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A direct method for model updating with incomplete measured data and without spurious modes[☆]

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Abstract

This paper presents a new method for finite element matrix updating problem in an undamped model. This method has a computationally convenient feature of handling the difficulty of the incomplete measured data in an algorithmic way without using standard modal expansion or reduction techniques. The completed eigenvector matrix is such that it is mass normalised with respect to the analytical mass matrix. The method is also capable of preserving the large number of eigenvalues and eigenvectors of the finite element model that are not affected by updating. The latter has an effect of preventing the appearance of spurious modes in the frequency range of interest, which is a common concern with most direct updating methods.

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1. Introduction

The model updating problem (MUP) is a practical industrial problem that arises in vibration industries, including automobile, space and aircraft industries, and others. In these industries a theoretical finite element model often needs to be updated using a few measured eigenvalues from a real-life structure. The reason for doing so is that very often theoretical finite element data does not match well with the measured data from an experimental or a real-life structure. In this situation a vibration engineer needs to update the theoretical model so that inaccurate modeling assumptions can be corrected in the original finite element model and the updated model then can be used in future design with confidence. Such an updated analytical model is an important tool for a vibration engineer in the design, analysis, and construction of mechanical and structural systems.

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Nomenclature

n	number of DOF of the analytical modal
m	number of measured degrees of freedom
p	number of experimental modes
M	mass matrix ($n \times n$)
K	stiffness matrix ($n \times n$)
X	the analytical eigenvector matrix ($n \times n$)
X_1	the matrix of analytical eigenvectors that need to be updated ($n \times p$)
X_2	the analytical eigenvector matrix that is to remain invariant after updating ($n \times (n - p)$)
Σ_1^2	the measured eigenvalue matrix ($p \times p$)
Λ	the analytical eigenvalue matrix ($n \times n$)
Λ_1	the matrix of analytical eigenvalues that need to be updated ($p \times p$)
Λ_2	the matrix of analytical eigenvalues that are to remain invariant after updating ($(n - p) \times (n - p)$)
Y_1	the experimental eigenvector matrix ($n \times p$), corresponding to X_1
Y_{11}	the experimental eigenvector matrix ($m \times p$) measured at retained DOF
Y_{12}	the experimental eigenvector matrix ($(n - m) \times p$) at slave DOF

The model updating problem has also applications in damage detection and health-monitoring of the structures, such as bridges, highways, etc. (see for example, [2–4]), and in controlling resonance vibrations in the above structures (see, for example, [5–8]). The MUP has been well studied and there now exists a wealth of information. Many papers have been written and a complete book [9] has been devoted to the subject.

The existing methods can be broadly classified into three classes: (i) direct matrix model updating methods, (ii) iterative methods, and (iii) frequency response methods. *In this paper, we are concerned only with direct matrix updating methods.* These methods aim at updating directly the mass, stiffness, and damping matrices in such a way that the updated model remains symmetric and reproduces the measured data as accurately as possible. Again numerous papers have been written on the direct matrix updating problem. These include [10–37] and many others. Most of these methods are optimisation based methods, although in many cases explicit formulas for updating are available (see [9]). There are some computational and engineering issues with most of these methods. See discussion on this aspect later in this paper, taken from the above book.

In this paper, we give a close look at some of these issues and then *propose a new method for updating the stiffness matrix in an undamped model.* The method is direct and can deal with, and also seems to overcome, some of these issues effectively in an algorithmic and mathematical way.

In particular, (i) the method does not need any model reduction or modal expansion techniques to handle the problem of incomplete experimental data, (ii) the incomplete part of the experimental modes which cannot be measured, are supplied by means of an effective numerical procedure in such a way that the completed eigenvector matrix satisfies the mass-orthogonality relation, (for this reason, the method is termed as “*incomplete data handling method*” (IDHM)), (iii) the method also works exclusively with only a small number of finite element frequencies and mode shapes, and (iv) the method is capable of preserving the frequencies and mode shapes that are not affected by updating.

The last feature (iv) has an effect on preventing the appearance of *spurious modes in the frequency range of interests*. When resonance is controlled using a model updating technique and it is known a priori from engineering experience that the frequencies and mode shapes outside the resonant frequency range are acceptable, a procedure of this type will be useful. Of course, if the model size is small, then all the frequencies and mode shapes of the updated model can be computed explicitly using the existing computational techniques to see if the spurious modes have appeared. Unfortunately, however, this is not possible if the model size is large; because, the state-of-the-art techniques of matrix computations are capable of computing only a few extremal eigenvalues and eigenvectors of a large quadratic or a linear matrix pencil (see [38]). Since the updated models are generally large, such direct verification is not computationally feasible.

Our computational procedure for completing the incomplete experimental data uses only low-dimensional matrices, even though the finite element model might be very large. It is also numerically effective, since it is composed of numerically stable matrix computations tools, such as the solutions of low-order Sylvester equations, linear systems, QR factorisation and the SVD (singular value decomposition). Numerically effective high-quality computational techniques now do exist for all these matrix problems (see [38–40]).

The proposed method is illustrated first by a small example, and then a numerical comparison is made with the well-known method of Berman and Nagy [12] by means of a case study on a free–free beam, using the experimental data generated at NIU's vibration laboratory. It is observed that both the methods can preserve the symmetry and reproduce the measured data accurately. In addition, in contrast with the Berman–Nagy method the proposed method is capable of retaining the eigenvalues and eigenvectors of the original finite element model that are not affected by updating.

