Finite Element Model Updating, Eigenstructure Assignment and Eigenvalue Embedding Techniques for Vibrating Systems.

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Abstract

This paper presents a brief overview of the existing direct methods for the FEM (Finite Element Model) updating problem and reviews the recently developed "Direct and Partial Modal" approach for the partial eigenstructure assignment problem and the "Eigenvalue Embedding" techniques for vibrating systems, in the context of the solution of this problem.

1. Introduction

Vibrating structures such as automotives, bridges, highways, building etc., are distributed parameter systems. However, due to lack of viable computational methods to handle distributed parameter systems, such systems are usually discretized to a matrix second-order system of the form:

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0 \tag{1.1}$$

by using finite element techniques. Such a system in vibration literature is called a Finite Element Model (FEM). The matrices M, K and D are, respectively, known as the mass, stiffness and damping matrices. These are $n \times n$ symmetric matrices and very often in practice, M is assumed to be positive definite (denoted by $M > 0$) and K is positive semidefinite (denoted by $K \geq 0$). While the mass and stiffness matrices are very often accurately measurable, it is hard to estimate damping matrix in practice (Inman (1989)).

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Separation of variables

$$x(t) = ze^{\lambda t},\, z \text{ a constant vector}$$

in equation (1.1), leads to the quadratic eigenvalue problem:

$$P(\lambda_k)z_k = 0,\;\; k = 1, 2, \cdots, 2n,$$

where $P(\lambda) = \lambda^2 M + \lambda D + K$. The numbers $\lambda_1, \cdots, \lambda_{2n}$ are the eigenvalues and the vectors $z_1, \cdots, z_{2n}$ are the eigenvectors.

Finite Element Models are generally of very high degrees of freedom; usually several thousands. Unfortunately, the FEM only accurately predicts about one-third of the natural frequences and mode shapes (eigenvalues and eigenvectors). To ensure the validity of the entire FEM, a few small number of eigenvalues and eigenvectors are experimentally measured from a realized practical structure and these measured eigenvalues and eigenvectors are then compared with those of the FEM. Frequently, there is a disagreement between these two sets.

A vibration engineer then faces the problem of updating the existing FEM with minimal changes so that the updated FEM has the following properties:

1. The measured eigenvalues and eigenvectors are also the eigenvalues and eigenvectors of the FEM.
2. The remaining eigenvalues and eigenvectors remain unchanged.
3. The exploitable properties such as symmetry, definiteness, sparsity and bandedness, etc., of the original model are preserved.

Such a problem is called **Finite Element Model Updating** (FEMU) problem. There are different variations of the updating problem: direct updating of the mass, stiffness and damping matrices, updating of only certain parameters, and updating based on frequency response functions.

In this paper, we are concerned only with the direct updating problem. For details of other variations and their existing solution techniques, the readers are referred to the authoritative book in this area by Friswell and Mottershead (1995).

Mathematically, the problem of direct updating of the mass, stiffness and damping matrices may be defined as follows:

Given

1. The finite element generated symmetric matrices M, K, and D:

$$M = M^T > 0, \quad K = K^T \geq 0 \text{ and } D = D^T$$

2. A set of measured eigenvalues $\{\mu_1, \cdots, \mu_m\}$ and the eigenvectors $\{y_1, \cdots, y_m\}$ from a real-life structure.

Find the **symmetric updates** $\Delta M, \Delta K$, and ΔD such that the quadratic eigenvalue pencil associated with the updated model

$$P_U(\lambda) = \lambda^2 \tilde{M} + \lambda \tilde{K} + \tilde{D} = 0,$$

where

$$\tilde{M} = M + \Delta M$$
$$\tilde{K} = K + \Delta K$$

and

$$\tilde{D} = D + \Delta D$$

has the spectrum $\{\mu_1, \cdots, \mu_m;\ \lambda_{m+1}, \cdots, \lambda_{2n}\}$ and the eigenvectors of $P_U(\lambda)$ are $\{y_1, y_2, \cdots, y_m;\ z_{m+1}, \cdots z_{2n}\}$.

The above is an ideal mathematical definition of the direct FEM updating problem. In certain practical instances, some of the constraints imposed by the above definition (such as the invariance of the unmeasured eigenvalues and eigenvectors) might be relaxed, depending upon what the purpose of updating is. For detailed discussions, see Friswell and Mottershead (1995).

The author believes that the updating problem, as defined above, has not been satisfactorily solved yet. However, there now exist many partial solutions to the problem. These include *optimization techniques* using measured and modal data and *eigenstructure assignment* techniques from control theory. There also have been some recent developments on the solutions of the *eigenvalue embedding problem*, which is closely related to the model updating problem.

