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Theory and Computations of Some Inverse Eigenvalue Problems for the Quadratic Pencil

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Abstract. The paper reviews, both from theoretical and computational view points, current developments on the partial modal approach for certain inverse eigenvalue problems for the quadratic pencil associated with a linear control system modeled by a system of matrix second-order differential equations. The paper concludes with some future research problems.

1. Introduction

An inverse eigenvalue problem for a matrix A is the problem of finding A given the complete or a part of the spectrum and/or eigenvectors. There are many different forms of inverse eigenvalue problems and they arise in various applications (see the recent expository paper of [**MTC**]).

In this paper, we focus on certain types of inverse eigenvalue problems associated with a quadratic matrix pencil arising in feedback control of a matrix second-order system. To define our problems, let's consider the dynamical system

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) \quad = \quad f(t), \tag{1.1}$$

where M, D, and K are symmetric matrices; M is positive definite (denoted by $M > 0$), and $\dot{x}(t)$ and $\ddot{x}(t)$, respectively, denote the first and second derivatives of the time dependent vector $x(t)$. The system of the type (1.1) arises in a wide range of applications, especially in the design and analysis of vibrating structures, such as bridges, highways, buildings, airplanes, etc. In vibration analysis, the matrices M, K, and D are known, respectively, as the *mass*, *stiffness* and *damping* matrices.

Upon separation of variables, the system gives rise to the quadratic eigenvalue problem for the pencil

$$P(\lambda) \quad = \quad \lambda^2 M + \lambda D + K. \tag{1.2}$$

The pencil (1.2) has $2n$ eigenvalues which are the roots of the equation

$$\det(P(\lambda)) = 0$$

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and $2n$ corresponding eigenvectors. If (1.1) represents a vibrating system, then the eigenvalues of $P(\lambda)$ are related to the *natural frequencies* of the homogeneous system

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) \quad = \quad 0,$$

and the eigenvectors are referred to as the *modes* of vibration of the system (see [**BNDa**], [**DJI**]).

Dangerous oscillations (called resonance) will occur when one or more eigenvalues of the pencil (1.2) became equal or close to the frequency of the external force. To avoid such unwanted oscillations of the vibratory system modeled by (1.1), a control force $f = Bu(t)$, where B is an $n \times m$ matrix and $u(t)$ is a time dependent $m \times 1$ vector needs to be applied to (1.1). Let $u(t)$ be chosen as

$$u(t) \quad = \quad F^T\dot{x}(t) + G^Tx(t), \tag{1.3}$$

where F and G are constant matrices, then the system (1.1) becomes

$$M\ddot{x}(t) + (D - BF^T)\dot{x}(t) + (K - BG^T)x(t) \quad = \quad 0. \tag{1.4}$$

Mathematically, the problem is then to choose the matrices F and G such that the eigenvalues of the associated closed-loop pencil

$$P_c(\lambda) \quad = \quad \lambda^2 M + \lambda(D - BF^T) + K - BG^T \tag{1.5}$$

can be altered as required to combat the effects of resonances or to ensure and improve the stability of the system.

In a realistic situation, however, only a few eigenvalues are "troublesome"; so it makes more sense to alter only those "troublesome" eigenvalues, while keeping the rest of the spectrum invariant. This leads to the following inverse eigenvalue problem, known as the *partial eigenvalue assignment problem* for the pencil (1.2).

Problem 1.1. Given

1. Real $n \times n$ matrices $M = M^T > 0$, $D = D^T$, $K = K^T$.
2. The $n \times m$ ($m \leq n$) control matrix B.
3. The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$, $p < n$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the corresponding eigenvector set $\{x_1, \ldots, x_p\}$.
4. The self-conjugate set $\{\mu_1, \ldots, \mu_p\}$ of numbers.

Find *real* feedback matrices F and G such that the spectrum of the closed-loop pencil (1.5) is $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

While Problem 1.1 is important in its own right, it is to be noted that, if the system response needs to be altered by feedback, both eigenvalue assignment as well as eigenvector assignment should be considered. This is because, the eigenvalues determine the rate at which system response decays or grows while the eigenvectors determine the shape of the response. Such a problem is called the *eigenstructure assignment problem.* Unfortunately, the eigenstructure assignment problem, in general, is not solvable if the matrix B is given apriori (see [**IK**]). This consideration leads to the following more tractable (but practical) inverse eigenstructure assignment problem for the quadratic pencil (1.2), known as the *partial eigenstructure assignment problem* for the pencil (1.2).

Problem 1.2. Given

1. Real $n \times n$ matrices $M = M^T > 0$, $D = D^T$, $K = K^T$.

2. The self-conjugate subset $\{\lambda_1, \dots, \lambda_p\}$, $p < n$ of the open-loop spectrum $\{\lambda_1, \dots, \lambda_p; \lambda_{p+1}, \dots, \lambda_{2n}\}$ and the corresponding eigenvector set $\{x_1, \dots, x_p\}$.
3. The self-conjugate sets of numbers and vectors $\{\mu_1, \dots, \mu_p\}$ and $\{y_1, \dots, y_p\}$, such that $\mu_j = \overline{\mu_k}$ implies $y_j = \overline{y_k}$.

Find a *real* control matrix B of order $n \times m$ $(m < n)$, and *real* feedback matrices F and G of order $n \times m$ such that the spectrum of the closed-loop pencil (1.5) is $\{\mu_1, \dots, \mu_p; \lambda_{p+1}, \dots, \lambda_{2n}\}$ and the eigenvector set is $\{y_1, \dots, y_p; x_{p+1}, \dots, x_{2n}\}$, where $x_{p+1}, \dots, x_{2n}$ are the eigenvectors of (1.2) corresponding to $\lambda_{p+1}, \dots, \lambda_{2n}$.

An obvious approach for the above problems is to recast the problem in terms of a first-order reformulation and then apply one of the many well-established techniques for full-order eigenvalue assignment of a first-order system (see e.g. [**BNDb**]) or, more appropriately, the partial pole placement technique of [**YS**]. There are some computational difficulties with this approach. If the standard first order transformation

$$\dot{z}(t) = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ M^{-1}B \end{pmatrix} u(t), \quad \text{where} \quad z(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

is used, then the matrix M has to be inverted, and, if it is ill-conditioned, the state matrix will not be computed accurately. Furthermore, all the exploitable properties such as definiteness, sparsity, bandness, etc. of the coefficient matrices M, D, and K, usually offered by a practical problem, will be completely destroyed. The use of a nonstandard first-order transformation, such as

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \dot{z}(t) \quad = \quad \begin{pmatrix} 0 & M \\ -K & -D \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ B \end{pmatrix} u(t)$$

will give rise to a descriptor system of the form $E\dot{z}(t) = Az(t) + \hat{B}u(t)$, and the eigenvalue assignment methods for the descriptor systems, especially, when the matrix E is ill-conditioned, are not well developed.