2. Problem statement, background, and motivations

2.1. Updating in an undamped model (MMUP)

A finite-element model of a vibrating structure may be represented as

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0, \quad (2.1)$$

where M, C and K , each of order n , are called, respectively, the *mass*, *damping* and *stiffness matrices*. In general, it is assumed that $M = M^T > 0$, $K = K^T \geq 0$, and $C = C^T$. Assuming a solution $x(t) = ue^{\lambda t}$, one obtains the quadratic eigenvalues problem:

$$(\lambda^2 M + \lambda C + K)u(t) = 0 \quad (2.2)$$

The matrix $P(\lambda) = \lambda^2 M + \lambda C + K$, called the *quadratic matrix pencil*, has $2n$ eigenvalues and $2n$ eigenvectors. For the sake of convenience, this pencil is denoted by the matrix triplet (M, C, K) . In the undamped case, the matrix $C = 0$, and the pencil is denoted by (M, K) .

In this paper, we consider matrix updating of the undamped finite element model (M, K) ; that is, the model has the form:

$$M\ddot{x}(t) + Kx(t) = 0. \quad (2.3)$$

The matrix M is not updated; only the stiffness matrix K is updated. If M and K are both $n \times n$, this model still has $2n$ eigenvalues and eigenvectors. Indeed, all the $2n$ eigenvalues of the associated quadratic pencil $\lambda^2 M + K$ are all purely imaginary in this case. Note that, on the other hand the eigenvalues and eigenvectors of a symmetric definite linear pencil $K - \lambda M$ are real.

Let $\{\lambda_1, \dots, \lambda_p; \lambda_{p+1}, \dots, \lambda_{2n}\}$ and $\{x_1, \dots, x_p; x_{p+1}, \dots, x_{2n}\}$ be the $2n$ eigenvalues and eigenvectors of the model (2.3), and let $\{\mu_1, \dots, \mu_p\}$ and $\{y_1, \dots, y_p\}$ be a set of p ($p \ll 2n$) eigenvalues and eigenvectors measured from an experimental structure. Mathematically, then the *matrix updating problem* (MUP) problem may be defined as follows:

2.2. Statement of MUP

Given (i) a set of p ($p \ll 2n$) computed eigenpairs $\{\lambda_k, x_k\}$, $k = 1, \dots, p$ of an $n \times n$ undamped finite-element second order model:

$$M\ddot{x}(t) + Kx(t) = 0$$

and (ii) another set of p measured eigenpairs $\{\mu_k, y_k\}$, $k = 1, \dots, p$ from an experimental or a real-life structure.

Find an updated symmetric model (M, K_U) such that the eigenpairs of the updated model are $\{\mu_1, \dots, \mu_p; \lambda_{p+1}, \dots, \lambda_{2n}; y_1, \dots, y_p; x_{p+1}, \dots, x_{2n}\}$ that is, the updated model has the following properties:

- (i) it is symmetric,
- (ii) it produces the measured eigenvalues and eigenvectors accurately, and
- (iii) the eigenvalues and eigenvectors corresponding to the unmeasured ones remain unchanged.

3. Some practical issues of model updating and the existing techniques to handle them

As noted in the Introduction, there exists a voluminous work on MUP. Most of these existing methods deal with updating the linear pencil $K - \lambda M$, rather than the quadratic pencil $P(\lambda)$. These methods preserve the symmetry, and the measured eigenvalues, and eigenvectors are incorporated rather accurately into the updated model; however, they “cannot guarantee that extra, spurious modes are not introduced into the frequency range of interest” [9, pp. 127].

Another practical difficulty with all these methods is that either a *model reduction technique* has to be applied to the FEM or *measured mode shapes have to be expanded* to the full length of the FEM. This is because the data (eigenvectors) measured experimentally from a real-life structure are very often *incomplete* in the sense that, due to hardware limitations, it can be measured only at a subset of the degrees of freedom of the finite element model. While the finite element models can be of several thousand degrees of freedom or more, the experimental eigenvectors from a practical structure can be measured at lengths of, at most a couple of hundreds. Therefore, comparing the analytical eigenvectors with those from a real-life structure that have different lengths becomes impossible in practice unless some measures are taken to fill in the missing entries or the order of the model is reduced. Although good computational algorithms for model reduction in first-order state-space systems exist [39,41,42], numerically effective model-reduction algorithms that work directly in matrix second-order models are virtually non-existent. Several attempts are currently underway.

A comparison by Friswell and Mottershead [9] of the best model reduction techniques available in vibration literature on a simple mass-spring systems show that the popular *Guyan* reduction technique “never reproduces any of the natural frequencies of the original model and all the frequencies are higher than those of the full model.” An improved version of the Guyan method, called the improved reduced system (IRS), “has produced natural frequencies which are very close to those of the full model”.

The *system equivalent reduction (SEREP)* method produces the natural frequencies correctly; but, like all the other methods, *the method has the computational disadvantage of requiring the computation of pseudo inverse*. The modal expansion techniques also rely on computing certain pseudo inverses. The experimental results using these techniques again on a mass, spring and a damper system show that the *expanded modes do not correctly satisfy the orthogonality relation* [9]. In this paper also, we have verified this fact by means of a 5×5 example, (Example IV.1) with a modal expansion method using partitioned mass and stiffness matrices (see [9] for a description of this expansion method).

For some more recent work on model reduction and expansion techniques, see [33,36,43].

4. A new method for MUP

In this section, we report our, *incomplete measured data handling (IMDH)* method. First, we establish some notations for compact eigenstructure representations of the undamped quadratic pencil $\lambda^2 M + K = 0$.

4.1. A compact eigenstructure representation

Assume that the matrices M and K are symmetric *positive definite*. Let (λ, x) be an eigenpair of the undamped pencil $P(\lambda) = \lambda^2 M + K$. Then we have,

$$\lambda^2 x^T M x = -x^T K x.$$

The eigenvalues are purely imaginary occurring in conjugate pairs. Suppose that corresponding to each pair $\lambda = \pm i\alpha$, an eigenvector x is presented only once. Then the number λ^2 is a non-positive real number and the eigenvector x is a real vector.