We present a brief overview of these existing and recently developed methods. Our discussions clearly show that further research is still needed in this area. Our own idea of solving the problem via eigenstructure assignment technique and the discussion of the eigenvalue embedding techniques in this context are included.

2. Practical Limitations

Though the FEMU problem is easy to state mathematically, there are some severe practical limitations which make it difficult to solve the problem in practice. Some of the principal limitations are:

i. **Incompleteness of the Measured Data:**
The data (eigenvalues and eigenvectors) measured experimentally from a real-life structure are very often incomplete in the sense that, because of hardware limitations, the data can be measured only at a subset of the degrees of freedom of the finite element model. While the finite element models can be of several thousand of degrees of freedom, the measured mode shapes from a practical structure can be of lengths of at most couple hundred.
Then comparing the analytical modes with those from a real-life structure which have different lengths become impossible in practice; unless some measures are taken to fill-in the missing entries of the modes of the practical structure or reduce the number of mode shape components of the FEM.

ii. **Complex-Nature of the Measured Modes.**
Experimentally measured modes can be complex, since they are obtained from a real-life structure which is damped. On the other hand, damping is rarely included in a finite element analysis and, as a result, very often the finite element generated modes are real.

Comparison of these types of modes, real modes from an undamped finite element model and complex modes from a damped real-life structure is extremely difficult; unless the complex modes are converted into the corresponding real ones using some mathematical techniques.

3. The Current Engineering Practice to Overcome Practical Limitations

The difficulty of the incomplete data is overcome either

i. by reducing the number of degrees of freedom in the finite element model (**Model Reduction**) or

ii. by expanding the measured modes by estimating the missing entries corresponding to the unmeasured degree of freedom or

iii. by combining these two approaches.

There are several methods for model reduction available in the vibration literature. Among them, the method, called **"System Equivalent Reduction Expansion Process"** (SEREP) (O'Callahan et.al., 1989) is known to reproduce the lower natural frequencies of the full-order model very accurately (Friswell and Mottershead (1995)). In this method, the reduced-order mass and stiffness matrices, denoted by M_R and K_R are given by

$$M_R = T_u^T M T_u, \quad K_R = T_u^T K T_u$$

where the transforming matrix T_u is defined by

$$T_u = \begin{pmatrix} \phi_m \\ \phi_s \end{pmatrix} \phi_m^\dagger.$$

The matrices ϕ_m and ϕ_s are the matrices resulting from the partitioning of the eigenvector matrix

$$\phi = \begin{pmatrix} \phi_m \\ \phi_s \end{pmatrix}$$

of the FEM, corresponding to the master and slave coordinates. The matrix $\phi_m^\dagger$ denotes the generalized inverse of ϕ_m (Friswell and Mottershead (1995)).

Implementation of this process requires knowledge of the complete mass and stiffness matrices in FEM; unfortunately, many times FEM packages do not store this data completely. **Thus, this process may be difficult to implement in practice.**

The expansion of the measured modes may be achieved either by using the mass and stiffness matrices or by using the modal data itself. We state the process below that uses the modal data.

The process of expansion of the measured modes using the modal data is based on the assumption that the transformation matrix T relating the finite element modes $[\phi_{\mathrm{FEM}}]_m$ at the measured degrees of freedom and the measured modes ϕ_m is given by

$$\phi_m = [\phi_{\mathrm{FEM}}]_m T.$$

Since both the matrices ϕ_m and $[\phi_{\mathrm{FEM}}]_m$ are known, T can be computed as

$$T = [\phi_{\mathrm{FEM}}]_m^{\dagger} \phi_m,$$

where $\dagger$ denotes the generalized inverse. The matrix of the modes at unmeasured degrees of freedom ϕ_{um} is then computed as

$$\phi_{um} = [\phi_{\mathrm{FEM}}]_{um} T,$$

where $[\phi_{\mathrm{FEM}}]_{um}$ is the matrix of the finite element modes at unmeasured degrees of freedom.

The problem of complex modes can perhaps be best handled by relating the matrix of real modes ϕ_R to the matrix ϕ_C of complex modes by the complex transformation T given by

$$\phi_R = \phi_C T.$$

Equating real and imaginary parts and setting the real part of T as the identity matrix, it is easy to see that

$$\phi_R = Re(\phi_C) + I_m(\phi_C)(Re(\phi_C)^T Re(\phi_C))^{-1} Re(\phi_C)^T I_m(\phi_C),$$

where $Re(\cdot)$ and $Im(\cdot)$ denote the real and complex parts, respectively (see Friswell and Mottershead (1995)).

