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Chapter 1

Introduction

Topology is the study of continuity and convergence in a general, abstract, context. We use as our starting point the intuitions obtained from the study of advanced calculus and extend the basic ideas to encompass much more general situations. In the process, we will find that many of the proofs are simplified and that the connection between ideas that are equivalent in the setting of the real numbers are clarified.

So, let's start with a familiar definition:

Definition 1.1 *Let A be a set of real numbers and let x_0 be an element of A. Suppose $f : A \to \mathbb{R}$ is a function. We say that f is continuous at x_0 if for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever x is in A and $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.*

There are several things to notice about this definition. The first is that we are dealing with *sets* and *functions*. This will require us to describe exactly what we mean by these concepts. Another observation, and a basic one for our subject, is that we are really interested in *distances* between the points x and x_0 and also between $f(x)$ and $f(x_0)$.

For those that have had a class in multi-variable advanced calculus, it can be pointed out that continuity of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined very similarly. Once again, it is the notion of the distance between points that is at the heart

of the notion of continuity.

This will lead us to the idea of a metric space. But first, we will review some basic properties of sets.

1.1 Sets

Almost all of mathematics is founded on the study of the properties of sets. In this course, we will not give formal axioms for set theory and will use 'naive' set theory, while pointing out the problems of this theory.

For our purposes, a *set* will be a collection of mathematical objects. We consider sets to be mathematical objects, so the collections can have sets as members. If A is a set and x is a mathematical object, we write $x \in A$ if x is in the collection A and $x \notin A$ otherwise. If $x \in A$, we say that x is an *element* of A. There are two main ways of describing sets. The first is to simply list the elements between braces. So, $A = \{1, 2, 3\}$ is a set whose elements are exactly the objects 1, 2, and 3. The second way is to describe some property that is obeyed by all elements of the set. So, if $P(x)$ is some property of x that can be either true or false for each object x, we write $A = \{x : P(x)\}$ to be the set of all objects where $P(x)$ is true. In particular, we always have $A = \{x : x \in A\}$ (why?).

Some important examples of sets are the following:

The empty set

$$\emptyset = \{\} = \{x : x \neq x\}$$

Notice that this set has no elements at all.

The set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

The set of integers:

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$$

The set of rational numbers:

$$\mathbb{Q} = \{x : \text{there exist } m, n \in \mathbb{Z}, n \neq 0, \text{ such that } x = \frac{m}{n}\} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$$

(notice the use of notation in the last equality.

The set of real numbers:

$$\mathbb{R}$$

If $a < b$ in $\mathbb{R}$, we define:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} = \{x : x \in \mathbb{R}, a < x < b\}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$

$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$

Notice that we do not give a precise description of the set $\mathbb{R}$. We assume it has the properties described in advanced calculus courses. In particular, we can add, multiply, subtract, and divide by non-zero real numbers. We can also compare to real numbers: for $x, y \in \mathbb{R}$, the expression $x < y$ can be either true or false.

Definition 1.2 *Let A and B be sets. We define the following:*

a) we say A is a subset of B if for all x, $x \in A$ implies that $x \in B$. We write $A \subseteq B$.

b) the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$, the intersection of A and B.

c) the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$, the union of A and B.

d) the set $A \setminus B = \{x : x \in A, x \notin B\}$

e) the set $\mathcal{P}(A) = \{x : x \subseteq A\}$, the power set of A.

Notice that $A = B$ when A and B have the same elements. This happens when $x \in A$ if and only if $x \in B$. This, in turn, is the same as saying that $A \subseteq B$ and $B \subseteq A$.