A second approach, popularly known in the engineering literature as the *independent modal space control* (IMSC) *approach*, also suffers from some serious computational difficulties and is almost impossible to implement in practice. The basic idea here is to decouple the problem into a set of n independent problems, solve each of these independent problems separately, and then piece the individual solutions together to obtain a solution of the given problem. The implementation of this idea requires knowledge of the complete spectrum and associated eigenvectors of the pencil $P(\lambda)$. Unfortunately, numerical methods for the quadratic eigenvalue problem are not well developed, especially for large and sparse problems. The state-of-the-art computational techniques are capable of computing only a few selected extremal eigenvalues and eigenvectors (see [**PC**] and [**SBFV**]). Furthermore, for decoupling of the right hand sides of the associated modal equations, some stringent conditions on the control vector need to be imposed (see [**DJI**]). Specifically, if the matrices M, D and K are simultaneously diagonalized by the matrix S, then for a decoupling of the right hand side of the associated modal equations, the following commutativity relations must be satisfied

$$BFM^{-1}D = DM^{-1}BF \text{ and } BGM^{-1}D = KM^{-1}BG;$$

assuming that BF and BG are both symmetric (see [**DJI**]).

In view of these statements, it is natural to wonder if solutions of the above problems can be obtained using only a partial knowledge of eigenvalues and eigenvectors, and without resorting to a first-order reformulation. A solution technique of this type will be called a *direct partial modal approach*. It is "direct", because the solution is obtained directly in the second-order setting without any types of reformulations. It is "partial modal", because only a part of the spectral data is needed for the solution. Such a direct partial modal approach for the Problem 1.1 and Problem 1.2 have been recently proposed in [**DERb**], [**DS**] and [**DERS**]. The solutions are obtained using only those small number of eigenvalues and the corresponding eigenvectors that are to be reassigned and directly in terms of the coefficient matrices M, D and K. Variation of Problem 1.1 have also been solved this way in [**DR**], [**DERa**] and [**CD**].

The partial modal solutions for Problem 1.1 (both for the single-input and multiple-input cases) and for Problem 1.2 are described in Section 3 and 4, respectively. Indeed, a unified treatment of solutions to both these problems is given, in the sense, that the results of existences and uniqueness (Theorem 3.1, Theorem 3.3 and Theorem 4.1) are derived and the solutions are expressed in each case using a single matrix Z_1 given by

$$Z_1 \quad = \quad \Lambda_1' Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1, \tag{1.6}$$

where

$$\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p),\ \Lambda_1' = \mathrm{diag}(\mu_1, \ldots, \mu_p) \tag{1.7}$$

and

$$X_1 = (x_1, \ldots, x_p),\ Y_1 = (y_1, \ldots, y_p). \tag{1.8}$$

Furthermore, in this paper the above Theorems are derived using a weaker condition than originally used in [**DERb**] to solve the single-input case of Problem 1.1. The constructive proofs of Theorems 3.1, 3.3 and 4.1, lead, respectively, to Algorithms 3.2, 3.5 and 4.2. Algorithms 3.5 and 4.2 are illustrated with a numerical example in Section 5. In Section 2, three important orthogonality relations between the eigenvectors of a quadratic pencil are stated and proved. One of these relations plays a key role in our derivation of the direct modal approach for Problems 1.1 and 1.2. However, these relations are of independent interest. Based on our discussions and observations in this paper, a few future research problems are stated in the concluding Section 6. Discussions pertaining justification of each of the problems are given and in one case (case (iii) in Section 6) our idea on a possible approach for its solution is stated. The numerical example in Section 5 supports our idea. Some more definitive work, however, should be done.

2. Orthogonality Relations of the Eigenvectors of Quadratic Matrix Pencil

In this section, we derive three orthogonality relations (due to [**DERb**]) between the eigenvectors of a symmetric definite quadratic pencil. One of these results plays a key role in our later developments. These results generalize the well-known results on orthogonality between the eigenvectors of a symmetric matrix and these of a symmetric definite linear pencil (see [**BNDa**]) of the form $K - \lambda M$.

Theorem 2.1. (Orthogonality of the Eigenvectors of Quadratic Pencil). *Let* $P(\lambda) = \lambda^2 M + \lambda D + K$, *where* $M = M^T > 0$, $D = D^T$, *and* $K = K^T$. *Let*

X and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{2n})$ be, respectively, the eigenvector and the eigenvalue matrix of the pencil (1.2). Assume that the eigenvalues $\lambda_1, \dots, \lambda_n$ are all distinct and different from zero. Then there exist diagonal matrices D_1, D_2, and D_3 such that

$$\Lambda X^T M X \Lambda - X^T K X = D_1 \tag{2.1}$$

$$\Lambda X^T D X \Lambda + \Lambda X^T K X + X^T K X \Lambda = D_2 \tag{2.2}$$

$$\Lambda X^T M X + X^T M X \Lambda + X^T D X = D_3 \tag{2.3}$$

Furthermore

$$D_1 = D_3 \Lambda \tag{2.4}$$

$$D_2 = -D_1 \Lambda \tag{2.5}$$

$$D_2 = -D_3 \Lambda^2. \tag{2.6}$$

Proof. By definition, the pair (X, Λ) must satisfy the $n \times 2n$ system of equations (called the *eigendecomposition* of the pencil $P(\lambda) = \lambda^2 M + \lambda D + K$):

$$M X \Lambda^2 + D X \Lambda + K X = 0. \tag{2.7}$$

Isolating the term in D, we have from above

$$-D X \Lambda = M X \Lambda^2 + K X.$$

Multiplying this on the left by ΛX^T gives

$$-\Lambda X^T D X \Lambda = \Lambda X^T M X \Lambda^2 + \Lambda X^T K X$$

Taking the transpose gives

$$-\Lambda X^T D X \Lambda = \Lambda^2 X^T M X \Lambda + X^T K X \Lambda$$

Now, subtracting the latter from the former gives, on rearrangement,

$$\Lambda X^T M X \Lambda^2 - X^T K X \Lambda = \Lambda^2 X^T M X \Lambda - \Lambda X^T K X$$

or

$$(\Lambda X^T M X \Lambda - X^T K X)\Lambda = \Lambda(\Lambda X^T M X \Lambda - X^T K X). \tag{2.8}$$

Thus, the matrix $\Lambda X^T M X \Lambda - X^T K X$ which we denote by D_1, must be diagonal since it commutes with a diagonal matrix, the diagonal entries of which are distinct. We thus have the *first orthogonality relation (2.1):*

$$\Lambda X^T M X \Lambda - X^T K X = D_1$$

Similarly, isolating the term in M of the eigendecomposition equation, we get

$$-M X \Lambda^2 = D X \Lambda + K X,$$

and multiplying this on the left by $\Lambda^2 X^T$ gives

$$-\Lambda^2 X^T M X \Lambda^2 = \Lambda^2 X^T D X \Lambda + \Lambda^2 X^T K X.$$

Taking the transpose, we have

$$-\Lambda^2 X^T M X \Lambda^2 = \Lambda X^T D X \Lambda^2 + X^T K X \Lambda^2.$$

Subtracting the last equation from the previous one and adding $\Lambda X^T K X \Lambda$ to both sides gives, after some rearrangement,

$$\Lambda(\Lambda X^T D X \Lambda + \Lambda X^T K X + X^T K X \Lambda) = (\Lambda X^T D X \Lambda + \Lambda X^T K X + X^T K X \Lambda)\Lambda$$

Again, this commutativity property implies, since Λ has distinct diagonal entries, that

$$\Lambda X^T DX\Lambda + \Lambda X^T KX + X^T KX\Lambda = D_2$$

is a diagonal matrix. This is the *second orthogonality relation (2.2)*.