Let us write a compact representation (Λ, X) of the finite eigenstructure of the pencil $P(\lambda)$ above as

$$MX\Lambda^2 + KX = 0,$$

where $X \in \mathbf{R}^{n \times n}$ and $\Lambda \in \mathbf{R}^{n \times n}$, Λ^2 is a diagonal matrix of eigenvalues and its diagonal entries are non-positive.

The matrices X and A can be partitioned as

$$X = (X_1, X_2), \quad A^2 = \begin{pmatrix} A_1^2 & 0 \\ 0 & A_2^2 \end{pmatrix},$$

where $X_1 \in \mathbf{R}^{n \times p}$, $A_1^2 \in \mathbf{R}^{p \times p}$.

Note that with these notations, we have $MX_2A_2^2 + KX_2 = 0$.

An analogous partitioning of the eigenvalue and eigenvector matrices of the experimental structure is also possible.

Thus, these matrices from the experiment can be partitioned, respectively, as

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \quad \text{and} \quad Y = (Y_1, Y_2),$$

where $\Sigma_1^2 \in \mathbf{R}^{p \times p}$ and $Y_1 \in \mathbf{R}^{n \times p}$.

4.2. Orthogonality relations with the compact eigenstructure

With the compact eigenstructure notations just described, the following orthogonality relations hold and can be easily proved.

Theorem IV.1. (a) Let $P(\lambda) = \lambda^2 M + K$ be a symmetric semidefinite pencil with distinct nonzero eigenvalues and let (A, X) be a compact representation of the finite eigenstructure of this pencil. Then the matrices D_1 and D_2 defined by

$$D_1 = X^T M X, \quad D_2 = X^T K X$$

are diagonal and $D_2 = -D_1 A^2$.

(b) Furthermore, suppose if A_1 and A_2 do not have a common eigenvalue, then

$$X_1^T M X_2 = 0 \quad \text{and} \quad X_1^T K X_2 = 0.$$

Theorem 4.1 (Eigenstructure preserving updating). Assume that A_1 and A_2 do not have a common eigenvalue. Then, for every symmetric matrix ψ , the updated pencil $P_U(\lambda) = \lambda^2 M + K_U$, where

$$K_U = K - M X_1 \psi X_1^T M \tag{4.1}$$

is such that

$$M X_2 A_2^2 + K_U X_2 = 0. \tag{4.2}$$

That is, the eigenvalues and eigenvectors of the original finite element model that are not to be affected by updating, remain unchanged.

Proof. The proof is immediate by substituting Eq. (4.1) into Eq. (4.2) and noting the orthogonality relation

$$X_1^T M X_2 = 0 \tag{4.3}$$

and the eigenvalue–eigenvector relation

$$M X_2 A_2^2 + K X_2 = 0. \tag{4.4}$$

Knowing that the updating formula (4.1) preserves these eigenvalues and eigenvectors that are not affected by updating, we would like to use this formula so that the updated model also reproduces the measured eigenvalue and eigenvectors from the experimental structure. For that to happen, a relationship between X_1 and Y_1 must hold; that is, in order to simultaneously preserve the unmeasured eigenvalues and eigenvectors, and reproduce the measured ones in the updated model by means of the single updating formula (4.1), X_1 and Y_1 must be related. Specifically, the following result has been proved in [44]. $\square$

Theorem 4.2. *There exists a symmetric matrix ψ such that $MY_1\Sigma_1^2 + K_U Y_1 = 0$, where K_U is given by (4.1), if and only*

$$Y_1 = X_1 T \quad \text{for some nonsingular matrix } T.$$

Furthermore, T can be written as $T = VD_1$, where D_1 is nonsingular diagonal matrix and V is orthogonal.

We will also assume that only the part Y_{11} of the experimental eigenvector matrix $Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix}$ has been measured, where $Y_{11} \in \mathbb{R}^{m \times p}$. In what follows, we will now present an algorithm to compute Y_1 (given Y_{11}) from without explicitly computing the matrix T ; because such computation might lead to an ill-conditioned transformation.

Remark. The condition of Theorem 4.2 somehow restricts the scope of the use of the algorithm; but, it seems that in practice, this condition is almost always satisfied. This has been demonstrated both by means of an illustrative example and a case-study in this paper.

4.3. Computing the incomplete measured data and incorporating the measured data into the updated model

In the following, we will show how to compute the unknown part Y_{12} of Y_1 algorithmically and using this Y_1 , how to choose the parametric matrix ψ such that it is symmetric, and the updated model (4.1) reproduces the measured eigenvalues and eigenvectors. That is, in matrix notation, we will show that

$$MY_1\Sigma_1^2 + K_U Y_1 = 0. \quad (4.5)$$

Substituting the expressions for K_U from (4.1) and Y_1 in Eq. (4.5), we have

$$M \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix} \Sigma_1^2 + K \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix} = MX_1 \psi X_1^T M \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix}. \quad (4.6)$$

Assume that MX_1 has full rank. Then the QR factorisation (see [38]) of this product defines orthogonal matrices $U_1 \in \mathbb{R}^{n \times m}$ and $U_2 \in \mathbb{R}^{n \times (n-m)}$, and an upper triangular matrix $Z \in \mathbb{R}^{p \times p}$ satisfying

$$MX_1 = [U_1 \quad U_2] \begin{bmatrix} Z \\ 0 \end{bmatrix}. \quad (4.7)$$

Let $M = [M_1 \quad M_2]$ and $K = [K_1 \quad K_2]$, where $M_1, K_1 \in \mathbb{R}^{n \times m}$ and $M_2, K_2 \in \mathbb{R}^{n \times (n-m)}$.