Proposition 1.3 *Let A, B, C be sets. Then*

a) $\emptyset \subseteq A$

b) $A \cap B \subseteq A \subseteq A \cup B$

c) $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

d) $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$

e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

f) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

g) $A \subseteq B$ if and only if $A \cap B = A$ if and only if $A \cup B = B$

h) $A \setminus (A \setminus B) = A \cap B$

i) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

j) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Now, suppose that $I \neq \emptyset$ is a set and for each $\alpha \in I$, we have a set A_α. We say that I is an index set for the family $\{A_\alpha\}_{\alpha \in I}$. We define the following:

Definition 1.4 *If $\{A_\alpha\}_{\alpha \in I}$ is a family of sets, define*

a) $\bigcup_{\alpha \in I} A_\alpha = \{x : \text{there exists } \alpha \in I \text{ such that } x \in A_\alpha\}$

b) $\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$

In particular, if we set $I = \{1, 2\}$, $A_1 = A$ and $A_2 = B$, then $\bigcup_{\alpha \in I} A_\alpha = A \cup B$ and $\bigcap_{\alpha \in I} A_\alpha = A \cap B$. So we have generalized the notions of union and intersection.

It is quite common to have the index set be the set of natural numbers, $\mathbb{N}$. In this case, we write $\bigcup_1^\infty A_n$ and $\bigcap_1^\infty A_n$ for $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$.

As an example, $\bigcup_1^\infty (-n, n) = \mathbb{R}$ and $\bigcap_1^\infty (-\frac{1}{n}, \frac{1}{n}) = \{0\}$.

We then have

Proposition 1.5 *For an indexed family $\{A_\alpha\}_{\alpha\in I}$, and any set B, we have*

a) $B \setminus (\bigcup_{\alpha\in I} A_\alpha) = \bigcap_{\alpha\in I}(B \setminus A_\alpha)$

b) $B \setminus (\bigcap_{\alpha\in I} A_\alpha) = \bigcup_{\alpha\in I}(B \setminus A_\alpha)$

Proof: First, suppose that $x \in B \setminus (\bigcup_{\alpha\in I} A_\alpha)$. Then $x \in B$ and $x \notin \bigcup_{\alpha\in I} A_\alpha$. So there does not exist and $\alpha \in I$ such that $x \in A_\alpha$. But this means that $x \notin A_\alpha$ for all $\alpha \in I$. Hence, for each α, $x \in B$ and $x \notin A_\alpha$, so $a \in B \setminus A_\alpha$. But this happens for all $\alpha \in I$, so $x \in \bigcap_{\alpha\in I}(B \setminus A_\alpha)$.

Conversely, suppose that $x \in \bigcap_{\alpha\in I}(B \setminus A_\alpha)$. Since $I \neq \emptyset$, $x \in B$. Also, for each α, we have $x \notin A_\alpha$. But this means there does not exist an α such that $x \in A_\alpha$, which means that $x \notin \bigcup_{\alpha\in I} A_\alpha$. In other words, $x \in B \setminus (\bigcup_{\alpha\in I} A_\alpha)$.

The other equality is demonstrated by similar logic. ♦

Definition 1.6 *Suppose that x and y are two mathematical objects. We define the ordered pair $(x, y) = \{\{x\}, \{x, y\}\}$.*

This is an unusual definition. The only reason we use it is that it gives the following result.

Proposition 1.7 *If x, y, z, w are mathematical objects, then $(x, y) = (w, z)$ if and only if $x = w$ and $y = z$.*

Definition 1.8 *Suppose that X and Y are sets. We define the cartesian product of X and Y by*

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

We can also define ordered n-tuples inductively as follows:

Definition 1.9 *If $x_1, x_2, \cdots, x_n$ are mathematical objects, we define the n-tuple by*

$$(x_1, x_2, \cdots, x_n) = (x_1, (x_2, \cdots, x_n))$$

Again, the only aspect of this definition we need is the following:

Proposition 1.10 *If $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ are mathematical objects, then $(x_1, x_2, \cdots x_n) = (y_1, y_2, \cdots, y_n)$ if and only if $x_1 = y_1$, $x_2 = y_2, \cdots x_n = y_n$.*

In a similar way, we can define the product of sets $X_1, X_2, \cdots, X_n$ inductively to be $X_1 \times (X_2 \times \cdots \times X_n)$. If all the sets are the same, $X_i = X$, we write X^n for the product $X_1 \times \cdots X_n$. Hence X^n is the set of all ordered n-tuples from the set X.