4. Direct Methods for Model Updating

In this section, we briefly review some of the existing and newly developed direct methods for FEM updating.

The direct methods can be broadly classified into the following groups:

i. Optimization Techniques on Modal Matrix (Baruch and Bar-Itzhack (1978), Baruch (1978)).

ii. Optimization Techniques using Measured Data (Berman and Nagy (1983), Caesar (1986, 1987)).

iii. Partial Eigenstructure Assignment Methods (Datta, Elhay, Ram and Sarkissian (2000), Datta and Sarkissian (2001), Minas and Inman (1990), Zimmerman and Widengren (1990)).

iv. Quadratic Eigenvalue Embedding Methods (Carvalho, Datta, Lin and Wang (2001), Ferng, Lin, Pierce and Wang (2000), Fritchman and Pierce (1996)).

4.1 Optimization Techniques on Modal Matrix Element Data

As the title suggests these methods use optimization techniques to correct the modal matrix ϕ assuming that one of the finite element matrices M or K is accurately known. We state one such method where the mass matrix M is assumed to be exact. The method comes in two steps. In the first step, the modal matrix ϕ is corrected using a minimization technique with constraints on the orthogonality of the mass matrix M and then the stiffness matrix K is corrected with constraints on the symmetry of K and on the orthogonality between the eigenvectors of the linear pencil $K - \lambda M$.

We remind the readers of the orthogonality relations between the eigenvectors (mass normalized) of the symmetric definite linear pencil: $K - \lambda M,\ M = M^T > 0$, and $K = K^T \geq 0$:

Let ϕ be the matrix of eigenvectors (known as the *modal matrix*) and Λ be the matrix of eigenvalues. Then the following orthogonality relations hold (Datta (1995, 2002)):

$$\phi^T M \phi = I$$

and

$$\phi^T K \phi = \Lambda.$$

Algorithm 4.1 *Finite Element Updating using Optimization of the Modal Matrix*

Inputs:	ϕ_m	- the measured eigenvector matrix
	W_ϕ	- some suitably chosen weight for the modal matrix ϕ
	M	- finite element generated mass matrix
	K	- finite element generated stiffness matrix
	m	- the number of measured eigenvalues and eigenvectors
	n	- the number of degrees of freedom
	Λ	- the diagonal matrix containing the measured eigenvalues
	W_K	- some suitably chosen weight for the stiffness matrix K
Output:	ϕ	- the corrected modal matrix
	$\tilde{K}$	- the corrected stiffness matrix

Step 1. (Updating of the Modal Matrix)
Determine the matrix ϕ such that

$$J_\phi = \|W_\phi(\phi - \phi_m)\|_2$$

is minimized, subject to the constraint:

$$\phi^T M \phi = I_m.$$

Step 2. (Updating of the Stiffness Matrix)
Determine the matrix $\tilde{K}$ by minimizing

$$J_K = \|W_K(\tilde{K} - K)W_K\|_2$$

subject to the constraints:

$$\tilde{K}^T = \tilde{K} \text{ (symmetry)}$$

and

$$\tilde{K}\phi = M\phi\Lambda$$

Computational Remarks: There are some freedom to the designer to choose the weighting matrices. Baruch and Bar-Itzhack (1978), Baruch (1978) have proposed

$$W_\phi = M^{\frac{1}{2}}, \text{ and } W_K = M^{-\frac{1}{2}}.$$

But other convenient choices are possible.

Minimizations of both J_ϕ and J_K can be done using the method of Lagrange multipliers. If the Lagrange multiplier matrix is made unique by imposing symmetry on it, then it can be shown that the updated stiffness matrix is unique which is the unique solution of the associated minimization problem.

4.2 Optimization Techniques using Measured Data

In these techniques, both the mass and stiffness matrices are updated using the measured eigenvector matrix in the constraints. The matrices M and K are updated one at a time in which the first updated matrix is used in the updating of the second. A wide variety of such methods were suggested by Caesar (1986, 1987). We state one such method where the mass matrix is updated first.