Pre-multiplying Eq. (4.6) by $[U_1 \quad U_2]^T$ and using Eq. (4.7) with the above partitioning of M and K , Eq. (4.6) can be rewritten as

$$\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} (M_1 Y_{11} + M_2 Y_{12}) \Sigma_1^2 + \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} (K_1 Y_{11} + K_2 Y_{12}) = \begin{bmatrix} Z \\ 0 \end{bmatrix} \psi X_1^T M Y_1. \quad (4.8)$$

Therefore, a solution $\psi \in \mathbb{R}^{p \times p}$ to Eq. (4.6) exists only if

$$U_2^T (M_1 Y_{11} + M_2 Y_{12}) \Sigma_1^2 = -U_2^T (K_1 Y_{11} + K_2 Y_{12}),$$

which is equivalent to

$$U_2^T M_2 Y_{12} \Sigma_1^2 + U_2^T K_2 Y_{12} = -U_2^T (K_1 Y_{11} + M_1 Y_{11} \Sigma_1^2). \quad (4.9)$$

Eq. (4.9) is a *descriptor Sylvester equation* for Y_{12} .

Once Y_{12} is obtained by solving Eq. (4.9), we can form the matrix $Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix}$, and then compute $\psi \in \mathbb{R}^{p \times p}$ by solving the following systems of equations:

$$Y_1^T M Y_1 \Sigma_1^2 + Y_1^T K Y_1 = (Y_1^T M X_1) \psi (Y_1^T M X_1)^T, \quad (4.10)$$

which was obtained by pre-multiplying Eq. (4.6) by Y_1^T . In principle, Eq. (4.10) gives just a least-squares solution of Eq. (4.9). However, once Eq. (4.9) is satisfied, Eq. (4.10) gives an exact solution of Eq. (4.6).

However, the symmetry of the solution ψ is not guaranteed yet. We will now prove the symmetry of ψ by imposing the mass normalisation condition; that is Y_1 will be such that

$$Y_1^T M Y_1 = D, \quad (4.11)$$

where D is a diagonal matrix.

This means that the matrix Y_1 will have to be updated before the computation of ψ , and a matrix $G \in R^{p \times p}$ must be found such that, after the update $Y_1 \leftarrow Y_1 G$, Eq. (4.11) is satisfied. We will show that G can be chosen as an orthogonal matrix. *We must remark that the invariant subspace spanned by this matrix would still be the same after the update; only the basis vectors would be modified.*

We now show that the matrix ψ satisfying Eq. (4.10) is symmetric. First note that Σ_1^2 is a diagonal matrix. So, using Eq. (4.11) and noting that $Y_1^T M Y_1$ is also a diagonal matrix, we see that the left-hand side of Eq. (4.10) is symmetric. Thus, the right-hand side matrix

$$(Y_1^T M X_1) \psi (Y_1^T M X_1)^T$$

of Eq. (4.10) is symmetric, implying that ψ is symmetric.

When this ψ is used in forming K_U in Theorem 4.1, the spectrum of the updated pencil becomes

$$\Omega(\lambda^2 M + K_U) = \{\mu_1, \dots, \mu_m; \lambda_{m+1}, \dots, \lambda_{2n}\}.$$

4.4. A remark on descriptor Sylvester equation

Eq. (4.9) is of the form

$$XA + BXE = J,$$

where A, B, E and J are given matrices of appropriate dimensions and the matrix X needs to be determined. This type of equation, which is a generalisation of the classical Sylvester equation, naturally arises in the design and analysis of descriptor control systems of the form

$$E\dot{x} = Ax + Bu.$$

See [39] for details.

Three numerical methods—one, a *Hessenberg-triangular method*, second, a SVD (*singular value decomposition*) method, and the third, a *Parametric QR* method, have been developed to solve this type of equation in the Ph.D. dissertation [35] of Joao Carvalho [available from the web page of Biswa Datta: www.math.niu.edu/~datta].

Based on our discussions above, we now state our IDHM for the matrix updating problem in algorithmic form as follows:

Algorithm 4.1. *Modeling updating of an undamped symmetric positive semi definite model with incomplete measured data and without spurious modes.*

Step 1: Form the matrices $\Sigma_1^2 \in R^{p \times p}$ and $Y_{11} \in R^{m \times p}$ from the measured experimental data. Form the corresponding matrices $\Lambda_1^2 \in R^{p \times p}$ and $X_1 \in R^{n \times p}$ from the FEM.

Step 2: Compute the matrices $U_1 \in R^{n \times m}$, $U_2 \in R^{n \times (n-m)}$ and $Z \in R^{p \times p}$ from the QR factorisation of the $n \times p$ matrix MX_1 :

$$MX_1 = [U_1 \quad U_2] \begin{bmatrix} Z \\ 0 \end{bmatrix}.$$

Step 3: Partition $M = [M_1 \quad M_2]$; $K = [K_1 \quad K_2]$; where $M_1, K_1 \in R^{n \times m}$

Step 4: Solve the following descriptor Sylvester equation to obtain $Y_{12} \in R^{(n-m) \times p}$,

$$U_2^T M_2 Y_{12} \Sigma_1^2 + U_2^T K_2 Y_{12} = -U_2^T (K_1 Y_{11} + M_1 Y_{11} \Sigma_1^2)$$

and then form the matrix

$$Y_1 = \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix}.$$

Step 5: Compute the orthogonal matrices P and $Q \in R^{p \times p}$ and the diagonal matrix $S \in R^{p \times p}$ by computing the singular value decomposition of $Y_1^T M Y_1 : PSQ^T = Y_1^T M Y_1$.

Update the matrix Y_1 by $Y_1 \leftarrow Y_1(P)^T$.

Step 6: Compute $\psi \in R^{p \times p}$ by solving the following $m \times m$ system of equations:

$$(Y_1^T M X_1)\psi(Y_1^T M X_1)^T = Y_1^T M Y_1(\Sigma_1)^2 + Y_1^T K Y_1.$$

Step 7: Update $K_U = K - M X_1 \psi X_1^T M$.