The first and second orthogonality relations together easily imply the *third orthogonality relation (2.3):*

$$\Lambda X^T MX + X^T MX\Lambda + X^T DX = D_3$$

To prove (2.4) we multiply the last equation on the right by Λ giving

$$\Lambda X^T MX\Lambda + X^T MX\Lambda^2 + X^T DX\Lambda = D_3\Lambda,$$

which, using the eigendecomposition equation, becomes

$$\Lambda X^T MX\Lambda + X^T(-KX) = D_3\Lambda.$$

So, from the first orthogonality relation (2.1) we see that

$$D_1 = D_3\Lambda$$

Next, using the eigendecomposition equation (2.7), we rewrite the second orthogonality relation (2.2) as

$$\begin{aligned} D_2 &= \Lambda X^T(DX\Lambda + KX) + X^T KX\Lambda \\ &= \Lambda X^T(-MX\Lambda^2) + X^T KX\Lambda = (-\Lambda X^T MX\Lambda^+ X^T KX)\Lambda \end{aligned}$$

By the first orthogonality relation we then have

$$D_2 = -D_1\Lambda$$

Finally, from $D_1 = D_3\Lambda$ and $D_2 = -D_1\Lambda$ we have

$$D_2 = -D_3\Lambda^2.$$

We remind the reader that matrix and vector transposition here does not mean conjugation for complex quantities. □

REMARK 2.2. If the condition "the eigenvalues $\lambda_1, \dots, \lambda_{2n}$ are all distinct and different from zero" in Theorem 2.1 is replaced by the weaker condition "the sets $\{\lambda_1, \dots, \lambda_p\}$ and $\{\lambda_{p+1}, \dots, \lambda_{2n}\}$ are disjoint" then the weaker version of the first orthogonality relation (2.1) holds:

$$\Lambda_1 X_1^T MX_2\Lambda_2 - X_1^T KX_2 = 0, \tag{2.9}$$

$$\Lambda_2 X_2^T MX_1\Lambda_1 - X_2^T KX_1 = 0, \tag{2.10}$$

where

$$\Lambda_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_p),\ \Lambda_2 = \operatorname{diag}(\lambda_{p+1}, \dots, \lambda_{2n}) \tag{2.11}$$

and

$$X_1 = (x_1, \dots, x_p),\ X_2 = (x_{p+1}, \dots, x_{2n}); \tag{2.12}$$

Indeed, if (2.8) is written as

$$N\Lambda = \Lambda N,\ \text{where } N = (n_{ij}) = \Lambda X^T MX\Lambda - X^T KX,$$

then $n_{ij}\lambda_i = \lambda_j n_{ij}$ implies $n_{ij} = 0$ if $1 \le i \le p < j \le 2n$ or $1 \le j \le p < i \le 2n$.

Therefore, (2.9) and (2.10) follow from (2.8).

3. Solution to Problem 1.1

In this section, we present a direct partial modal approach for the solution of Problem 1.1. We consider the single-input and multi-input cases separately.

3.1. Case 1. Single-input Case. In the single-input case, Problem 1.1 reduces to the following problem:

Given $n \times n$ matrices $M = M^T > 0$, $D = D^T$, $K = K^T$, the $n \times 1$ $(m = 1)$ control vector b; a part of the spectrum $\{\lambda_1, \ldots, \lambda_p\}$ of the open-loop pencil $P(\lambda)$, and the set $\{\mu_1, \ldots, \mu_p\}$, both closed under complex conjugation, find real feedback vectors f and g such that the spectrum of $P_c(\lambda) = \lambda^2 M + \lambda(D - bf^T) + K - bg^T$ is precisely the set $\{\mu_1, \mu_2, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$, where $\lambda_{p+1}, \ldots, \lambda_{2n}$ are the remaining eigenvalues of the pencil $P(\lambda)$.

As before, let (X, Λ) be the eigenvector-eigenvalue matrix pair of the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda D + K$.

Partition X and Λ in the form

$$X = (X_1, X_2), \Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2),$$

where Λ_1, Λ_2 and X_1, X_2 are defined by (2.11) and (2.12), respectively. Define the vectors f and g by:

$$f = MX_1\Lambda_1\beta \text{ and } g = -KX_1\beta,$$

where β is an arbitrary $p \times 1$ vector.

We first show that with this choice of f and g, the $(2n - p)$ eigenvalues $\lambda_{p+1}, \cdots, \lambda_{2n}$ of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - bf^T) + (K - bg^T)$ remain unchanged by feedbacks; that is, they are the same as those of the open-loop pencil: $P(\lambda) = \lambda^2 M + \lambda D + K$.

In terms of the eigenvalue and eigenvector matrices, this amounts to proving that

$$MX_2\Lambda_2^2 + (D - bf^T)X_2\Lambda_2 + (K - bg^T)X_2 = 0.$$

To prove this, we consider the eigendecomposition equation again:

$$MX\Lambda^2 + DX\Lambda + KX = 0. \tag{3.1}$$

From this, we obtain

$$\begin{aligned} &MX_2\Lambda_2^2 + (D - bf^T)X_2\Lambda_2 + (K - bg^T)X_2 \\ &\quad = MX_2\Lambda^2 + DX_2\Lambda_2 + KX_2 - b\beta^T(\Lambda_1 X_1^T MX_2\Lambda_2 - X_1^T KX_2) \\ &\quad = -b\beta^T(\Lambda_1 X_1^T MX_2\Lambda_2 - X_1^T KX_2). \end{aligned}$$

Indeed, (3.1) implies $MX_2\Lambda_2^2 + DX_2\Lambda_2 + KX_2 = 0$ and, furthermore, if we assume that $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$, then by (2.9) in Remark 2.2, we have

$$\Lambda_1 X_1^T MX_2\Lambda_2 - X_1^T KX_2 = 0.$$

Thus $MX_2\Lambda_2^2 + (D - bf^T)X_2\Lambda_2 + (K - bg^T)X_2 = 0$.

Choosing β for Partial Assignment of Eigenvalues. In order to solve Problem 1.1 completely, we still need to choose β which will move eigenvalues $\{\lambda_j\}_{j=1}^p$ of the pencil $P(\lambda)$ to $\{\mu_j\}_{j=1}^p$ in $P_c(\lambda)$, if that is possible. If there is such a vector β, then there exists an eigenvector matrix Y_1 of ordfer $n \times p$:

$$Y_1 = (y_1, y_2, \ldots, y_p),\ y_j \neq 0,\ j = 1, 2, ..., m,$$

such that

$$MY_1(\Lambda_1')^2 + (D - bf^T)Y_1\Lambda_1' + (K - bg^T)Y_1 = 0,$$

where $\Lambda_1' = \text{diag}(\mu_1, \mu_2, \dots, \mu_p)$.

Substituting for f, g and rearranging, we have

$$\begin{aligned} MY_1(\Lambda_1')^2 + DY_1\Lambda_1' + KY_1 &= b\beta^T(\Lambda_1 X_1^T MY_1\Lambda_1' - X_1^T KY_1) \\ &= b\beta^T Z_1^T = bc^T, \end{aligned}$$

where Z_1 is given by (1.6) and $c = Z_1\beta$ is a vector that will depend on the scaling chosen for the eigenvectors in Y_1. If we assume that the open-loop pencil (1.2) is partially controllable with respect to $\mu_1, \dots, \mu_p$ then we can solve for each of the eigenvectors y_i using the equations

$$(\mu_j^2 M + \mu_j D + K)y_j = b, \qquad j = 1, 2, ..., p \tag{3.2}$$

to obtain Y_1. This corresponds to choosing the vector $c = (1, 1, ..., 1)^T$, so, having computed the eigenvectors we could solve the $p \times p$ linear system

$$Z_1\beta = (1, 1, ..., 1)^T \tag{3.3}$$

for β, and hence determine the vectors f and g.