Algorithm 4.2: *Finite Element Updating using Measured Data*

Inputs:	ϕ_m	- the measured eigenvector matrix
	W_M	- some suitable weight on the mass matrix
	M	- the finite element generated mass matrix
	K	- the finite element generated stiffness matrix
	m	- the number of measured eigenvalues and eigenvectors
	n	- the number of degrees of freedom
	Λ	- the diagonal matrix containing the measured eigenvalues
	W_K	- some suitably chosen weight on the stiffness matrix
Outputs:	$\tilde{M}$	- the updated mass matrix
	$\tilde{K}$	- the updated stiffness matrix

Step 1. (Updating the Mass Matrix)
Compute the updated mass matrix $\tilde{M}$ by minimizing

$$J_M = \frac{1}{2}\|W_M(\tilde{M} - M)W_M\|_2$$

subject to the constraints:

$$\phi_m^T \tilde{M}\phi_m = I_m$$

$$\tilde{M}^T = \tilde{M} \text{ (symmetry)}$$

Step 2. (Updating of the Stiffness Matrix)
Compute the updated stiffness matrix $\tilde{K}$ by minimizing

$$J_K = \|W_K(\tilde{K} - K)W_K\|_2$$

subject to the constraints:

$$\tilde{K}^T = \tilde{K} \text{ (symmetry)}$$

$$\tilde{K}\phi_m = \tilde{M}\phi_m\Lambda$$

Computational Remark
Again, the weighting matrices can be chosen by the designer. It has been suggested in the literature to use $W_M = M^{-\frac{1}{2}}$ and $W_K = K^{-\frac{1}{2}}$ so that equal weighting may be placed on both mass and stiffness matrices.

4.3 Partial Eigenstructure Assignment and Model Updating

In this section, we discuss the idea of using the concept of eigenstructure assignment from control theory to solve the model updating problem.

Consider the second-order control system with a control force $Bu(t)$:

$$\begin{aligned} &M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t) \\ &y(t) = C_0x(t) + C_1\dot{x}(t) \end{aligned} \tag{4.3.1}$$

Here $y(t)$ is the output vector, and $u(t)$ is the control or input vector.

In control theory, the matrices B, C_0 and C_1 are given, but in structural dynamics for the model updating problem, these matrices can be chosen (Friswell and Mottershead (1995)).

When the matrices B, C_0, and C_1 are arbitrarily given a priori, a solution to the eigenstructure assignment may not exist (Andry et al. (1983)).

A well-known classical problem that arises in control system design is the **Partial Eigenstructure Assignment Problem** . The aim of this problem is to modify the stiffness and damping matrices by using feedback so that the modified second-order system will contain a number of given eigenvalues and eigenvectors while the other eigenvalues and eigenvectors remain unchanged.

Specifically, if the state-feedback is used, then the control law of the form

$$u(t) = F^T\dot{x}(t) + G^Tx(t) \tag{4.3.2}$$

is sought so that the closed-loop quadratic matrix pencil

$$P_C^S(\lambda) = \lambda^2M + \lambda(D - BF^T) + (K - BG^T) \tag{4.3.3}$$

associated with the updated system

$$M\ddot{x}(t) + (D - BF^T)\dot{x}(t) + (K - BG^T)x(t) = 0 \tag{4.3.4}$$

will have a given set of eigenvalues and eigenvectors while the remaining eigenvalues and eigenvectors of the pencil $P_C^S(\lambda)$ remain the same as those of the original pencil $P(\lambda) = \lambda^2 M + \lambda K + D$.

In the output-feedback case, a control law of the form

$$u(t) = Hy(t) \tag{4.3.5}$$

is sought so that the closed-loop pencil

$$P_C^O(\lambda) = \lambda^2 M + \lambda(D - BHC_1) + (K - BHC_0) \tag{4.3.6}$$

will have a given set of eigenvalues and the eigenvectors while the remaining eigenvalues and eigenvectors are the same as those of the original pencil $P(\lambda)$.
Clearly, the eigenstructure assignment problem is closely related to the FEM updating problem. In fact, the FEM updating problem is the constrained partial eigenstructure assignment problem under the constraint that the closed-loop pencil $P_C^S(\lambda)$ or $P_C^O(\lambda)$ is symmetric.

We now state a recent result on the eigenstructure assignment by state feedback and show how it can possibly be used to solve the FEM updating problem.