1.2 Functions

Now we come to the concept of a function. This is perhaps the most important concept we have defined so far.

Definition 1.11 *Let X and Y be sets. A function is a subset $f \subseteq X \times Y$ with the following properties:*

a) For each $x \in X$, there is a $y \in Y$ such that $(x, y) \in f$

b) If $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$.

In other words, for each $x \in X$ there is exactly one $y \in Y$ with $(x, y) \in f$. For this y, we write $y = f(x)$. If f is a function from X to Y, we write $f : X \to Y$. We call the set X the domain of f and Y the co-domain of f.

Definition 1.12 *Suppose that $f : X \to Y$.*

a) We say f is one-to-one if whenever $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, we may conclude $x_1 = x_2$.

b) We say f is onto Y if whenever $y \in Y$, there is an $x \in X$ such that $f(x) = y$.

c) If $A \subseteq X$, we define the set $f[A] = \{f(a) : a \in A\}$, the image of A under f.

d) If $B \subseteq Y$, we define the set $f^{-1}[B] = \{x \in X : f(x) \in B\}$, the pre-image of B under f.

Alternatively, a function $f : X \to Y$ is one-to-one if whenever $x_1 \neq x_2$ in X, we have $f(x_1) \neq f(x_2)$. Be sure you understand why these two descriptions are equivalent to each other.

Definition 1.13 *Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. We define the composition of f and g to be the function $g \circ f : X \to Z$ defined by $g \circ f(x) = g(f(x))$.*

Proposition 1.14 *Suppose that $f : X \to Y$ is a function. Then we have the following:*

a) For any $A \subseteq X$, $A \subseteq f^{-1}[f[A]]$. Equality need not hold.

b) Equality holds in the previous if f is one-to-one.

c) For any $B \subseteq Y$, $f[f^{-1}[B]] \subseteq B$. Equality need not hold.

d) Equality holds in the previous if f is onto.

e) If $B \subseteq Y$, then $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$.

f) If $\{A_\alpha\}$ is a family of subsets of X, then $f[\bigcup_{\alpha\in I} A_\alpha] = \bigcup_{\alpha\in I} f[A_\alpha]$.

g) If $\{A_\alpha\}$ is a family of subsets of X, then $f[\bigcap_{\alpha\in I} A_\alpha] \subseteq \bigcap_{\alpha\in I} f[A_\alpha]$. Equality need not hold.

h) Equality holds in the previous if f is one-to-one.

i) If $\{B_\alpha\}$ is a family of subsets of Y, then $f^{-1}[\bigcup_{\alpha\in I} B_\alpha] = \bigcup_{\alpha\in I} f^{-1}[B_\alpha]$.

j) If $\{B_\alpha\}$ is a family of subsets of Y, then $f^{-1}[\bigcap_{\alpha\in I} B_\alpha] = \bigcap_{\alpha\in I} f^{-1}[B_\alpha]$.

Proposition 1.15 *Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. If $A \subseteq X$, then $g\circ f[A] = g[f[A]]$. Also, if $C \subseteq Z$, then $(g\circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$.*

Next, we define a few almost trivial functions. None-the-less, they turn out to be quite useful.

Definition 1.16 *Let X be a set. We define the identity function $i_X : X \to X$ by $i_X(x) = x$. If $A \subseteq X$, we define the inclusion function $i_A : A \to X$ by $i_A(a) = a$. Note that these notations are consistent.*

If X and Y are sets, we define $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. These are called the projections to X and Y resp.

Another way to get new functions from old ones is to use restrictions:

Definition 1.17 *Suppose that $F : X \to Y$ is a function and $A \subseteq X$. Define the restriction of f to A by $f|_A(a) = f(a)$. Then, $f|_A : A \to Y$.*

Notice that $f|_A = f \circ i_A$.

Finally, we get an equivalence of one-to-one and of onto for functions in terms of the existence of one-sided inverses.