We now show that the vectors f and g obtained this way are real vectors. Since the set $\{\lambda_1, \dots, \lambda_p\}$ is self-conjugate and the coefficient matrices M, D and K of the open-loop pencil $P(\lambda)$ are real, we know that $\lambda_j = \overline{\lambda_k}$ implies that $x_j = \overline{x_k}$ (conjugate eigenvectors correspond to conjugate eigenvalues). Therefore, there exists a nonsingular permutation matrix T such that

$$\overline{X_1} = X_1 T \text{ and } \overline{X_1\Lambda_1} = X_1\Lambda_1 T.$$

Similarly, there is a permutation matrix T' such that

$$\overline{Y_1} = Y_1 T' \text{ and } \overline{Y_1\Lambda_1'} = Y_1\Lambda_1' T'.$$

Thus, conjugating (1.6), we obtain

$$\overline{Z_1} = (T')^T\Lambda_1' Y_1^T MX_1\Lambda_1 T - (T')^T Y_1^T KX_1 T \quad = \quad (T')^T Z_1 T,$$

and conjugation of (3.3) gives

$$\overline{Z_1\beta} = ((T')^T Z_1 T)\overline{\beta} = (T')^T(1, 1, \dots, 1)^T,$$

implying that $\overline{\beta} = T^T\beta$.

Therefore,

$$\overline{f} = M(X_1\Lambda_1 T)(T^T\beta) = f \text{ and } \overline{g} = -K(X_1 T)(T^T\beta) = g$$

which shows that f and g are real vectors.

Theorem 3.1. (Solution to Single-input Partial Eigenvalue Assignment Problem for a Quadratic Pencil).
If $\{\lambda_1, \dots, \lambda_p\} \cap \{\lambda_{p+1}, \dots, \lambda_{2n}\} = \emptyset$ *then*

(i) *For any arbitrary vector β, the feedback vectors f and g defined by*

$$f = MX_1\Lambda_1\beta \tag{3.4}$$

$$\text{and } g = -KX_1\beta. \tag{3.5}$$

are such that $2n - p$ eigenvalues $\lambda_{p+1}, \dots, \lambda_{2n}$ of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - bf^T) + K - bg^T$ are the same as these of the open-loop pencil $P(\lambda) = \lambda^2 M + \lambda D + K$.

(ii) *Let $y_1, \ldots, y_p$ be the set of p vectors such that for each $k = 1, 2, \ldots, p$,*

$$\begin{pmatrix} y_k \\ 1 \end{pmatrix} \in \text{null}(\mu_k^2 M + \mu_k D + K, -b).$$

*(*Equivalently, the pencil $P(\lambda)$ is partially controllable with respect to $\mu_1, \ldots, \mu_p$*).*
Define

$$Z_1 = \Lambda_1' Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1$$

as in (1.6).
Then Problem 1.1 has a solution in the form (3.4)-(3.5) if and only if the system of equations

$$Z_1 \beta = (1, 1, \ldots, 1)^T.$$

has a solution.

Based on Theorem 3.1 we can state the following algorithm.

ALGORITHM 3.2. *(An Algorithm For the Single-input Partial Eigenvalue Assignment Problem for the Quadratic Pencil).*

Inputs:
1. The $n \times n$ matrices M, K, and D; $M = M^T > 0$, $D = D^T$ and $K = K^T$.
2. The $n \times 1$ control (input) vector b.
3. The set $\{\mu_1, \cdots, \mu_p\}$, closed under complex conjugation.
4. The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the associated eigenvector set $\{x_1, \ldots, x_p\}$.

Outputs: The feedback vectors f and g such that the spectrum of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - bf^T) + (K - bg^T)$ is $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

Assumptions:
1. The quadratic pencil is (partially) controllable with respect to the "eigenvalues to be assigned" $\mu_1, \ldots, \mu_p$.
2. $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$.

Step 1. Form $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$ and $X_1 = (x_1, \ldots, x_p)$.

Step 2. Solve for $y_1, \ldots, y_p$:

$$(\mu_j^2 M + \mu_j D + K) y_j = b, \; j = 1, \ldots, p.$$

Step 3. Form

$$Z_1 = \Lambda_1' Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1,$$

where $Y_1 = (y_1, \ldots, y_p)$ and $\Lambda_1' = \text{diag}(\mu_1, \ldots, \mu_p)$. If Z_1 is ill-conditioned, then warn the user that the problem is ill-posed.

Step 4. Solve for β:

$$Z_1 \beta = (1, 1, \cdots, 1)^T.$$

Step 5. Form

$$\begin{aligned} f &= M X_1 \Lambda_1 \beta \\ g &= -K X_1 \beta. \end{aligned}$$

3.2. Case 2. Multi-input case. In the multi-input case, we obtain the following generalization of Theorem 3.1.

THEOREM 3.3. (Solution to Multi-input Partial Eigenvalue Assignment Problem for a Quadratic Pencil).
If $\{\lambda_1, \dots, \lambda_p\} \cap \{\lambda_{p+1}, \dots, \lambda_{2n}\} = \emptyset$. *then*

(i) *For any arbitrary matrix* Φ*, the feedback matrices* F *and* G *defined by*

$$F = MX_1\Lambda_1\Phi^T \tag{3.6}$$

$$\text{and } G = -KX_1\Phi^T. \tag{3.7}$$

are such that $2n - p$ *eigenvalues* $\lambda_{p+1}, \dots, \lambda_{2n}$ *of the closed-loop pencil* $P_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + K - BG^T$ *are the same as those of the open-loop pencil* $P(\lambda) = \lambda^2 M + \lambda D + K$.

(ii) *Let* $\{y_1, \dots, y_p\}$ *and* $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$ *be the two sets of vectors chosen in such a way that* $\mu_j = \overline{\mu_k}$ *implies* $\gamma_j = \overline{\gamma_k}$ *and for each* $k = 1, 2, \dots, p$,

$$\begin{pmatrix} y_k \\ \gamma_k \end{pmatrix} \in \operatorname{null}(\mu_k^2 M + \mu_k D + K, -B) \tag{3.8}$$

(equivalently, the pair $(P(\lambda), B)$ *is* partially controllable *with respect to the modes* $\mu_1, \dots, \mu_p$*).*

Define Z_1 *and* Y_1 *as in Theorem 3.1. Then Problem 1.1 (in the multi-input case) has a solution with* F *and* G *given by (3.6), and (3.7), respectively, provided that* Φ *satisfies the linear system of equations:*

$$\Phi Z_1^T = \Gamma, \quad \text{where } \Gamma = (\gamma_1, \dots, \gamma_p). \tag{3.9}$$

PROOF. Using the first orthogonality relation (2.1), it is easy to verify that

$$\begin{aligned} &MX_2\Lambda_2^2 + (D - BF^T)X_2\Lambda_2 + (K - BG^T)X_2 \\ &\quad = (MX_2\Lambda_2^2 + DX_2\Lambda_2 + KX_2) - B\Phi(\Lambda_1 X_1^T MX_2\Lambda_2 - X_1^T KX_2) = 0, \end{aligned}$$

which proves Part (i).