4.3.1 State-Feedback Eigenstructure Assignment in Finite Element Model

Let
$\Lambda_1 =$ diag $(\lambda_1, \cdots, \lambda_p)$ - the matrix of the finite element generated eigenvalues to be reassigned.
$X_1 = (x_1, \cdots, x_p)$ - the matrix of the finite element generated eigenvectors to be reassigned.
$\Lambda_1' =$ diag $(\mu_1, \cdots, \mu_p)$ - the matrix of the measured eigenvalues.
$Y_1 = (y_1, \cdots, y_p)$ - the matrix of the measured eigenvectors.
Define

$$\Phi = \Lambda_1' Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1. \tag{4.3.7}$$

Then, the following result can be proved (Datta, Elhay, Ram and Sarkissian (2000)):

Theorem 4.1 (Partial Eigenstructure Assignment in Finite Element Model)
Let $\Lambda_1, X_1, \Lambda_1'$ and Y_1 be the same as above. Then

i. The triplet $(\hat{B}, \hat{F}, \hat{G})$ defined by

$$\begin{aligned} \hat{B} &= MY_1(\Lambda_1')^2 + DY_1\Lambda_1' + KY_1 \\ \hat{F} &= MX_1\Lambda_1\Phi^{-1}, \end{aligned} \tag{4.3.8}$$

and

$$\hat{G} = -KX_1\Phi^{-1}$$

constitutes a solution (possibly complex) of the state-feedback eigenstructure assignment problem

ii. Given a possibly complex solution triplet $(\hat{B}, \hat{F}, \hat{G})$, the real solution triplet (B, F, G) is computed from $\hat{B}, \hat{F}$, and $\hat{G}$, as follows:

Let $H = \hat{B}(\hat{F}^T|G^T)$ (real), and let $LR = H$ be a factorization of H, where $L \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times 2n}$, obtained by either the QR factorization or the Singular Value decomposition (Datta (1995)).

Then

$$\begin{aligned} B &\equiv L \in \mathbb{R}^{n\times m} \\ F^T &\equiv \text{ the first } n \text{ columns of } R \\ G^T &\equiv \text{the last } n \text{ columns of } R \end{aligned} \tag{4.3.9}$$

Remark: A more general case (assuming only that the mass, stiffness and damping matrices are symmetric and the mass matrix is invertible) of both the partial eigenvalue and eigenstructure assignment problems has been treated in the recent Ph.D. dissertation of Daniil Sarkissian (2001).

A proposed solution to the FEMU problem using Theorem 4.1

The updated matrices $(K-BG^T)$ and $(D-BF^T)$ are, in general, not symmetric. To bring the symmetric back to these matrices, as much as possible, an optimization procedure will now be proposed.
Our strategy will be to update the matrix D first, using an optimization technique that will make the matrices BF^T and BG^T as symmetric as possible, in the sense that the sum of the norms of their non-symmetric parts will be minimized. After the matrix D is updated to a new symmetric matrix D^*, Theorem 4.1 will then be used to construct updates $\tilde{D} = D^* - BF^T$, $\tilde{K} = K - BG^T$.
Since the damping matrix D is treated as a variable in the optimization procedure (described below) and the matrices B, X_1 and Λ_1 depend on D (B depends explicitly and X_1, Λ_1 depend implicitly), we will denote these quantities now by B_D, X_1^D and Λ_1^D to emphasize their dependence on D. Furthermore, since the matrices Φ, F and G also depend on X_1 and Λ_1 (and therefore on D), we will also denote them by Φ^D, F_D and G_D, respectively. Thus, we write the equations (4.3.7) and (4.3.8) as follows:

$$\Phi^D = \Lambda_1' Y_1^T M X_1^D \Lambda_1^D - Y_1^T K X_1^D \tag{4.3.10}$$

$$B_D = M Y_1 (\Lambda_1')^2 + D Y_1 \Lambda_1' + K Y_1 \tag{4.3.11}$$

$$F_D = M X_1^D \Lambda_1^D (\Phi^D)^{-1} \tag{4.3.12}$$

$$G_D = -K X_1^D (\Phi^D)^{-1} \tag{4.3.13}$$

Also, we have the eigendecomposition

$$M X_1^D (\Lambda_1^D)^2 + D X_1^D \Lambda_1^D + K X_1^D = 0. \tag{4.3.14}$$

Based on the above discussions, we propose the following algorithm for FEM updating problem.

Algorithm 4.3 Finite Element Updating using State-Feedback Eigenstructure Assignment

Inputs:	M, D and K	- FEM generated matrices
	Λ_1' and Y_1	- matrices of measured eigenvalues and corresponding eigenvectors
Outputs:	$\tilde{D}, \tilde{K}$	- updated matrices such that $MY_1(\Lambda_1')^2 + \tilde{D}Y_1\Lambda_1' + \tilde{K}Y_1 = 0$.

Step 1: Compute a matrix D^* that minimizes the functional

$$J(D) = \|B_D F_D^T - F_D B_D^T\|_F + \|B_D G_D^T - G_D B_D^T\|_F \ , \tag{4.3.15}$$

where matrix functions B_D, F_D and G_D are defined by (4.3.10)- (4.3.14), subject to the constraint

$$D = D^T. \tag{4.3.16}$$

Step 2: Compute matrices $\hat{B}, \hat{F}$ and $\hat{G}$ using (4.3.10)-(4.3.14) with $D = D^*$.

