with only one rank-m_r symmetric update. The process will not only be more efficient than that which assigns one eigenvalue at a time, but it will be rich in BLAS-3 operations, such as matrix-matrix multiplications, rank-r updates, etc., which will make it suitable for high performance computing.

To develop formula (5.1a), and considering the m_r-th iteration of (5.2), we observe that

$$
\begin{aligned}
(5.4)\quad M_{m_r} &= M_0 - \sum_{s=1}^{m_r} \varepsilon_s \lambda_s M_{s-1} y_s y_s^\top M_{s-1} \\
&= M_0 - [M_0 y_1, \cdots, M_{m_r-1} y_{m_r}] \begin{bmatrix} \varepsilon_1 \lambda_1 & & \\ & \ddots & \\ & & \varepsilon_{m_r} \lambda_{m_r} \end{bmatrix} [M_0 y_1, \cdots, M_{m_r-1} y_{m_r}]^\top .
\end{aligned}
$$

We also observe that, for $s = 1, \ldots, m_r$,

$$
\begin{aligned}
(5.5)\qquad \theta_s &= y_s^\top M_{s-1} y_s \\
&= y_s^\top \left[M_{s-2} - \varepsilon_{s-1} \lambda_{s-1} M_{s-2} y_{s-1} y_{s-1}^\top M_{s-2} \right] y_s \\
&= y_s^\top M_{s-2} y_s - \varepsilon_{s-1} \lambda_{s-1} \left(y_s^\top M_{s-2} y_{s-1} \right) \left(y_{s-1}^\top M_{s-2} y_s \right)
\end{aligned}
$$

and

$$
\begin{aligned}
(5.6)\qquad M_{s-1} y_s &= \left[M_{s-2} - \varepsilon_{s-1} \lambda_{s-1} M_{s-2} y_{s-1} y_{s-1}^\top M_{s-2} \right] y_s \\
&= M_{s-2} y_s - \varepsilon_{s-1} \lambda_{s-1} \left(y_{s-1}^\top M_{s-2} y_s \right) M_{s-2} y_{s-1} .
\end{aligned}
$$

Therefore, formula (5.1a) can be derived from (5.4) by letting

$$
W = [M_0 y_1, \cdots, M_{m_r-1} y_{m_r}] \quad \text{and} \quad D_M = \begin{bmatrix} \varepsilon_1 \lambda_1 & & \\ & \ddots & \\ & & \varepsilon_{m_r} \lambda_{m_r} \end{bmatrix} .
$$

In addition, matrices D_M and W can be determined by using recursions (5.5) and (5.6).

Similarly, formulae (5.1b) and (5.1c) can be obtained for the appropriate matrices U, D_K and D_C.

We now provide an algorithm for the simultaneous assignment of several real eigenvalues.

ALGORITHM 5.1. (Simultaneous Assignment of Real Eigenvalues)

Input:

- *A set of real numbers* $\{\mu_i\}_{i=1}^r$*;*
- *A set of unwanted real eigenpairs* $\{(\lambda_i, y_i)\}_{i=1}^r$*;*
- *Symmetric matrices* M*,* C *and* K *such that* M*, and* K *are positive definite.*

Output: *Integer* m_r*, symmetric matrices* M_{new}*,* C_{new} *and* K_{new}*.*

Step 1. *Compute* $m_i = My_i$, $k_i = Ky_i$, $i = 1, \ldots, r$.
Step 2. *Compute* $\alpha_{ij} = y_i^\top m_j$, $\beta_{ij} = y_i^\top k_j$, $j = i, \ldots, r$, $i = 1, \ldots, r$.
Step 3. $\eta_1 = \sqrt{y_1^\top K y_1}$*;*
Update $\alpha_{1j} = \alpha_{1j}/\eta_1$, $\beta_{1j} = \beta_{1j}/\eta_1$, $j = 1, \ldots, r$.
Step 4. *Set* $\varepsilon_1 = \frac{\lambda_1 - \mu_1}{1 - \lambda_1 \mu_1 \alpha_{11}}$.
Step 5. *For* $s = 2, \ldots, r$,
For $i = s, \ldots, r$,
For $j = i, \ldots, r$,
Update $\alpha_{ij} = \alpha_{ij} - \varepsilon_{s-1} \lambda_{s-1} \alpha_{s-1,i} \alpha_{s-1,j}$, $\beta_{ij} = \beta_{ij} - \frac{\varepsilon_{s-1}}{\lambda_{s-1}} \beta_{s-1,i} \beta_{s-1,j}$
End for j