Remarks. (i) The algorithm above can also be used when a complete measured data is available. In this case, Steps 2–4 are skipped. We recommend doing this when the measurements of Σ_1 and Y_1 are highly accurate.

(ii) The matrix equation to be solved in Step 6 is of the form

$$AXA^T = C.$$

This equation is a special case of the descriptor Sylvester equation and therefore the numerical methods developed for this equation in the Ph.D. thesis of Carvalho [35] can be used to solve this equation.

4.5. Computational and engineering features of the algorithm

- The algorithm offers an opportunity to conveniently exploit the structural properties of the model, such as, the symmetry, positive definiteness, sparsity and bandedness of the matrices M and K , while implementing Step 2 and Steps 4–7 of the algorithm.
- The algorithm can be implemented with the knowledge of only a few eigenvalues that need to be reassigned to the measured ones, and the corresponding eigenvectors (note that the state-of-the-art matrix algorithms can compute only a few extremal eigenvalues and eigenvectors of a large quadratic matrix pencil, see [38]).
- No model reduction is necessary, no matter how large the model may be. The difficulty of incomplete measured data is handled by the algorithm itself without using any model reduction or modal expansion techniques and the completed data satisfies the mass orthogonality relation.
- The frequencies and mode shapes that correspond to the unmeasured ones do remain unaffected. This fact is established by means of a mathematical result, which is important, because it is impossible to verify this fact even with the state-of-the-art computational techniques, if the model size is large, which is the case very often in practice.
- The computational requirements of the algorithm are minimal. All it needs are (i) solutions of small linear systems, and (ii) computations of the QR factorisation and the singular value decomposition (SVD) of small matrices.

4.6. Example IV.1: an illustrative example

Consider the positive semi definite model (M, K) , where the matrices M and K are given by

$$M = \begin{bmatrix} 1.2940 & 0 & 0 & 0 & 0 \\ 0 & 1.2940 & 0 & 0 & 0 \\ 0 & 0 & 1.2940 & 0 & 0 \\ 0 & 0 & 0 & 1.2940 & 0 \\ 0 & 0 & 0 & 0 & 1.2940 \end{bmatrix},$$

$$K = \begin{bmatrix} 1188.5000 & 196.6000 & 0 & 0 & -642.4000 \\ 196.6000 & 626.3000 & 0 & -555.6000 & 0 \\ 0 & 0 & 1188.5000 & -196.6000 & -546.1000 \\ 0 & -555.6000 & -196.6000 & 626.3000 & 196.6000 \\ -642.4000 & 0 & -546.1000 & 196.6000 & 4019.1000 \end{bmatrix}.$$

This model has two finite eigenvalues. In this example, $n = 5$, $m = 3$, and $p = 2$.

Step 1. The modal matrices are

$$A_1^2 = \begin{bmatrix} -3297.1 & 0 \\ 0 & -23.648 \end{bmatrix}, \quad X_1 = \begin{bmatrix} -0.17754 & 0.12529 \\ -0.018246 & -0.61176 \\ -0.15356 & -0.085635 \\ 0.056719 & -0.61158 \\ 0.84507 & 0.0386 \end{bmatrix}.$$

The corresponding matrices of measured frequencies and mode shapes are taken as

$$\Sigma_1^2 = \begin{bmatrix} -3297.6 & 0 \\ 0 & -23.148 \end{bmatrix}, \quad Y_{11} = \begin{bmatrix} -0.1749 & 0.1101 \\ 0.0133 & -0.5821 \\ -0.1416 & -0.0890 \end{bmatrix}.$$

Also, note that

$$A_2^2 = \begin{bmatrix} -968.03 & 0 & 0 \\ 0 & -942.69 & 0 \\ 0 & 0 & -679.39 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -0.11595 & 0.6424 & 0.54723 \\ -0.51934 & 0.24413 & -0.26249 \\ -0.41414 & -0.54545 & 0.52236 \\ 0.54443 & -0.033487 & 0.31309 \\ -0.14737 & 0.043366 & 0.1832 \end{bmatrix}.$$

Step 2. From the QR factorisation of MX_1 to obtain U_1 and U_2 , and Z :

$$U_1 = \begin{bmatrix} -0.2020 & 0.1425 & -0.1633 \\ -0.0208 & -0.6959 & -0.1209 \\ -0.1747 & -0.0974 & 0.9680 \\ 0.0645 & -0.6957 & -0.0391 \\ 0.9613 & 0.0439 & 0.1416 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} 0.1307 & 0.9461 \\ -0.0858 & 0.1742 \\ -0.0289 & 0.1485 \\ 0.7142 & 0.0132 \\ -0.0405 & 0.2286 \end{bmatrix} \quad \text{and}$$

$$Z = \begin{bmatrix} 1.1375 & 0 \\ 0 & 1.1375 \end{bmatrix}.$$

Step 3. The partitioning of the matrices M and K is straightforward.