Step 3. Compute B, F, and G from $\hat{B}, \hat{F}$ and $\hat{G}$ using (4.3.9).

Step 4: Compute $\tilde{D} = D^* - BF^T, \tilde{K} = K - BG^T$.

Return

4.3.2 Output-Feedback Eigenstructure Assignment and Finite Element Model Updating.

The idea of using partial output-feedback eigenstructure assignment to solve the model updating problem has been used in two important papers: Minas and Inman (1990) and Zimmerman and Widengren (1990).

Let $\Lambda_m =$ diag $(\lambda_1, \cdots, \lambda_m)$ and ϕ_m be the matrices of measured eigenvalues and eigenvectors. Then the ϕ_m and Λ_m satisfy the following matrix equation:

$$M\phi_m\Lambda_m^2 + D\phi_m\Lambda_m + K\phi_m = BHC_1\phi_m\Lambda_m + BHC_0\phi_m$$

Since M is invertible, we obtain

$$\phi_m\Lambda_m^2 + M^{-1}D\phi_m\Lambda_m + M^{-1}K\phi_m = M^{-1}BH[C_1\phi_m\Lambda_m + C_0\phi_m].$$

An explicit expression for the output-feedback matric H now can be computed:

$$H = (M^{-1}B)^\dagger(\phi_m\Lambda_m^2 + M^{-1}D\phi_m\Lambda_m + M^{-1}K\phi_m)(C_1\phi_m\Lambda_m + C_0\phi_m)^\dagger,$$

where '†' denotes the pseudo inverse. Using this expression of H, the updates for the stiffness and damping matrices can be calculated. The updated damping and stiffness matrices are given by:

$$\hat{D} = D - BHC_1$$

and

$$\hat{K} = K - BHC_0$$

Unfortunately, the updated matrices $\hat{D}$ and $\hat{K}$ are not necessarily symmetric. An optimization procedure now needs to be performed to bring the symmetry back. This is an unconstrained optimization problem.

Let $A_1 = BHC_1, \quad A_2 = BHC_0$. Define $A_3 = A_1 - A_1^T, \quad A_4 = A_2 - A_2^T$. Then Minas and Inman (1990) have suggested to minimize the objective function J as the sum of the squares of the elements of the skew-symmetric matrices A_3 and A_4. The minimization is performed over the entries of the matrices C_0 and C_1.

An explicit expression for the objective function J in terms of the matrices $M,\ K,\ D,\ \Lambda_m,\ \phi_m$ and the unknown matrices C_0 and C_1 can be written down (see Minas and Inman (1990)).

The optimization procedure, in general, will improve the symmetry of the updated damping and stiffness matrices, but will not make them perfect symmetric. Minas and Inman (1990) have proposed an iterative technique to overcome this practical difficulty. The matrices C_0 and C_1 are kept fixed, but the original damping and stiffness matrices are now replaced by the symmetric parts of $\hat{D}$ and $\hat{K}$ and the eigenstructure assignment process is repeated.

After every iteration, the model is updated and checked for convergence of the eigenvalues and eigenvectors of the updated model to the assigned data.

Remarks: Both the state-feedback and output-feedback methods recover the measured eigenvalues and eigenvectors, the updated matrices are nearly symmetric; however, **there is no guarantee that the unmeasured eigenvalues of the original system remain unchanged after updating.** Zimmerman and Widengren (1990) have used the idea of algebraic Riccati equations to produce the symmetric updates directly. We refer the readers to their paper for details. See also Wei (1989).

5. Model Updating by Eigenvalue Embedding

In a recent paper, Ferng, Lin, Pierce and Wang (2001) have considered the updating of a slightly more general model using nonequivalence transformation of matrix polynomials.

The second-order model considered by them is of the form

$$M\ddot{x}(t) + (D_1 + d(s)D_2)\dot{x}(t) + (K_1 + d(s)K_2)x(t) = H(t,s). \tag{5.1}$$

The system (5.1) arises in the construction of aircraft structures models for dynamic load analysis. Choosing $d(s) = \dfrac{\rho(s-\omega)+\rho}{s-\omega}$, with constants $\rho \neq 0$ and $\omega \neq 0$, the above model can be written in the standard state-space form: $\dot{z} = Az + Bu(t,s)$, where

$$A = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 \\ I & 0 & 0 & 0 \\ 0 & \rho I & \omega I & 0 \\ \rho I & 0 & 0 & \omega I \end{pmatrix} \in \mathbb{R}^{4n\times 4n}$$

$$M_1 = -M^{-1}(D_1 + \rho D_2)$$

$$M_2 = -M^{-1}(K_1 + \rho K_2)$$

$$M_3 = -M^{-1}K_2$$

$$M_4 = -M^{-1}D_2$$

and

$$z = \begin{pmatrix} z_1 \\ x \\ z_3 \\ z_4 \end{pmatrix}$$

and $Bu(t)$ represents $H(t,s)$.