End for i
Compute $\varepsilon_s = \frac{\lambda_s - \mu_s}{1-\lambda_s\mu_s\alpha_{ss}}$.
If $\beta_{ss} > 0$, *then*
Compute $\eta_s = \sqrt{\beta_{ss}}$.
Update $\alpha_{i,s} = \alpha_{i,s}/\eta_s, \quad \beta_{i,s} = \beta_{i,s}/\eta_s, \quad i = 1, \ldots, s.$
Update $\alpha_{s,j} = \alpha_{s,j}/\eta_s, \quad \beta_{s,j} = \beta_{s,j}/\eta_s, \quad j = 1, \ldots, s.$
Compute $m_r = s$.
Else Exit Loop.
End for s.

Step 6. *Normalize* $m_i = m_i/\eta_i,\ k_i = k_i/\eta_i,\ i = 1, \ldots, m_r$.

Step 7. *For* $s = 2, \ldots, m_r$,
For $i = s, \ldots, m_r$,
Update $m_i = m_i - \varepsilon_{s-1}\lambda_{s-1}\alpha_{s-1,i}m_{s-1},\ k_i = k_i - \frac{\varepsilon_{s-1}}{\lambda_{s-1}}\beta_{s-1,i}k_{s-1}$.
End for i.
End for s.

Step 8. *Set* $W = [m_1, m_2, \cdots, m_{m_r}]$, $U = [k_1, k_2, \cdots, k_{m_r}]$, $D_M = diag(\varepsilon_1\lambda_1, \varepsilon_2\lambda_2, \cdots, \varepsilon_{m_r}\lambda_{m_r})$,
$D_C = diag(\lambda_1, \lambda_2, \cdots, \lambda_{m_r})$ *and* $D_K = diag\left(\frac{\varepsilon_1}{\lambda_1}, \frac{\varepsilon_2}{\lambda_2}, \cdots, \frac{\varepsilon_{m_r}}{\lambda_{m_r}}\right)$.

Step 9. *Update*

$$\begin{aligned} M_{new} &= M - WD_MW^\top, \\ C_{new} &= C + UD_CW^\top + WD_CU^\top, \\ K_{new} &= K - UD_KU^\top. \end{aligned}$$

Return

To show the efficiency of the simultaneous assignment process, we compare the *flop counts* of Algorithm 5.1, with those of the successive assignment strategy by using non-equivalence transformation (2.3). In table 5.1, we list the *flop counts* of these two methods.

TABLE 5.1
Approximate flop counts for embedding r ($r \ll n$) real eigenvalues.

strategy	parameters	M_{new}, C_{new}, K_{new}	total
r sequential assignment	$6n^2r$	$\frac{7n^2r}{2}$	$\frac{19n^2r}{2}$
simultaneous assignment	$4n^2r + 4nr^2$	$3n^2r + 3nr$	$7n^2r + 4nr^2$

From Table 5.1, we observe that the simultaneous assignment method is more efficient than the successive assignment method.

6. NUMERICAL RESULTS. In this section, we illustrate the efficiency and reliability of non-equivalence transformations using two examples. The first example calculates a 6×6 artificial quadratic matrix pencil. The second calculates a set of experimental data arising from the aerospace industry [5]. All numerical implementations were performed on a IBM Pentium III machine using MATLAB 5.3.

6.1. Example 1: Real Multi-Embedding, $n = 6$.. The system matrices are evaluated by letting

$$M = \begin{bmatrix} 3.5870 & 0.2170 & 0.2250 & -1.3460 & 0.1700 & 1.7140 \\ 0.2170 & 3.3970 & -0.1280 & -0.4700 & -0.3040 & -1.0940 \\ 0.2250 & -0.1280 & 4.6260 & -0.6600 & 0.7070 & -0.5020 \\ -1.3640 & -0.4700 & -0.6600 & 1.9520 & 0.1920 & 0.1960 \\ 0.1700 & -0.3040 & 0.7070 & 0.1920 & 4.9550 & -0.1060 \\ 1.7140 & -1.0940 & -0.5020 & 0.1960 & -0.1060 & 3.7070 \end{bmatrix},$$