To prove Part (ii), we note using (3.6) and (3.7), that

$$\begin{aligned} P_c(\mu_k)y_k &= \left(\mu_k^2 M + \mu_k \left(D - B\sum_{j=1}^{p} \phi_j \lambda_j x_j^T M\right) + \left(K + B\sum_{j=1}^{p} \phi_j x_j^T K\right)\right) y_k \\ &= B\gamma_k - B\left(\sum_{j=1}^{p} \phi_j x_j^T (\mu_k \lambda_j M - K)\right) y_k = B\left(\gamma_k - \sum_{j=1}^{p} \phi_j z_{kj}\right), \end{aligned}$$

where $\Phi = (\phi_1, \phi_2, \dots, \phi_p)$ and z_{kj}'s are the elements of the matrix Z_1. Then $P_c(\mu_k)y_k = 0$ for $k = 1, 2, \dots, p$ can be written in the form of the single matrix equation

$$\Phi Z_1^T = \Gamma,$$

which proves (3.9).

We now show that the matrices F and G obtained this way are real matrices. Since, if $\gamma_1, \dots, \gamma_p$ are chosen in such a way that $\mu_j = \overline{\mu_k}$ implies $\gamma_j = \overline{\gamma_k}$, then this also implies $y_j = \overline{y_k}$ and, then, as in the proof of Theorem 3.1, there exist permutation matrices T and T' such that

$$\overline{X_1} = X_1 T,\ \overline{X_1\Lambda_1} = X_1\Lambda_1 T,\ \overline{\Gamma} = \Gamma T',\ \overline{Y_1} = Y_1 T' \text{ and } \overline{Y_1\Lambda_1'} = Y_1\Lambda_1' T'.$$

Thus, conjugating (1.6) gives $\overline{Z_1} = (T')^T Z_1 T$ and, conjugating (3.9), we get

$$\overline{\Phi} T^T Z_1 T' = \Gamma T',$$

which implies that $\overline{\Phi} = \Phi T$.

Therefore,

$$\overline{F} = M(X_1\Lambda_1 T)(T^T\Phi^T) = F \text{ and } \overline{G} = -K(X_1T)(T^T\Phi^T) = G$$

showing that F and G are real matrices. □

Remark 3.4. The results of Theorem 3.3 provide a parametric solution to Problem 1.1 in the multi-input case. The freedom of choosing these parameters can be conveniently exploited to obtain a solution with certain desirable properties such as the one having minimal norm, etc. See Section 6 for further discussions.

Based on the Theorem 3.3 we can state the following algorithm.

Algorithm 3.5. *(An Algorithm For the Multi-input Partial Eigenvalue Assignment Problem for the Quadratic Pencil).*

Inputs:

1. The $n \times n$ matrices M, K, and D; $M = M^T > 0$, $K = K^T$ and $D = D^T$.
2. The $n \times m$ control (input) matrix B.
3. The set $\{\mu_1, \cdots, \mu_p\}$, closed under complex conjugation.
4. The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the associated eigenvector set $\{x_1, \ldots, x_p\}$.

Outputs: The feedback matrices F and G such that the spectrum of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + (K - BG^T)$ is $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

Assumptions:

1. The quadratic pencil is (partially) controllable with respect to the "eigenvalues to be assigned" $\mu_1, \ldots, \mu_p$.
2. $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$.

Step 1. Form $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and $X_1 = (x_1, \ldots, x_p)$.

Step 2. Choose arbitrary vectors $\gamma_1, \ldots, \gamma_p$ in such a way that $\mu_j = \overline{\mu_k}$ implies $\gamma_j = \overline{\gamma_k}$ and solve for $y_1, \ldots, y_p$:

$$(\mu_j^2 M + \mu_j D + K)y_j = B\gamma_j,\ j = 1, \ldots, p.$$

Step 3. Form

$$Z_1 = \Lambda_1' Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1,$$

where $Y_1 = (y_1, \ldots, y_p)$ and $\Lambda_1' = \mathrm{diag}(\mu_1, \ldots, \mu_p)$. If Z_1 is ill-conditioned, then return to Step 2 and select different vectors $\gamma_1, \ldots, \gamma_p$.

Step 4. Form $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ and solve for Φ:

$$\Phi Z_1^T = \Gamma.$$

Step 5. Form

$$\begin{aligned} F &= M X_1 \Lambda_1 \Phi^T \\ G &= -K X_1 \Phi^T. \end{aligned}$$

4. Solution to Problem 1.2

The solution process consists of two stages:

Stage I. Determine matrices $\hat{B}$, $\hat{F}$, and $\hat{G}$ (generally complex) which satisfy

$$MY(\Lambda')^2 + (D - \hat{B}\hat{F}^T)Y\Lambda' + (K - \hat{B}\hat{G}^T)Y \quad = \quad 0, \tag{4.1}$$

where $Y = (Y_1, X_2)$; $Y_1 = (y_1, y_2, \ldots, y_p)$, $X_2 = (x_{p+1}, \ldots, x_{2n})$, and $\Lambda' = (\Lambda_1', \Lambda_2) = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n})$.

Stage II. From $\hat{B}$, $\hat{F}$, and $\hat{G}$ in Stage I, find real matrices B, F, and G such that

$$BF^T = \hat{B}\hat{F}^T \text{ and } BG^T = \hat{B}\hat{G}^T;$$

solving Problem 1.2.

Let's first focus on Stage I. Let $\Lambda_1' = \operatorname{diag}(\mu_1, \ldots, \mu_p)$. Suppose that the triplet $(\tilde{B}, \tilde{F}, \tilde{G})$ is a solution. Then (4.1) implies

$$MY_1(\Lambda_1')^2 + DY_1\Lambda_1' + KY_1 \quad = \quad \tilde{B}\left(\tilde{F}^T Y_1 \Lambda_1' + \tilde{G}^T Y_1\right). \tag{4.2}$$

Note that if $\tilde{B}$, $\tilde{F}$, and $\tilde{G}$ constitute a solution to Problem 1.2, then for any invertible W, $\hat{B} = \tilde{B}W$, $\hat{F} = \tilde{F}W^{-T}$, and $\hat{G} = \tilde{G}W^{-T}$ also constitute a solution; because $\tilde{B}\tilde{F}^T = \hat{B}\hat{F}^T$ and $\tilde{B}\tilde{G}^T = \hat{B}\hat{G}^T$. Denote

$$W \quad = \quad \tilde{F}^T Y_1 \Lambda_1' + \tilde{G}^T Y_1. \tag{4.3}$$

Then, provided that W is invertible, $\hat{B} = \tilde{B}W$ is admissible for some $\hat{F}$ and $\hat{G}$. Thus we can take

$$\hat{B} \quad = \quad MY_1(\Lambda_1')^2 + DY_1\Lambda_1' + KY_1 \tag{4.4}$$

by virtue of (4.2) and (4.3). Relations (4.4) and (4.1) together imply that

$$\hat{F}^T Y_1 \Lambda_1' + \hat{G}^T Y_1 \quad = \quad I. \tag{4.5}$$

It was shown in Theorem 3.3 that for any Φ, the matrices

$$\hat{F} = MX_1\Lambda_1\Phi^T \quad \text{and} \quad \hat{G} = -KX_1\Phi^T \tag{4.6}$$

satisfy

$$MX_2\Lambda_2^2 + (D - \hat{B}\hat{F}^T)X_2\Lambda_2 + (K - \hat{B}\hat{G}^T)X_2 \quad = \quad 0.$$

Substituting (4.6) into (4.5), we obtain

$$\Phi \quad = \quad \left(\Lambda_1 X_1^T M Y_1 \Lambda_1' - X_1^T K Y_1\right)^{-1}$$

from which $\hat{F}$ and $\hat{G}$ can be determined.