Proposition 1.18 *Suppose that $f : X \to Y$ with $X \neq \emptyset$. Then*

a) *f is one-to-one if and only if there is a function $g : Y \to X$ such that $g \circ f = i_X$.*

b) *f is onto if and only if there is a function $g : Y \to X$ such that $f \circ g = i_Y$.*

The second statement is actually equivalent to the Axiom of Choice.

1.3 Equivalence of Sets

Definition 1.19 *We say a function $f : X \to Y$ is an equivalence if f is both one-to-one and onto.*

Immediately, we have the following (even without the axiom of choice).

Proposition 1.20 *A function f is an equivalence if and only if there is a function $g : Y \to X$ such that $g \circ f = i_X$ and $f \circ g = i_Y$. We denote this function by $g = f^{-1}$.*

Proof: First suppose that f is an equivalence. We have that $f \subseteq X \times Y$. Define $g = \{(y, x) : (x, y) \in f\}$. We claim that g is a function from X to Y. Clearly, $g \subseteq Y \times X$. Also, if $y \in Y$, the fact that f is onto implies that there exists an $x \in X$ such that $y = f(x)$. In other words, $(x, y) \in f$, so $(y, x) \in g$.

Now, suppose that $(y, x_1), (y, x_2) \in g$. Then $(x_1, y), (x_2, y) \in f$, so $f(x_1) = y$ and $f(x_2) = y$. But now, the fact that f is one-to-one shows that $x_1 = x_2$. Hence g satisfies the definition of a function $g : Y \to X$.

Now, if $x \in X$, $f(x) \in Y$ is such that $(x, f(x)) \in f$. Also, $g \circ f(x) = g(f(x)) \in X$ is such that $(f(x), g \circ f(x)) \in g$. But this means that $(g \circ f(x), f(x)) \in f$. Again, by the fact that f is one-to-one, we get that $x = g \circ f(x)$. Hence $g \circ f = 1_X$.

The fact that $f \circ g(y) = y$ for all $y \in Y$ follows in a similar fashion. ♦

Beware! If $B \subseteq Y$, we define $f^{-1}[B]$ even in the case where f is not an equivalence. So f^{-1} may not be defined as a function in such a situation. In particular, it would be wrong to declare in general that $f^{-1}[B] = \{f^{-1}(y) : y \in B\}$.

Definition 1.21 *We say two sets X and Y are equivalent if there exists an equivalence $f : X \to Y$.*

It should be pointed out that only one equivalence is required to show that X and Y are equivalent sets. In general, if one equivalence exists, many do. Also, other functions from X to Y need not be equivalences.

Proposition 1.22 *Equivalence of sets is an equivalence relation. In other words,*

a) Any set X is equivalent to itself.

b) If X is equivalent to Y, then Y is equivalent to X.

c) If X is equivalent to Y and Y is equivalent to Z, then X is equivalent to Z.

Proof:

a) The function 1_X is an equivalence from X to X.

b) If $f : X \to Y$ is an equivalence, then $f^{-1} : Y \to X$ is also.

c) If $f : X \to Y$ and $g : Y \to Z$ are equivalences, then $g \circ f : X \to Z$ is also.

♦

Definition 1.23 *We say a set X is finite if it is either empty or if there is an $n \in \mathbb{N}$ such that X is equivalent to the set $\{1, 2, \cdots, n\}$. We say X is countable if it is either finite or equivalent to $\mathbb{N}$.*

Theorem 1.24 *Every subset of $\mathbb{N}$ is countable.*

Theorem 1.25 *We have the following:*

a) A set X is countable if and only if there is a one-to-one function $f : X \to \mathbb{N}$.

b) A set X is countable if and only if there is an onto function $f : \mathbb{N} \to X$.

c) The set $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$.

d) Subsets of countable sets are countable.

e) If $I \neq \emptyset$ is countable and A_α is countable for each $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha$ is countbale.