Step 4. The solution of the descriptor Sylvester equation in Step 4 gives

$$Y_{12} = \begin{bmatrix} 0.0845 & -0.5782 \\ 0.8009 & 0.0789 \end{bmatrix}.$$

Step 5. Computing the SVD of $Y_1^T M Y_1$ and updating Y_1 , we obtain

$$Y_1 = \begin{bmatrix} -0.1101 & -0.1749 \\ 0.5821 & 0.0133 \\ 0.0890 & -0.1416 \\ 0.5782 & 0.0845 \\ -0.0789 & 0.8009 \end{bmatrix}.$$

Step 6. The symmetric matrix ψ obtained by solving the matrix equation in Step 6 is

$$\psi = \begin{bmatrix} 3264.9 & -171.87 \\ -171.87 & -3264.9 \end{bmatrix}.$$

Step 7. The updated stiffness matrix is

$$K_U = \begin{bmatrix} 1089.2 & -209.52 & -208.86 & -330.54 & 232.75 \\ -209.52 & 2676.9 & 298.57 & 1488.7 & -193.78 \\ -208.86 & 298.57 & 1107.3 & 162.96 & 122.72 \\ -330.54 & 1488.7 & 162.96 & 2633.5 & -342.6 \\ 232.75 & -193.78 & 122.72 & -342.6 & 141.82 \end{bmatrix}.$$

4.7. Verification

- (i) $\|MY_1 \Sigma_1^2 + K_U Y_1\|_F = 2.0206 \times 10^{-12}$ (*Reproduction of the measured data*).
- (ii) $\|MX_2 A_1^2 + K_U X_2\|_F = 8.5767 \times 10^{-13}$ (*Invariance of the unmeasured eigenstructure*).
- (iii) $Y_1^T M Y_1 = \begin{bmatrix} 0.9051 & 0.0001 \\ 0.0001 & 0.9051 \end{bmatrix}$ (*Orthogonality check*).
- (iv) $MAC = \begin{bmatrix} 0.9972 & 0.0028 \\ 0.0028 & 0.9972 \end{bmatrix}$ (*Correlation between eigenvectors*).

Therefore, we conclude that computationally:

- The updated model reproduces the measured data accurately.
- The frequencies and mode shapes of the original model that did not take part in updating process did not change.
- The completed measured eigenvector matrix is mass normalised.

4.8. Comparison with a modal expansion technique

The incomplete data handling part of our algorithm; namely Steps 2–6 of Algorithm 4.1 could be substituted by means of a modal expansion scheme. In order to see how effective this would be in comparison with our scheme, we applied the well-known expansion scheme using mass and stiffness matrices to the data of our example, and the following results were obtained. (For a description of this scheme, see the book [9].)

4.8.1. Computation of Y_{12}

The matrix Y_{12} with expansion technique using mass and stiffness matrices is:

$$Y_{12} = \begin{bmatrix} -0.0033 & -0.5546 \\ -0.0228 & 0.0326 \end{bmatrix}.$$

Step 6. Matrix ψ using Step 6 of Algorithm 4.1

$$\psi = \begin{bmatrix} 27953 & -89.525 \\ -89.525 & -3275.5 \end{bmatrix}.$$

Step 7. The updated stiffness matrix is

$$K_U = \begin{bmatrix} -207.41 & -359.44 & -1335.5 & 68.432 & 6421.4 \\ -359.44 & 2666.7 & 170.51 & 1541.3 & 514.58 \\ -1335.5 & 170.51 & 128.99 & 511.66 & 5497.9 \\ 68.432 & 1541.3 & 511.66 & 2516.7 & -2253.5 \\ 6421.4 & 514.58 & 5497.9 & -2253.5 & -29389 \end{bmatrix}.$$

4.8.2. Verification

- (i) $\|MY_1\Sigma_1^2 + K_U Y_1\|_F = 122.5$ (Reproduction of the eigenvalues).
(The measured eigenvalues were not produced accurately.)
- (ii) $\|MX_2A_1^2 + K_U X_2\|_F = 1.7551 \times 10^{-12}$ (Invariance of the unmeasured eigenstructure).
- (iii) $Y_1^T M Y_1 = \begin{bmatrix} 0.8641 & 0 \\ 0 & 0.066083 \end{bmatrix}$ (Orthogonality of the measured data).
(The matrix Y_1 is not mass-normalised.)

5. Case study: a free–free rectangular beam

In this section, we present the results of our case study on a free–free rectangular aluminum beam having length $l = 0.40005$ m, width $w = 0.0508$ m, thickness $t = 6.456$ mm, Young's modulus $E = 72$ Gpa, and Poisson's ratio $\mu = 0.33$, with two direct methods: the popular *Berman and Nagy method (BNM)* (see [12]) and the proposed *incomplete data handling method (IDHM)*.

The analytical data obtained from the software package ANSYS is compared to and correlated with the data obtained from an experiment by using Bruel & Kjaer PULSE FFT analyzer. The finite element model of the beam has 33 DOF. The experiment is performed on this model with six DOF. For comparing and correlating analytical data with the test data, the *modal assurance criteria (MAC)* is used. To deal with the problem of incomplete measured data in the Berman–Nagy method, the system equivalent reduction expansion process (SEREP) due to O'Callahan et al. (see [9]) is applied. The results obtained from these methods are compared and conclusions are drawn.

5.1. The finite element model

The free–free rectangular aluminum beam is modeled in ANSYS with 10 Beam3 elements. Each Beam3 element has two nodes with three DOF (X, Y, RZ) at each node that combines to give a total of six DOF per element. The modeled rectangular beam has 33 DOF (10 elements, and 11 nodes). The output from ANSYS consists of the mass matrix $M_{33 \times 33}$, the stiffness matrix $K_{33 \times 33}$, the natural frequency vector $\lambda_{a33 \times 1}$; and the mode shape matrix $X_{133 \times 6}$. The set of first six strain modes $X_{133 \times 6}$ and corresponding frequencies $\lambda_{a6 \times 1}$ are computed.

5.2. The test model

The same beam is divided into five elements and six nodes. A vibration test is performed by placing an accelerometer on the 5th node and exciting the beam at all the six nodes with an impact hammer. The first

three elastic modes and natural frequencies are obtained. The test data was generated at the NIU vibration laboratory.

5.3. Correlation between the experimental and test data

The modal assurance criterion (MAC) is widely used to pair two sets of modeshapes; for example, those from analytical and experimental models. Specifically, the MAC between an analytical mode X_{1k} and an experimental mode Y_{1j} is defined as

$$MAC_{jk} = \frac{|Y_{1j}^T X_{1k}|^2}{|X_{1k}^T X_{1k}| |Y_{1j}^T Y_{1j}|}.$$

This value is between 0 and 1. The pairing between two modes is considered to be good if this value is close to 1. The MAC values between our finite-element data on a free–free beam and the experimental data from the NIU vibration laboratory, displayed in Table 1, show that those two sets of data are well correlated.