$$C = \begin{bmatrix} 5.4240 & 0.0520 & -0.6670 & 0.6660 & 0.2420 & 0.0780 \\ 0.0520 & 4.8520 & 0.1530 & 0.4400 & -0.4330 & 0.2010 \\ -0.6670 & 0.1530 & 4.8360 & -0.3960 & 0.0980 & -0.4510 \\ 0.6660 & 0.4400 & -0.3960 & 4.9830 & -0.0740 & -0.6460 \\ 0.2420 & -0.4330 & 0.0980 & -0.0740 & 5.0700 & 0.2550 \\ 0.0780 & 0.2010 & -0.4510 & -0.6460 & 0.2550 & 5.2560 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.8090 & 0.2850 & 0.2180 & 0.2260 & 0.6230 & -1.9020 \\ 0.2850 & 4.4470 & 0.5910 & 0.3360 & -0.9340 & -0.6080 \\ 0.2180 & 0.5910 & 4.5720 & 0.9170 & 0.8890 & 0.1960 \\ 0.2260 & 0.3360 & 0.9170 & 3.2480 & 0.8770 & 0.6790 \\ 0.6230 & -0.9340 & 0.8890 & 0.8770 & 5.1250 & -0.0810 \\ -1.9020 & -0.6080 & 0.1960 & 0.6790 & -0.0810 & 4.4910 \end{bmatrix}.$$

The pencil $F(\lambda) = \lambda^2 M + \lambda C + K$ has four real eigenvalues and four pair of complex conjugate eigenvalues. The four real eigenvalues and corresponding eigenvectors are

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-7.6759, -0.1151, -0.5974, -0.7853\},$$

$$[y_1, y_2, y_3, y_4] = \begin{bmatrix} 0.1540 & 1.0000 & 0.1012 & 0.1740 \\ -0.0231 & 0.0236 & 0.1533 & 0.1014 \\ 0.0055 & -0.0665 & 0.0115 & -0.0524 \\ 0.1821 & -0.1546 & -0.4940 & -0.1844 \\ -0.0200 & -0.0687 & 0.1641 & 0.0766 \\ -0.1087 & 0.5341 & 0.3397 & 0.2204 \end{bmatrix}.$$

Suppose that we assign all these four real eigenvalues to the set

$$\{\mu_1, \mu_2, \mu_3, \mu_4\} = \{-2, -4, -6, -8\}.$$

Algorithm 5.1 returns $m_r = 3$, i.e., no more than three eigenvalues can be assigned. The matrices W, U, and the diagonal matrices, D_M, D_C and D_K, produced by Algorithm 5.1 for assigning the first three eigenvalues are:

$$W = \begin{bmatrix} 0.2318 & 5.3897 & -0.9753 \\ -0.0128 & -0.2207 & 0.5054 \\ -0.0338 & -0.3873 & 0.6161 \\ 0.2659 & -0.9900 & -1.7750 \\ -0.0318 & -0.3940 & 0.9704 \\ -0.1611 & 3.4628 & 0.5725 \end{bmatrix}, \quad U = \begin{bmatrix} 1.0383 & 0.7810 & -1.8654 \\ 0.1847 & 0.0479 & 0.0863 \\ 0.3533 & -0.1613 & 0.0490 \\ 1.0863 & 0.0199 & -1.5705 \\ 0.3845 & 0.0281 & 0.1865 \\ -1.3080 & 0.3267 & 0.8683 \end{bmatrix},$$

$$D_M = \begin{bmatrix} -19.7295 & 0 & 0 \\ 0 & 0.1583 & 0 \\ 0 & 0 & 0.8708 \end{bmatrix}, \ D_M = \begin{bmatrix} 2.5702 & 0 & 0 \\ 0 & -1.0478 & 0 \\ 0 & 0 & -1.4576 \end{bmatrix}, \ D_M = \begin{bmatrix} -0.3348 & 0 & 0 \\ 0 & 6.9338 & 0 \\ 0 & 0 & 2.4400 \end{bmatrix}.$$