Proof:

a) If X is countable, then either $X = \emptyset$, in which case, let $f = \emptyset$, or $X \simeq \{1, \cdots, n\}$ for some $n \in \mathbb{N}$, in which case, let $f : X \to \{1, \cdots, n\} \to \mathbb{N}$ be the equivalence, or $X \simeq \mathbb{N}$ and let $f : X \to \mathbb{N}$ be the equivalence.

b) If there is a one-to-one function $f : X \to \mathbb{N}$, then there is a function $g : \mathbb{N} \to X$ with $g \circ f = 1_X$. But now, 1_X is onto, so g is onto. So there is an onto function from $\mathbb{N}$ to X.

   Conversely, if there is an onto function from $\mathbb{N}$ to X, that function has a right inverse which must be one-to-one.

c) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(m, n) = 2^m 3^n$. By the fundamental theorem of arithmetic, f is one-to-one. Hence, $\mathbb{N} \times \mathbb{N}$ is countbale. Notice that it is not finite, so it must be equivalent to $\mathbb{N}$.

d) If X is countable and $A \subseteq X$, let $f : X \to \mathbb{N}$ be one-to-one. Then $f \circ 1_A : A \to X \to \mathbb{N}$ is one-to-one, so A is countbale.

e) Let $f : \mathbb{N} \to I$ be onto. For each $\alpha \in I$, let $g_\alpha : \mathbb{N} \to A_\alpha$ be onto. Define $h : \mathbb{N} \times \mathbb{N} \to \bigcup A_\alpha$ by $h(m,n) = g_{f(n)}(m)$. Then h is onto. Now, if $k : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is onto, $h \circ k : \mathbb{N} \to \bigcup A_\alpha$ is onto, proving the result.

♦

Theorem 1.26 *(Schröder-Bernstein) Suppose that $f : X \to Y$ and $g : Y \to X$ are one-to-one functions. Then, X and Y are equivalent.*

Proof: Let $C_1 = X \setminus g[Y]$ and $C_{n+1} = g \circ f[C_n]$. Then let $C = \bigcup_{n=1}^{\infty} C_n$. Notice that if $x \notin C$, then $x \in g[Y]$. Since g is one-to-one, there is then a unique $y \in Y$ such that $g(y) = x$.

Define $h : X \to Y$ by $h(x) = f(x)$ if $x \in C$ and $h(x) = g^{-1}(x)$ if $x \notin C$. By the last paragraph, h IS well-defined. We claim that h is one-to-one and onto.

Suppose that $x_1, x_2 \in X$ with $h(x_1) = h(x_2)$. There are three cases: if $x_1, x_2 \in C$, then $f(x_1) = f(x_2)$, so $x_1 = x_2$ since f is one-to-one. If $x_1, x_2 \notin C$, then $g^{-1}(x_1) = g^{-1}(x_2)$, so applying g gives $x_1 = x_2$. Finally, if $x_1 \in C$ and $x_2 \notin C$ (the other case is the same), $f(x_1) = g^{-1}(x_2)$, so $g \circ f(x_1) = x_2$. But, if $x_1 \in C$, then $x_1 \in C_n$ for some $n \geq 1$ and so $g \circ f(x_1) \in C_{n+1}$, so $x_2 \in C_{n+1} \subseteq C$, a contradiction. Thus h is one-to-one.

Next, if $y \in Y$, there are two cases. If $y \in f[C]$, then there is an $x \in C$ with $h(x) = f(x) = y$. Otherwise, for $y \notin f[C]$, let $x = g(y)$. Since $x \in g[Y]$, $x \notin C_1$. But, now, if $x \in C_{n+1}$ for some $n \geq 1$, there is a $z \in C_n$ with $g \circ f(z) = x = g(y)$. Since g is one-to-one, $y = f(z) \in f[C]$, a contradiction. Hence $x \notin \cup_{n=1}^{\infty} C_n = C$, so $h(x) = g^{-1}(x) = y$. Hence h is onto. ♦

Theorem 1.27 *Let X be any set. Then there is no onto function from X to $\mathcal{P}(X)$. In particular, X and $\mathcal{P}(X)$ are not equivalent.*

As a consequence, $\mathcal{P}(\mathbb{N})$ is an infinite set that is not countable.