The idea is to transform, using a nonequivalent transformation, the matrix A to another matrix $\tilde{A}$ having the same structure as A such that an isolated real or a pair of complex conjugate eigenvalues of A are replaced by a given real or a pair of complex conjugate eigenvalues of $\tilde{A}$.

Since $\tilde{A}$ will have the same structure as A, this form of $\tilde{A}$ will then correspond to an updated matrix second-order system of the same form as (5.1), but the spectrum of this updated system will contain the given set of the eigenvalues and eigenvector set will contain the given eigenvectors.

Define a nonsingular matrix Q as

$$Q = \begin{pmatrix} I & -\omega I & 0 & 0 \\ 0 & I & -\omega I & 0 \\ 0 & 0 & \rho I & 0 \\ 0 & \rho I & 0 & I \end{pmatrix}$$

Then it is easy to see that

$$Q^{-1}AQ = \begin{pmatrix} A_2 & A_1 & A_0 & M_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & \omega I \end{pmatrix},$$

where

$$A_0 = -\omega M_2 + \rho M_3$$

$$A_1 = -\omega M_1 + M_2 + \rho M_4$$

$$A_2 = M_1 + \omega I.$$

Let (μ_1, y_1) be a real eigenpair of A, and μ_1 be replaced by η_1. Define

$$\tilde{A}_3 = A_3 = -I$$

$$\tilde{A}_2 = A_2 - \epsilon_1 A_3 y_1 z_1^T$$

$$\tilde{A}_1 = A_1 - \epsilon_1(\mu_1 A_3 + A_2) y_1 z_1^T$$

$$\tilde{A}_0 = A_0 - \epsilon_1(\mu_1^2 A_3 + \mu_1 A_2 + A_1) y_1 z_1^T,$$

where $\epsilon_1 = \eta_1 - \mu_1$ and z_1 is chosen such that $z_1^T y_1 = 1$.

When a complex conjugate pair of eigenvalues $\alpha_1 + \beta_1$ and $\alpha_1 - i\beta_1$ are replaced, respectively, by $\tilde{\alpha}_1 + i\tilde{\beta}_1$ and $\tilde{\alpha}_1 - i\tilde{\beta}$, then these matrices are given by

$$\begin{aligned}
\tilde{A}_3 &= A_3 = -I \\
\tilde{A}_2 &= A_2 - A_3 y_1 E_1 Z_1^T \\
\tilde{A}_1 &= A_1 - (A_3 y_1 \Lambda_1 + A_2 y_1) E_1 Z_1^T \\
\tilde{A}_0 &= A_0 - (A_3 y_1 \Lambda_1^2 + A_2 y_1 \Lambda_1 + A_1 Y_1) E_1 Z_1^T
\end{aligned}$$

where $E_1 = \Omega_1 - \Lambda_1$, with

$$\Lambda_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} \tilde{\alpha}_1 & \tilde{\beta}_1 \\ -\tilde{\beta}_1 & \tilde{\alpha}_1 \end{pmatrix}$$

and $Y_1 = [Re(y_1),\ I_m(y_1)]$, and Z_1 is chosen such that $Z_1^T y_1 = I_2$.
With these modified matrices $\tilde{A}_i,\ i = 0, 1, 2$, the new matrix $\tilde{A}$ having a new real or a pair of complex conjugate eigenvalues, is given by:

$$\tilde{A} = \begin{pmatrix} \tilde{M}_1 & \tilde{M}_2 & \tilde{M}_3 & \tilde{M}_4 \\ I & 0 & 0 & 0 \\ 0 & \rho I & \omega I & 0 \\ \rho I & 0 & 0 & \omega I \end{pmatrix}$$

where

$$\tilde{M}_1 = \tilde{A}_2 - \omega I$$

$$\tilde{M}_2 = \tilde{A}_1 + \omega \tilde{M}_1 - \rho \tilde{M}_4$$

$$\tilde{M}_3 = \frac{1}{\rho}(\tilde{A}_0 + \omega \tilde{M}_2)$$

$$\tilde{M}_4 = M_4.$$

Note: The eigenvectors will be modified when some eigenvalues are replaced. Ferng et al. (2001) have described a procedure which can be used to preserve certain eigenvectors and even to assign some given sets of eigenvectors as well. For details, see Ferng et al. (2001).