5.4. Comparison between the Berman–Nagy and the incomplete data handling method

In this section, we present the results of comparison between the two methods on our test data: the Berman–Nagy method (BNM) and the incomplete data handling methods (IDHM). For the sake of convenience, we state the updating formulas of these two methods below.

5.4.1. The Berman–Nagy updating formula [12]

$$M_U = M + MY_1(Y_1^T MY_1)^{-1}(I - Y_1^T MY_1)(Y_1^T MY_1)^{-1} Y_1^T M,$$

$$K_U = K - KY_1 Y_1^T M - MY_1 Y_1^T K + MY_1 Y_1^T KY_1 Y_1^T M + MY_1 \Sigma_1^2 Y_1^T M.$$

Table 1
Correlation of experimental data with analytical data

Original FEM frequency (Hz)	Test frequency (Hz)	Difference (Hz)	Difference (%)	MAC
211.2	212	0.8	0.39	0.9985
582.4	584	1.6	0.27	0.9990
1142.4	1140	−2.4	−0.21	0.9982
1891.3				
2832.8				
3972.9				

Table 2
Comparison of updated frequencies and mode shapes between the Berman–Nagy and incomplete data handling methods

Test frequency (Hz)	Original FEM frequency (Hz)	Updated frequency (Hz)		Difference (%)		MAC	
		BNM	IDHM	BNM	IDHM	BNM	IDHM
212	211.2	212	212	0	0	1.0000	0.9985
584	582.4	584	584	0	0	1.0000	0.9990
1140	1142.4	1140	1140	0	0	1.0000	1.0000
	1891.3	1890	1891.3	−1.3	0	–	–
	2832.8	2835	2832.8	2.2	0	–	–
	3972.9	3970	3972.9	−2.9	0	–	–

Table 3

Norms showing reproduction of measured data and changes (invariance) of the Berman–Nagy method and IDHM

Frobenius norm	BNM	IDHM
$\ MY_1\Sigma_1^2 + K_U Y_1\ _F$	1.4990e–08	1.4773e–07
$\ MX_2\mathcal{A}_2^2 + KX_2\ _F$	9.3287e+09	6.5998e–05

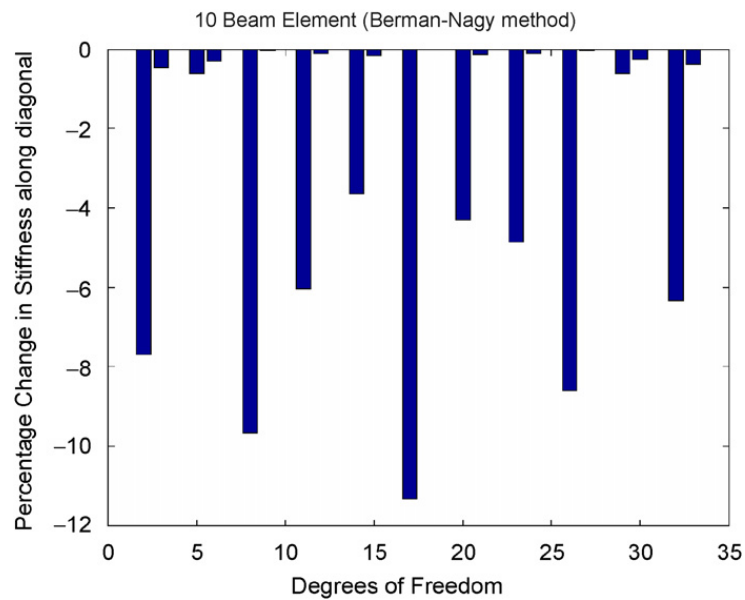


Fig. 1. Percentage change in stiffness along the diagonal by the Berman–Nagy method.

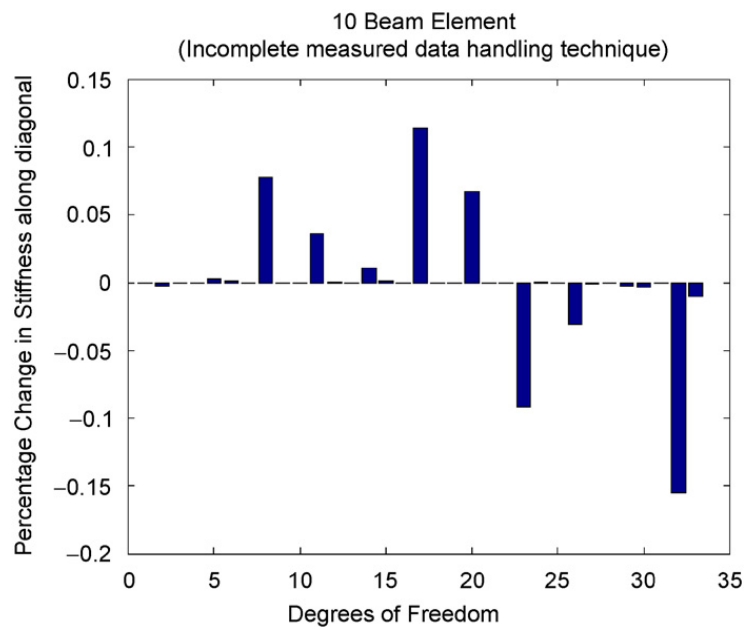


Fig. 2. Percentage change in stiffness along the diagonal by the incomplete measured data handling method.