Now, consider Stage II. Since $\mu_j = \overline{\mu_k}$ implies $y_j = \overline{y_k}$, as in the proof of Theorem 3.3, there exist permutation matrices T and T' such that

$$\overline{X_1} = X_1T,\ \overline{X_1\Lambda_1} = X_1\Lambda_1T,\ \overline{Y_1} = Y_1T',\ \overline{Y_1\Lambda_1'} = Y_1\Lambda_1'T' \text{ and } \overline{Y_1(\Lambda_1')^2} = Y_1(\Lambda_1')^2T'.$$

Thus, $\overline{Z_1} = (T')^T Z_1 T$ and using (4.4) and (4.6) we obtain

$$\begin{aligned}
\overline{\hat{B}} = MY_1(\Lambda_1')^2T' + DY_1\Lambda_1'T' + KY_1T' \quad &= \quad \hat{B}T', \\
\overline{\hat{B}\hat{F}^T} = \hat{B}T'(MX_1\Lambda_1T(T^TZ_1^{-1}T'))^T = \hat{B}MX_1\Lambda_1Z_1^{-1} \quad &= \quad \hat{B}\hat{F}^T \text{ and} \\
\overline{\hat{B}\hat{G}^T} = \hat{B}T'(-KX_1T(T^TZ_1^{-1}T'))^T = -\hat{B}KX_1Z_1^{-1} \quad &= \quad \hat{B}\hat{G}^T,
\end{aligned}$$

which implies that both $\hat{B}\hat{F}^T$ and $\hat{G}\hat{G}^T$ are real matrices.

Define now the real $n \times 2n$ matrix

$$H = \hat{B}\left(\hat{F}^T, \hat{G}^T\right)$$

and let $LR = H$ be a factorization of H; where L and R are, respectively, of order $n \times m$ and $m \times 2n$. Then we can take B to be L, the first n columns of R to be F^T, and the last n columns of R to be G^T. Either the economy size QR factorization of H or the economy singular value decomposition of H can be used to compute B, F, and G (see [**GVL**] or [**BNDa**]).

The above discussion leads to the following theorem.

THEOREM 4.1. (Solution to the Partial Eigenstructure Assignment for a Quadratic Pencil).
Let $X_1 = (x_1, x_2, \ldots, x_p)$, $Y_1 = (y_1, y_2, \ldots, y_p)$, $\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$, *and* $\Lambda_1' = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_p)$. *Then provided that the matrix*

$$Z_1 = \Lambda_1' Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1$$

is nonsingular, the triplet $(\hat{B}, \hat{F}, \hat{G})$ *defined by*

$$\begin{aligned}
\hat{B} &= MY_1(\Lambda_1')^2 + DY_1\Lambda_1' + KY_1, \\
\hat{F} &= MX_1\Lambda_1 Z_1^{-1}, \text{ and} \\
\hat{G} &= -KX_1 Z_1^{-1}
\end{aligned}$$

constitutes a (possibly complex) solution to the Problem 1.2.

A solution with real B, F, *and* G *is obtained from the triplet* $(\hat{B}, \hat{F}, \hat{G})$ *by taking the economy size QR factorization or the SVD of the real matrix* $H = \hat{B}\left(\hat{F}^T, \hat{G}^T\right)$.

If QR factorization is used and if $LR = H$ *is the economy QR factorization of* H, *then*

$$\begin{aligned}
B &= L, \\
F^T &= (r_1, r_2, \ldots, r_n) \text{ and} \\
G^T &= (r_{n+1}, r_{n+2}, \ldots, r_{2n});
\end{aligned}$$

where $R = (r_1, \ldots, r_{2n})$.

If SVD $H = U\Sigma V^T$ *is used then the above formulae could be used either with*

$$L = U,\ R = \Sigma V^T,$$

or with

$$L = U\Sigma,\ R = V^T.$$

Based on the Theorem 4.1 we can state the following algorithm.

ALGORITHM 4.2. *(An Algorithm For the Partial Eigenstructure Assignment Problem for a Quadratic Pencil).*

Inputs:

1. The $n \times n$ matrices M, K, and D; $M = M^T > 0$, $K = K^T$, $D = D^T$.
2. The set of p numbers $\{\mu_1, \cdots, \mu_p\}$ and the set of p vectors $\{y_1, \ldots, y_p\}$, both closed under complex conjugation.
4. The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the associated eigenvector set $\{x_1, \ldots, x_p\}$.

Outputs: The feedback matrices F and G such that the spectrum of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + (K - BG^T)$ is $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the eigenvectors corresponding to $\mu_1, \ldots, \mu_p$ are $y_1, \ldots, y_p$, respectively.

Assumptions:

1. $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$.

2. $\mu_j = \overline{\mu_k}$ implies $y_j = \overline{y_k}$ for all $1 \le j, k \le p$.

Step 1. Obtain the first p eigenvalues $\lambda_1, \ldots, \lambda_p$ that need to be reassigned and the corresponding eigenvectors $x_1, \ldots, x_p$.

Form $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $\Lambda_1' = \mathrm{diag}(\mu_1, \ldots, \mu_p)$, $X_1 = (x_1, \ldots, x_p)$ and $Y_1 = (y_1, \ldots, y_p)$.

Step 2. Form the matrices $\hat{B}$ and Z_1

$$\begin{aligned} \hat{B} &= MY_1(\Lambda_1')^2 + DY_1\Lambda_1' + KY_1, \\ Z_1 &= \Lambda_1' Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1. \end{aligned}$$

Step 3. Solve for $\hat{H} \in \mathbb{C}^{p \times 2n}$

$$Z_1^T \hat{H} = (\Lambda_1^T X_1^T M, -X_1^T K)$$

and form $H = \hat{B}\hat{H}$.

Step 4. Compute the economy size QR decomposition of $H = BR$.

Step 5. Partition $R \in \mathbb{R}^{m \times 2n}$ to get $F, G \in \mathbb{R}^{m \times n}$

$$R = (F^T, G^T).$$

(This step can also be implemented using the SVD of H, as shown in Theorem 4.1).

Note 4.3. MATLAB codes for Algorithm 3.5 and 4.2 are available from the authors upon request.