Exercises:

1. Prove that if A, B, and C are sets, then $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

2. Prove that if B is a set and $\{A_i\}_{i \in I}$ is an idexed family of sets, then $B \setminus (\bigcap_{i \in I} A_i) = \bigcup_{i \in I} (B \setminus A_i)$.

3. Suppose that $A, B \subseteq X$ and $C, D \subseteq Y$. Is it true that $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$? Either prove this result or give a counter-example.

4. Suppose that $A, B \subseteq X$ and $C, D \subseteq Y$. Is it true that $(A \times C) \cup (B \times D) = (A \cup B) \times (C \cup D)$? Either prove this result or give a counter-example.

5. Suppose that $A \subseteq X$ and $B \subseteq Y$. Is it true that $(X \times Y) \setminus (A \times B) = (X \setminus A) \times (Y \setminus B)$? Either prove this result or give a counter-example.

6. Suppose that $f : X \to Y$ is a function and $A, B \subseteq X$. Is it true that $f[A \cup B] = f[A] \cup f[B]$? Either prove this result or give a counter-example.

7. a) Suppose that $f : X \to Y$ is a function and $\{A_\alpha\}_{\alpha \in I}$ is a family of subsets of X. Show that $f[\bigcup_\alpha A_\alpha] = \bigcup_\alpha f[A_\alpha]$.

   b) Suppose that $f : X \to Y$ is a function and $\{B_\alpha\}_{\alpha \in I}$ is a family of subsets of Y. Show that $f^{-1}[\bigcup_\alpha B_\alpha] = \bigcup_\alpha f^{-1}[B_\alpha]$.

8. Suppose that $f : X \to Y$ and $A \subseteq X$. Is it true that $f[X \setminus A] = Y \setminus f[A]$? Either prove this result or give a counter-example.

9. a) Give and example of sets X and Y, a subset $A \subseteq X$ and a function $f : X \to Y$ such that $f^{-1}[f[A]] \neq A$.

   b) Give and example of sets X and Y, a subset $B \subseteq Y$ and a function $f : X \to Y$ such that $f[f^{-1}[B]] \neq B$.

10. Show that $f : X \to Y$ is one-to-one if and only if $f^{-1}[f[A]] = A$ for all $A \subseteq X$.

11. Show that $f : X \to Y$ is onto if and only if $f[f^{-1}[B]] = B$ for all $B \subseteq Y$.

12. Suppose that $f : X \to Y$ is a function and that $f[A \cap B] = f[A] \cap f[B]$ for all subsets $A, B \subseteq X$. Show that f must be one-to-one.

13. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions.

    a) If f is one-to-one, must $g \circ f : X \to Z$ be one-to-one?

    b) If both f and g are one-to-one, must $g \circ f$ be one-to-one?

c) If $g \circ f$ is one-to-one, must f be one-to-one? How about g?

Justify your answers either by a proof or by a counter-example.

14. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions.

    a) If f is onto, must $g \circ f$ be onto?

    b) If both f and g are onto, must $g \circ f$ be onto?

    c) If $g \circ f$ is onto, must f be onto? How about g?

    Justify your answers either by a proof or by a counter-exmaple.

15. Let $A = \{a, b, c\}$. List all elements of $\mathcal{P}(A)$.

16. Let X be a set and define $C : \mathcal{P}(X) \to \mathcal{P}(X)$ by setting $C(A) = X \setminus A$. Is C a one-to-one function? Is it onto?

17. Suppose that $f : X \to Y$ is a function. For $A \subseteq X$, define $\hat{f}(A) = f[A]$. Why does this define a function $\hat{f} : \mathcal{P}(X) \to \mathcal{P}(Y)$? Show that if f is one-to-one, then $\hat{f}$ is also. Is the converse true? What happens for f onto?

18. Show that $X \times Y$ is equivalent to $Y \times X$ as sets.

19. Suppose that X is equivalent to Y and Z is equivalent to W as sets. Show that $X \times Z$ is equivalent to $Y \times W$.