The updated matrices M_{new}, C_{new} and K_{new} are given by

$$M_{\text{new}} = M - WD_MW^\top = \begin{bmatrix} -0.7806 & 0.7762 & 0.9242 & -0.7923 & 1.1852 & -1.4921 \\ 0.7762 & 3.1701 & -0.4042 & 0.2096 & -0.7369 & -1.1844 \\ 0.9242 & -0.4042 & 4.2943 & 0.0543 & 0.1834 & -0.4894 \\ -0.7923 & 0.2096 & 0.0543 & 0.4483 & 1.4637 & 0.7784 \\ 1.1852 & -0.7369 & 0.1834 & 1.4637 & 4.1302 & -0.2728 \\ -1.4921 & -1.1844 & -0.4894 & 0.7784 & -0.2728 & 2.0350 \end{bmatrix},$$

$$C_{\text{new}} = C + UD_CW^\top + WD_CU^\top = \begin{bmatrix} -7.4637 & 1.5348 & 2.4259 & -4.3381 & 3.4539 & -3.0192 \\ 1.5348 & 4.7349 & -0.0060 & 1.9651 & -0.6938 & -0.6426 \\ 2.4259 & -0.0060 & 2.5558 & 1.1290 & -0.2562 & -0.5866 \\ -4.3381 & 1.9651 & 1.1290 & -1.6177 & 2.8416 & 1.8342 \\ 3.4539 & -0.6938 & -0.2562 & 2.8416 & 4.5028 & -1.1485 \\ -3.0192 & -0.6426 & -0.5866 & 1.8342 & -1.1485 & 2.5193 \end{bmatrix},$$

$$K_{\text{new}} = K - UD_KU^\top = \begin{bmatrix} -10.5496 & 0.4824 & 1.4372 & -6.6520 & 1.4533 & -0.1738 \\ 0.4824 & 4.4243 & 0.6561 & 0.7272 & -0.9588 & -0.9802 \\ 1.4372 & 0.6561 & 4.4276 & 1.2554 & 0.9436 & 0.3028 \\ -6.6520 & 0.7272 & 1.2554 & -2.3777 & 1.7276 & 3.4856 \\ 1.4533 & -0.9588 & 0.9436 & 1.7276 & 5.0842 & -0.7082 \\ -0.1738 & -0.9802 & 0.3028 & 3.4856 & -0.7082 & 2.4841 \end{bmatrix}.$$

The matrices $M_{\text{new}}, C_{\text{new}}, K_{\text{new}}$ are symmetric and it is easily verified that $\{\mu_1, \mu_2, \mu_3\} = \{-2.0000, -4.0000, -6.0000\} \in \text{spec}(F_{\text{new}}(\lambda))$, where $F_{\text{new}}(\lambda) = \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}$.
Furthermore, if we let $Y_1 = [y_1, y_2, y_3]$ and $\Omega_1 = \text{diag}\{\mu_1, \mu_2, \mu_3\}$, then

$$\|M_{\text{new}} Y_1 \Omega_1^2 + C_{\text{new}} Y_1 \Omega_1 + K_{\text{new}} Y_1\|_2 = 3.3229 \times 10^{-13}$$

which shows the accuracy of Algorithm 5.1 for Example 1.

6.2. Example 2. The test matrices K, C, M in this example arise from an aerospace industry problem in constructing aircraft structural models [4, 5].

In this example, we repeatedly apply Algorithm 3.1 ten times to replace the 10 pairs of unwanted complex conjugate eigenvalues by the prescribed conjugate numbers, which are the eigenvalues of the realized system. Figures 6.1–6.6 sketch the numerical implementation of Example 2. Figure 6.1 shows the absolute errors between matrices M and M_{new}. To examine these absolute errors in detail, we consider the matrix $\log M$ by letting

$$\log M(i,j) = \begin{cases} \log_{10} |M_{\text{new}}(i,j) - M(i,j)|, & \text{if } |M_{\text{new}}(i,j) - M(i,j)| > 1.e4, \\ 0, & \text{otherwise.} \end{cases}$$

The resulting matrix is sketched in Figure 6.2. Since the resulting matrix $\log M$ is symmetric, for brevity, only the upper triangular part of the matrix is plotted in Figure 6.2. Similarly, we also sketch matrices $|C_{\text{new}} - C|$, the logarithm of $|C_{\text{new}} - C|$, $|K_{\text{new}} - K|$ and the logarithm of $|K_{\text{new}} - K|$ in Figures 6.3–6.6, respectively.