5. Illustrative Numerical Examples

Example 5.1. We illustrate Algorithm 3.5 for the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ with random matrices M, D, K and B given by

$$M = \begin{pmatrix} 1.4685 & 0.7177 & 0.4757 & 0.4311 \\ 0.7177 & 2.6938 & 1.2660 & 0.9676 \\ 0.4757 & 1.2660 & 2.7061 & 1.3918 \\ 0.4311 & 0.9676 & 1.3918 & 2.1876 \end{pmatrix}, D = \begin{pmatrix} 1.3525 & 1.2695 & 0.7967 & 0.8160 \\ 1.2695 & 1.3274 & 0.9144 & 0.7325 \\ 0.7967 & 0.9144 & 0.9456 & 0.8310 \\ 0.8160 & 0.7325 & 0.8310 & 1.1536 \end{pmatrix}$$

$$K = \begin{pmatrix} 1.7824 & 0.0076 & -0.1359 & -0.7290 \\ 0.0076 & 1.0287 & -0.0101 & -0.0493 \\ -0.1359 & -0.0101 & 2.8360 & -0.2564 \\ -0.7290 & -0.0493 & -0.2564 & 1.9130 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0.3450 & 0.4578 \\ 0.0579 & 0.7630 \\ 0.5967 & 0.9990 \\ 0.2853 & 0.3063 \end{pmatrix}$$

The open-loop eigenvalues of $P(\lambda)$, computed via MATLAB, are

$$-0.0861 \pm 1.6242i, -0.1022 \pm 0.8876i, -0.1748 \pm 1.1922i \text{ and } -0.4480 \pm 0.2465i.$$

We will solve Problem 1.1, reassigning only the most unstable pair of the open-loop eigenvalues; namely, $-0.0861 \pm 1.6242i$ to the locations $-0.1 \pm 1.6242i$. That is, we want the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + K - BG^T$ to have the spectrum

$$\{-0.1 \pm 1.6242i, -0.1022 \pm 0.8876i, -0.1748 \pm 1.1922i, -0.448 \pm 0.2465i\}. \tag{5.1}$$

The em random choices of γ_1 and γ_2 produce matrix Z_1 with the condition number $\mathrm{Cond}_2(Z_1) = ||Z_1||_2||Z_1^{-1}||_2 = 1.64$ and the feedback matrices

$$F = \begin{pmatrix} 3.3599 & -2.4691 \\ -2.5437 & 1.2692 \\ 10.3080 & -6.8494 \\ -8.0702 & 5.6643 \end{pmatrix} \text{ and } G = \begin{pmatrix} -4.1868 & 1.4318 \\ -0.3352 & 0.4506 \\ -6.4921 & 1.2369 \\ 7.2495 & -2.2698 \end{pmatrix}$$

with the norms $||F||_2 = 16.6$ and $||G||_2 = 10.99$ such that the spectrum of the closed-loop pencil $P_c(\lambda)$ is precisely (5.1).

The method, essentially similar to the method 2/3 in [**KNVD**], that uses the freedom in chosing vectors γ_1 and γ_2 in order to improve the condition number of Z_1, converges after 3 steps, producing the matrix $Z_1^{(\mathrm{robust})}$ with the condition number $\mathrm{Cond}_2(Z_1^{(\mathrm{robust})}) = 1.1$ and the feedback matrices

$$F^{(\mathrm{robust})} = \begin{pmatrix} -0.6532 & 0.4781 \\ 0.7079 & -0.4183 \\ -2.2620 & 1.5349 \\ 1.6636 & -1.1733 \end{pmatrix} \text{ and } G^{(\mathrm{robust})} = \begin{pmatrix} 1.3988 & -0.7501 \\ -0.0075 & -0.0285 \\ 2.5185 & -1.2554 \\ -2.4964 & 1.3185 \end{pmatrix}$$

with the norms $||F^{(\mathrm{robust})}||_2 = 3.6$ and $||G^{(\mathrm{robust})}||_2 = 4.3$ such that the spectrum of the "robust" closed-loop pencil $P_c^{(\mathrm{robust})}(\lambda)$ is precisely (5.1).

We call the last closed-loop pencil "robust" because, aside from the mere reduction in the norm of the feedback matrices, our numerical experiments suggest that the eigenvalues of $P_c^{(\mathrm{robust})}(\lambda)$ are less affected by the random perturbations of feedback matrices. This is illustrated Figure 1 that plots the convex hulls of the closed-loop eigenvalues, when the feedback matrices F, G, $F^{(\mathrm{robust})}$ and $G^{(\mathrm{robust})}$ are perturbed, respectively, by ΔF, ΔG, $\Delta F^{(\mathrm{robust})}$ and $\Delta G^{(\mathrm{robust})}$, such that

$$||\Delta F||_2 < 0.01||F||_2, \; ||\Delta G||_2 < 0.01||G||_2$$

and

$$||\Delta F^{(\mathrm{robust})}||_2 < 0.01||F^{(\mathrm{robust})}||_2, \; ||\Delta G^{(\mathrm{robust})}||_2 < 0.01||G^{(\mathrm{robust})}||_2,$$

with 200 random perturbations.

Example 5.2. The same quadratic pencil is now used to illustrate Algorithm 4.2. We will solve Problem 1.2, reassigning again the most unstable pair of the open-loop eigenvalues; namely, $-0.0861 \pm 1.6242i$ to the same locations $-0.1 \pm 1.6242i$. That is, we want the closed-loop pencil to have the spectrum (5.1). Let the matrix of vectors to be assigned be:

$$Y_1 = \begin{pmatrix} 1.0000 & 1.0000 \\ 0.0535 + 0.3834i & 0.0535 - 0.3834i \\ 0.5297 + 0.0668i & 0.5297 - 0.0668i \\ 0.6711 + 0.4175i & 0.6711 - 0.4175i \end{pmatrix}.$$

Algorithm 4.2 produces the control matrix

$$B = \begin{pmatrix} -0.3814 & -0.5751 \\ -0.5555 & -0.4821 \\ -0.5191 & 0.4311 \\ -0.5258 & 0.5010 \end{pmatrix}$$

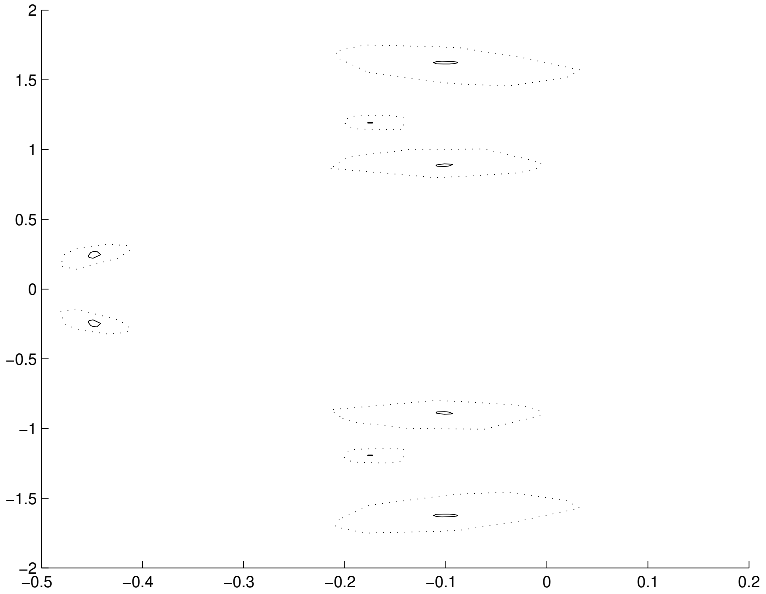

FIGURE 1. The convex hulls of closed-loop eigenvalues under 200 random 1% perturbation of the feedback matrices for quadratic pencils $P_c^{(\mathrm{robust})}(\lambda)$ (solid lines) and $P_c(\lambda)$ (dashed lines).

with $||B||_2 = 1$ and the feedback matrices

$$F = \begin{pmatrix} -8.1693 & -0.2320 \\ 2.5326 & -0.4130 \\ -20.6466 & 0 \\ 18.0019 & 0.2962 \end{pmatrix} \text{ and } G = \begin{pmatrix} 0.1688 & -1.3245 \\ 2.0584 & 0.2236 \\ -5.7223 & -3.0184 \\ 0.9821 & 2.4989 \end{pmatrix}$$

with the norms $||F||_2 = 28.6976$ and $||G||_2 = 7.07673$. The spectrum of the closed-loop pencil $P_c(\lambda)$ is precisely (5.1) and the columns of Y_1 are the eigenvectors, corresponding to the eigenvalues $-0.1 \pm 1.6242i$.