5.4.2. The IDHM updating formula

$$K_U = K - MX_1\psi X_1^T M,$$

where X_1 is the analytical eigenvector matrix corresponding to the natural frequencies that need to be updated and the symmetric matrix ψ is given by Eq. (4.10).

The results, displayed in Table 2, show that both the methods reproduced the measured data accurately; however the Berman–Nagy method produced changes in the finite element data which are not updated, while with the IDHM these data remained numerically same. These findings are confirmed by the norm-wise results displayed in Table 3.

Both methods, however, failed to maintain physical connectivity between the elements; i.e., elements in the stiffness matrix are populated after updating, whereas the original stiffness matrix is almost tridiagonal.

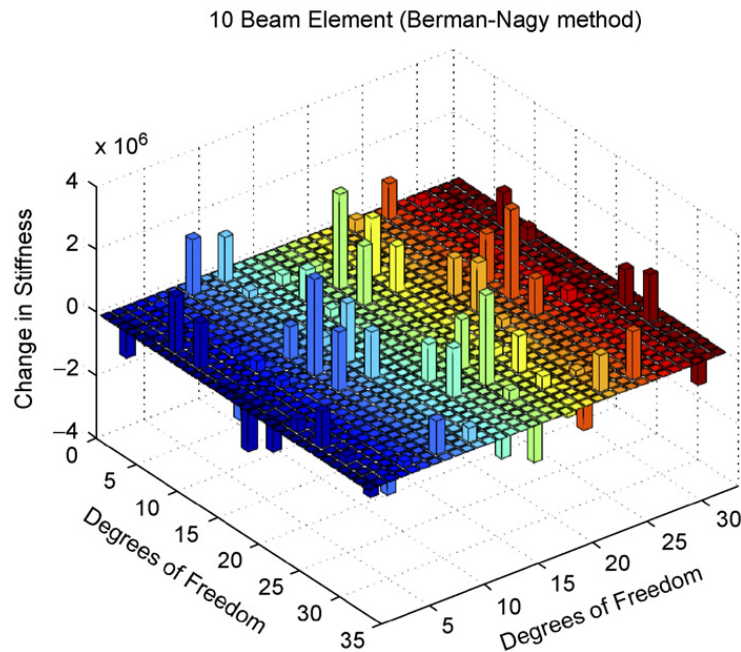


Fig. 3. Change in stiffness (the Berman–Nagy method).

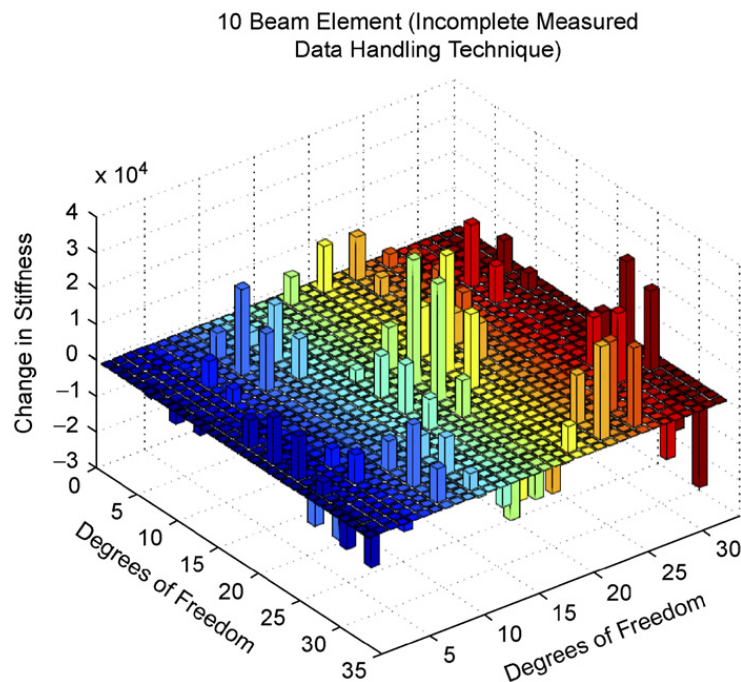


Fig. 4. Change in stiffness (incomplete measured data handling method).

The following figures (Figs. 1–4) show percent changes in stiffness along the diagonal and total stiffness changes produced by these two methods.

6. Summary and conclusion

Finite element matrix updating has received a considerable amount of attention by the engineering community and as a result, there now exists a voluminous work on this problem. In this paper, another new method for matrix updating is proposed. Unlike the most existing methods, the proposed method is neither formulated in an optimisation setting nor is solved using any optimisation techniques. The method has two distinguishing features:

No model reduction or expansion techniques are required. The missing components of the measured experimental eigenvectors are computed in an algorithmic way by using the sophisticated and stable matrix computational techniques, such as the QR factorisation and singular value decomposition, thus avoiding the potential drawback of ill-conditioned matrix transformation, which is usually encountered in most of the expansion schemes. (Note that, our algorithmic component of filling-in the incomplete experimental data might be mathematically equivalent to some of the existing expansion schemes (see [9]) or smoothing technique [45]), but it should be noted that two methods may be mathematically equivalent but might behave quite differently in a computational setting.

The method needs the knowledge of only a small number of eigenvalues and eigenvectors of the associated analytical quadratic matrix pencil, which are required to be reassigned to the measured data; while the remaining large number of eigenvalues and eigenvectors which cannot be computed even using the state-of-the-art algorithms and softwares, are guaranteed to remain invariant by means of a proven mathematical result. This feature of our algorithm makes it practically applicable even to large real-life structures.

Although it might not be necessary or desirable for practical engineering applications to keep the unmeasured eigenvalues and eigenvectors fixed in their positions, keeping them invariant has certainly the effect of preventing the appearance of spurious modes into the frequency range of interests, which is a major concern with all the direct methods for matrix updating. *To our knowledge, the proposed method is the first one to address this practical concern.*

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