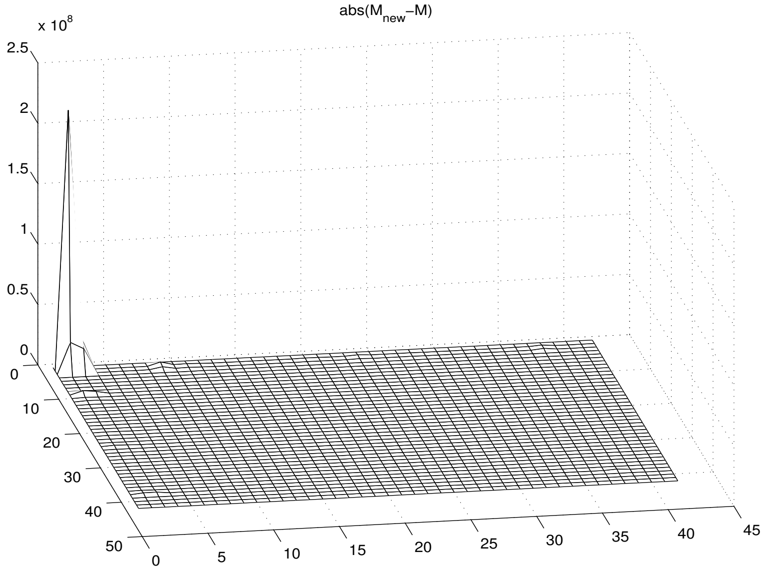

FIG. 6.1. *The absolute error of* $|M_{new} - M|$.

From these figures, we know that the non-equivalence transformations are suitable, and reliable, for model tuning problems in practice, because the figures show that only a few rows or columns require modification. Obviously, from these figures, we see that the larger errors always cluster around the third and 37-th rows/columns. This result is similar to the result in [4]. Hence, the effectiveness of the non-equivalence transformations satisfies engineering expectations.

Theoretically, the non-equivalence transformations (2.3) and (3.7) can be applied repeatedly until the product matrices K_{new} or M_{new} are no longer positive semi-definite. From the numerical implementation of Example 2, the order of the replacement of the unwanted eigenvalues affects the accuracy of the numerical results. Therefore, the order also affects the positivity of matrices K_{new} and M_{new}. Thus, the order of replacement of the unwanted eigenvalues decides the effectiveness of the low rank transformations.

7. CONCLUSIONS. In this paper, we have proposed low rank non-equivalence transformations for eigenvalue embedding problems in a quadratic vibrating system. The transformations equip with several characters that can be nicely fitted into real applications. First, the transformations preserve the symmetries and positive semi-definiteness of the matrices in the quadratic system. The approaches thus allowed us to embed desired spectrum in the quadratic system without changing other eigen-structures. Such embedding is especially important for the experiments that needed to be re-designed by adopting a backward transformation. Besides, when the updating eigenvalues are complex, the non-equivalence transformations need only real arithmetic environment. Furthermore, while a complex conjugate pair of eigenvalues is assigned, the error analysis suggests that the non-equivalence transformations are numerical backward stable. Finally, the numerical experiments demonstrated that the computational results satisfying the engineering requirements.

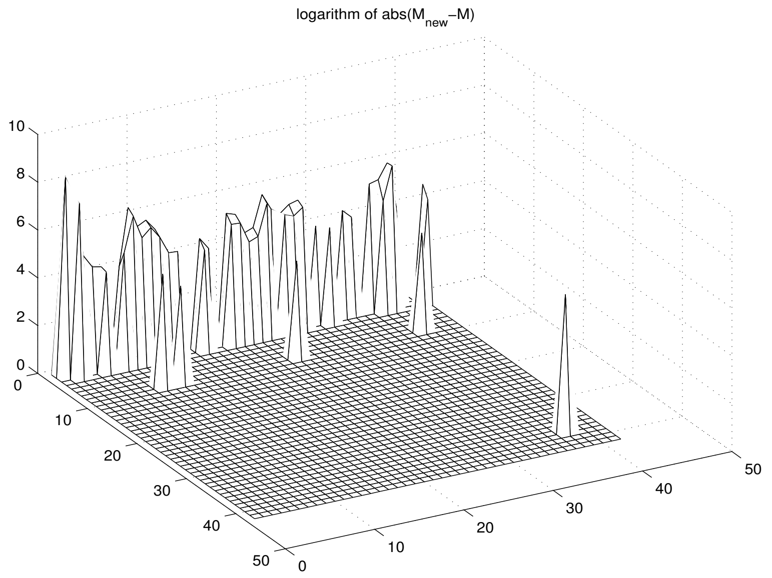

FIG. 6.2. *Logarithm of the absolute error of* $|M_{new} - M|$.