REMARK 5.3. Examples 5.1 and 5.2 are purely illustrative ones. A *real-life example* involving an 211×211 quadratic pencil with sparse matrices has been solved in [**DS**], using Algorithm 3.5.

6. Conclusions and Future Research

An uniform treatment of solutions, both theoretical and algorithmic, is presented for two important inverse eigenvalue problems for the quadratic pencil (1.2). The two problems are the problems of partial eigenvalue assignment and the partial eigenstructure assignment arising in feedback control of the matrix second-order control systems (1.1). The solutions have the following important practical features:

1. They are "*direct*" in the sense that they are obtained directly in the matrix second-order settings without resorting to a first-order reformulation, so that the important structures, such as sparsity, definiteness, symmetry, etc. can be exploited.

2. They are "*partial modal*", meaning that only a part of the spectrum (in fact, only that part that needs to be reassigned) and the corresponding eigenvectors are required.
3. No *spill-over* occurs; that is, the eigenvalues and eigenvectors that are not required to be altered do not become affected by application of feedback.
4. No explicit knowledge of damping matrix is needed in finding the feedback matrices. Damping is needed only to compute the small number of eigenvalues and the corresponding eigenvectors that need to be reassigned and in finding the matrix B in Problem 1.2.

Future Research. We conclude this section by mentioning some future research problems in this area.

Our discussions in this paper reveal that the "direct partial modal" approach for feedback problems is quite attractive for practical computations even for very large and sparse systems. Thus, some further studies on this approach for the problems under consideration and related problems are in order. The studies should include:

(i) Robust eigenvalues assignment for the quadratic pencil (1.1).
(ii) Finding a computational algorithm for minimizing the feedback norms, both for Problem 1.1 and Problem 1.2.
(iii) Finding an optimization-based algorithm, which allows one to choose the eigenvalues to be assigned from the specified stability subregion of the complex plane in such a way, that the conditioning of the closed-loop eigenvalues is a small as possible.
(iv) Extending the partial modal approach to the feedback problems of distributed parameter systems.

To justify the studies (i)-(iii), we first consider the following well-known fact: *Even if a feedback matrix is computed accurately by a numerically viable algorithm, there is no guarantee in practice that the eigenvalues of the closed-loop pencil are the same as those prescribed.*

There are several factors associated with this phenomenon (see [**BNDb**] for details):

(a) Conditioning of the eigenvector matrix of the closed-loop system.
(b) Large norm of the feedback matrix.
(c) Wide separation of the closed-loop and open-loop eigenvalues.
(d) Nearness to uncontrollability: *Small perturbations in the data can make the system uncontrollable.*

Factor (a) prompts the robust eigenvalue assignment. The robust eigenvalue assignment is concerned with choosing the eigenvector matrix of the closed-loop system in such a way that the condition number of the eigenvector matrix is as small as possible. Recall that the closed-loop eigenvector matrix for Problem 1.1 is the matrix

$$\begin{pmatrix} Y_1 & X_2 \\ Y_1\Lambda_1' & X_2\Lambda_2 \end{pmatrix}. \tag{6.1}$$

Since the matrices X_2 and Λ_2 are to remain unaltered, the problem then reduces to choosing the matrix $\begin{pmatrix} Y_1 \\ Y_1\Lambda_1' \end{pmatrix}$ in such a way that the condition number of (6.1) is minimized.

This can perhaps be done using the same type of technique as used in the well-known paper [**KNVD**] for the first-order system. Empirical results suggest that this is indeed a good thing to do and the condition number obtained this way, in each case of our numerical experiments, has turned out to be smaller then that obtained without applying any specific criterion for choosing $\begin{pmatrix} Y_1 \\ Y_1\Lambda_1' \end{pmatrix}$. Some more definitive work needs to be done. See also a related paper by [**CD**] where two numerical algorithms for robust eigenvalue assignment for a quadratic pencil have been proposed for the full-order eigenvalue assignment.

The consideration of factor (b) gives rise to (ii). There exists an algorithm due to [**KFB**] for minimizing the norm of the feedback matrix for the first-order system.

Factor (c) is related to (iii). An optimization-based algorithm has been recently proposed in [**CLRa**] and [**CLRb**] for the first-order model. An analogous algorithm for the quadratic pencil (1.2) is to be developed.

Finally, regarding (iv), we note that the second-order model (1.2) is just a discretized approximation (say, by the finite element method) of a distributed parameter system; thus, in spite of the fact that a second-order model is much used in practice for convenience, it has some severe limitations. For example, suppose that starting with a distributed model, first a second-order model is obtained by discretization and then Problems 1.1 and 1.2 are solved using the direct partial-modal approach of this paper. Even though Theorem 2.1 and Theorem 3.1 guarantee no *spill-over* of the $2n-p$ eigenvalues that are not required to be reassigned, there still remains obvious uncertainity with the remaining infinite number of eigenvalues of the infinite-order system.

It is, therefore, desirable (though extremely hard) to obtain solutions directly from the distributed model without going through a discretization procedure. Some attempts, however, have been made already in this direction. Generalizing the results of [**DERb**], a solution to a single-input version of Problem 1.1 for a distributed gyroscopic system entirely in terms of the distributed parameters has been recently obtained in [**DRS**] (see also [**YMR**]). Specifically, the following problem has been solved:

Given the self-adjoint positive definite operators M and K, a gyroscopic operator G, and a self-conjugate set $\Omega = \{\mu_1, \ldots, \mu_p\}$, find feedback functions $f(x)$ and $g(x)$ such that each member of Ω is an eigenvalue of the closed-loop operator system

$$M\frac{\partial^2 \nu(t,x)}{\partial t^2} + G\frac{\partial \nu(t,x)}{\partial t} + K\nu(t,x) \quad = \quad b(x)(f(x), \frac{\partial \nu(t,x)}{\partial t}) + b(x)(g(x), \nu(t,x)),$$

where $(\cdot,\cdot)$ is a scalar product, and the remaining infinite number of eigenvalues $\lambda_{p+1}, \lambda_{p+2}, \ldots$ remain the same as those of the open-loop operator system

$$M\frac{\partial^2 u(t,x)}{\partial t^2} + G\frac{\partial u(t,x)}{\partial t} + Ku(t,x) \quad = \quad 0.$$

The solution has been obtained in terms of the quantities given and entirely in the distributed parameter setting (that is, without any use of the discretization technique). The results obtained in this paper are the first and only results available for inverse eigenvalue problems for a quadratic operator pencil. Clearly, much remains to be done in this area.

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