Remarks: The proposed procedure preserves the matrix structure and assigns a given set of eigenvalues and some eigenvectors. **But, unfortunately, the method can not preserve the symmetry**.

A Symmetry Preserving Eigenvalue Embedding Method

The following method, recently proposed by Carvalho, Datta, Lin and Wang (2001), assigns a given set of real or complex conjugate pairs of eigenvalues, the associated eigenvectors are preserved, the eigenvalues and eigenvectors which are not to be reassigned remain unchanged and above all, the updated model remains symmetric. The updated matrices are obtained from the original finite element generated matrices using low rank symmetric perturbations of the mass, stiffness and damping matrices.

The following theorem shows how the updated matrices are computed to assign a real isolated eigenvalue of $P(\lambda)$.

Theorem 5.1 *Let (λ_1, y_1) be an unwanted real isolated eigenpair of $P(\lambda) = \lambda^2 M + \lambda D + K$ with $y_1^T K y_1 = 1$. Let λ_1 be reassigned to μ_1. Define $\theta_1 = y_1^T M y_1$ and assume that $1 - \lambda_1 \mu_1 \theta_1 \neq 0$ and $1 - \lambda_1^2 \theta_1 \neq 0$.*
Also, define $\epsilon = \dfrac{\lambda_1 - \mu_1}{1 - \lambda_1 \mu_1 \theta_1}$. Then the following updated matrix polynomial

$$P_U(\lambda) = \lambda^2 M_U + \lambda D_U + K_U$$

with

$$M_U = M - \epsilon_1 \lambda_1 M y_1 y_1^T M$$

$$D_U = D + \epsilon_1 (M y_1 y_1^T K + K y_1 y_1^T M)$$

$$K_U = K - \frac{\epsilon_1}{\lambda_1} K y_1 y_1^T K$$

is such that

i. The eigenvalues of $P_U(\lambda)$ are the same as those of $P(\lambda)$ except that λ_1 has been replaced by μ_1.

ii. y_1 is also an eigenvector of $P_U(\lambda)$ corresponding to the embedded eigenvalue μ_1.

iii. If (λ_2, y_2) is an eigenpair of $P(\lambda)$, where $\lambda_2 \neq \lambda_1$, then (λ_2, y_2) is also an eigenpair of $P_U(\lambda)$.

Remarks: In the context of the FEM updating problem, the pair (λ_1, y_1) is a finite element generated eigenpair and μ_1 is a measured eigenvalue.
Notes:

i. The method described in the above theorem has been extended to simultaneously assign a group of real eigenvalues.

ii. A theorem analogous to Theorem 5.1 on the assignment a pair of complex conjugate eigenvalues has been obtained; however, formulas for updating of the mass, stiffness and damping matrices in this case are more complicated. Details can be found in the papers by Carvalho et al.(2001).

iii. Simultaneous assignment of more than one pair of complex conjugate eigenvalues as well as assignment of a given set of eigenvectors are currently being investigated.

6. Conclusion

We have briefly discussed some of the existing approaches, some are well-known and some are recently developed, for the well-known **finite element model updating problem**. These include

i. Optimization Procedures

ii. Eigenstructure Assignment Procedures

iii. Eigenvalue Embedding Procedures

Unfortunately, none of these approaches completely solves the FEM updating problem, as stated here.

The optimization techniques in Section 4.2 deal with mass and stiffness matrices only; the damping matrix is not considered. **Thus, these techniques are not truly the model updating techniques of a damped second-order model**.

The state feedback and output-feedback approaches presented in **Section 4.3.1** and **Section 4.3.2**, respectively, address the problem of explicitly updating a damped second-order model. **But they can not guarantee the invariance of the unassigned eigenvalues and eigenvectors**. In particular, **Algorithm 4.3** has not been tested yet.

The eigenvalue embedding method of Ferng et al. (2001) does not preserve the symmetry.

Finally, the eigenvalue embedding method of Carvalho et al. (2001) assigns the measured eigenvalues, preserves the remaining eigenvalues and eigenvectors and above all, preserve the symmetry of the original FEM. **However, the method is not designed to assign a measured set of eigenvectors as well**. Attempt is now made to extend the method so that the assignment of a given set of eigenvectors can also be incorporated.

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