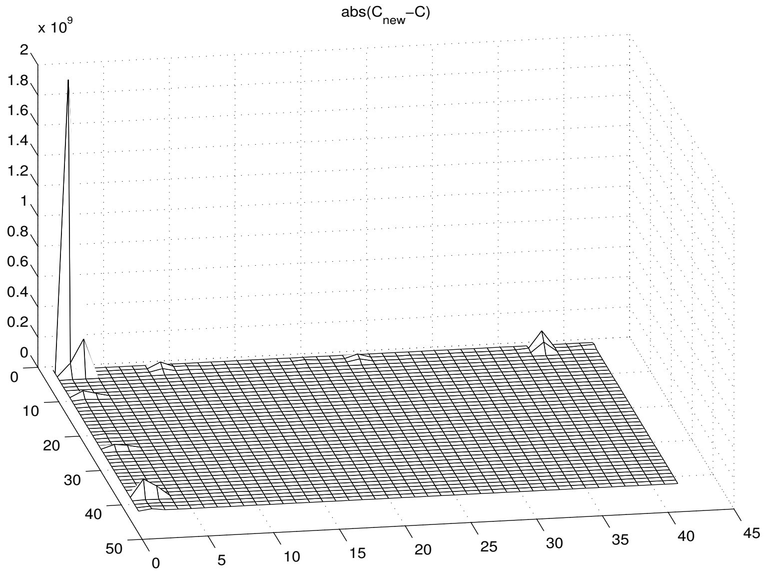

FIG. 6.3. *The absolute error of* $|C_{new} - C|$.

REFERENCES

[1] B. N. DATTA, S. ELHAY, AND Y. M. RAM, *Orthogonality and partial pole assignment for the symmetric definite quadratic pencil*, Lin. Alg. Appl., 257 (1997), pp. 29–48.
[2] B. N. DATTA, S. ELHAY, Y. M. RAM, AND D. R. SARKISSIAN, *Partial eigenstructure assignment for the quadratic pencil*, J. Sound and Vibration, 230 (2000), pp. 101–110.
[3] P. DEWILDE AND P. VANDEWALLE, *On the factorization of a nonsingular rational matrix*, IEEE Trans. Circuits and Systems, 22 (1975), pp. 637–645.
[4] W. R. FERNG, W.-W. LIN, D. J. PIERCE, AND C.-S. WANG, *Nonequivalence transformation of*

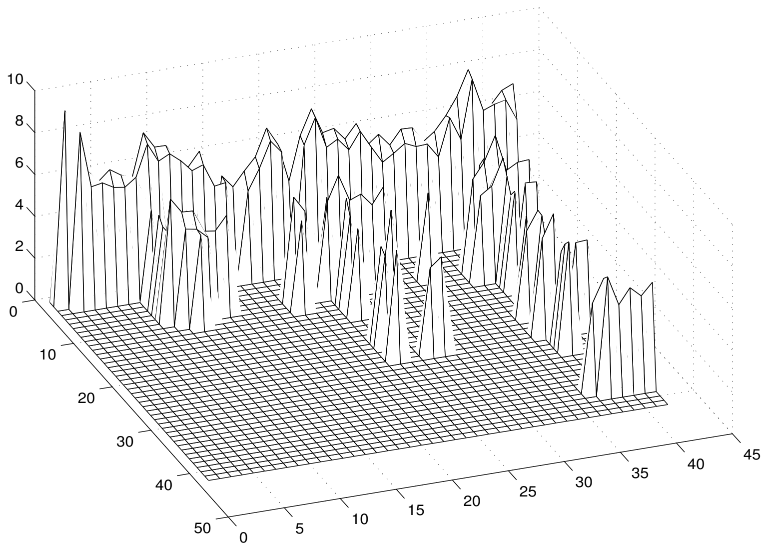

FIG. 6.4. *Logarithm of the absolute error of* $|C_{new} - C|$.

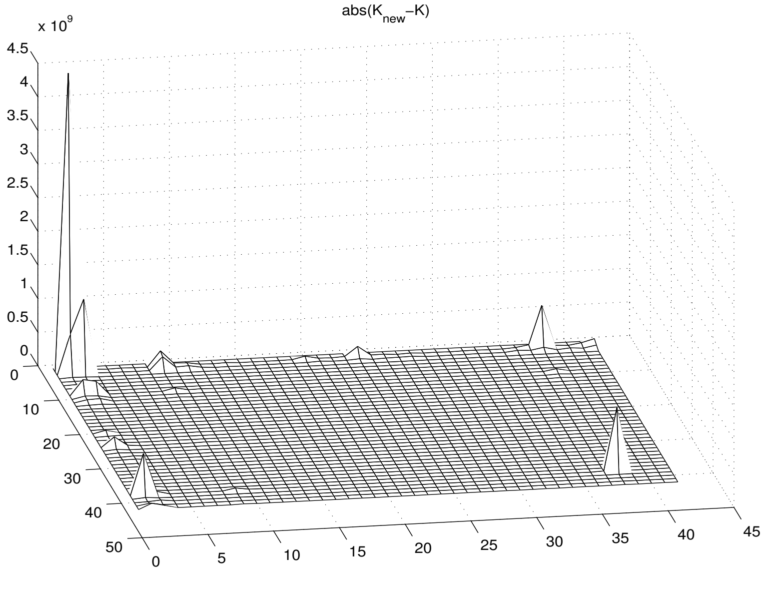

FIG. 6.5. *The absolute error of* $|K_{new} - K|$.

λ-matrix eigenproblems and model embedding approach to model tuning, Num. Lin. Alg. Appl., 8 (2001), pp. 53–70.

[5] B. FRITCHMAN AND D. J. PIERCE, *Model embedding approach to model tuning*, Tech. Rep. ISSTECH-96-026, The Boeing Company, Dec. 1996.

[6] J.-S. GUO, W.-W. LIN, AND C.-S. WANG, *Numerical solution for large sparse quadratic eigenvalue problem*, Lin. Alg. Appl., 225 (1995), pp. 57–89.

[7] ———, *Nonequivalent deflation for the solution of matrix latent value problems*, Lin. Alg. Appl., 231 (1995), pp. 15–45.

[8] W.-W. LIN, *An algorithm for computing the eigenstructure of a regular matrix polynomial*,

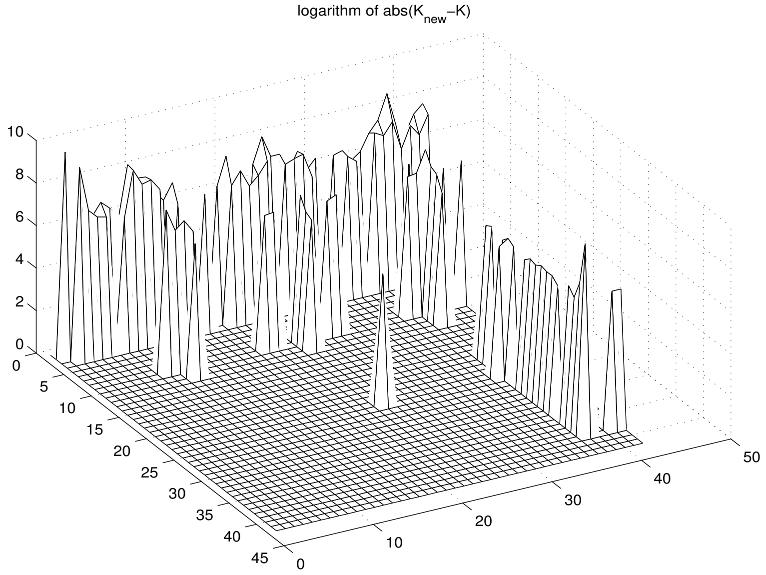

FIG. 6.6. *Logarithm of the absolute error of* $|K_{new} - K|$.

Lin. Alg. Appl., 91 (1990), pp. 195–211.

[9] C. MINAS AND D. INMAN, *Correcting finite element models with measured modal results using eigenstructure assignment methods*, in Proc. of the 6th International Modal Analysis Conference, Orlando, Florida, February 1988, pp. 583–587.

[10] ———, *Matching finite element models to model data*, Trans. ASME, 112 (1990), pp. 84–92.

[11] P. VAN DOOREN AND P. DEWILDE, *The eigenstructure of an arbitrary polynomial matrix: computational aspects*, Lin. Alg. Appl., 50 (1983), pp. 545–579.

[12] J. H. WILKINSON, The Algebraic Eigenvalue Problem, M.A. (Cantab.), Sc.D., D.Tech., F.R.S., 1965.

[13] D. ZIMMERMAN AND M. WIDENGREN, *Correcting finite element models using a symmetric eigenstructure assignment technique*, AIAA, 28 (1990), pp. 1670–1676.