20. Suppose that $f : X \to Y$ and $g : Y \to X$ are both onto functions. Show that X and Y are equivalent. Hint: do NOT attempt to mimic the proof of Schröder-Bernstein. Instead, use that result.

Chapter 2

Metric Spaces

2.1 Euclidean Spaces

Definition 2.1 *Recall that* $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\}$. *For* $x, y \in \mathbb{R}^n$, *we define the Euclidean distance from* x *to* y *to be*

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Lemma 2.2 *For* $x, y \in \mathbb{R}^n$, *we have*

$$\left| \sum_{i=1}^{n} x_i y_i \right|^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right)$$

Proof:

We define some notation to make the argument easier. If $x, y \in \mathbb{R}^n$, define $\langle x, y \rangle = \sum x_i y_i$. Then we need to show that

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

If either $x = 0$ or $y = 0$, the result holds since both sides are 0. So now assume that both $x \neq 0$ and $y \neq 0$.

For every real number $t \in \mathbb{R}$, we have that $x + ty \in \mathbb{R}^n$ and

$$0 \quad \leq \quad \langle x + ty, x + ty \rangle$$

$$\begin{aligned} &= \langle x,x\rangle + t\langle x,y\rangle + t\langle y,x\rangle + t^2\langle y,y\rangle \\ &= \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle \end{aligned}$$

Now, let

$$t = -\frac{\langle x,y\rangle}{\langle y,y\rangle}.$$

Since $y \neq 0$, $\langle y,y\rangle \neq 0$, so t is defined. Plug into the above inequality and rearrange to get the result. ♦

Proposition 2.3 *If $x, y, z \in \mathbb{R}^n$, then $d_2(x,z) \leq d_2(x,y) + d_2(y,z)$.*

Proof:

$$\begin{aligned} d_2(x,z)^2 &= \langle x-z, x-z\rangle \\ &= \langle x-y+y-z, x-y+y-z\rangle \\ &= \langle x-y, x-y\rangle + 2\langle x-y, y-z\rangle + \langle y-z, y-z\rangle \\ &\leq \langle x-y, x-y\rangle + 2|\langle x-y, y-z\rangle| + \langle y-z, y-z\rangle \\ &\leq \langle x-y, x-y\rangle + 2\langle x-y, x-y\rangle^{1/2}\langle y-z, y-z\rangle^{1/2} + \langle y-z, y-z\rangle \\ &= \left(\langle x-y, x-y\rangle^{1/2} + \langle y-z, y-z\rangle^{1/2}\right)^2 \\ &= \left(d_2(x,y) + d_2(y,z)\right)^2 \end{aligned}$$

giving the result. ♦

We now recall the definition of continuity given in advanced calculus courses for functions between Euclidean spaces:

Definition 2.4 *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ and $x_0 \in \mathbb{R}^m$. We say that f is continuous at x_0 if for each $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x \in \mathbb{R}^m$ with $d_2(x, x_0) < \delta$, we have $d_2(f(x), f(x_0)) < \varepsilon$.*

Sinilarly, we have convergence of sequences in $\mathbb{R}^n$ defined as follows:

Definition 2.5 *Suppose that $\{x_m\}_{m=1}^{\infty}$ is a sequence in $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. We say that $\{x_m\}$ converges to x_0 if for every $\varepsilon > 0$ there is a M such that whenever $m \geq M$ we have $d_2(x_m, x_0) < \varepsilon$.*

These are the two definitions we are going to generalize to a much broader context. This cannot be done in the general context of sets because there is no concept of distance between points in an arbitrary set. To remedy this, we add some extra structure called a metric.

Definition 2.6 *A metric on a set X is a function $d : X \times X \to \mathbb{R}$ which satisfies*

a) $d(x, y) \geq 0$ for all $x, y \in X$.

b) $d(x, y) = 0$ if and only if $x = y$.

c) $d(x, y) = d(y, x)$ for all $x, y \in X$.

d) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The last property is called the triangle inequality. If d is a metric on the set X, we say that the pair (X, d) is a metric space. If d is understood in context, we will simply say that X is a metric space.

If we have a metric d on a set X, we call $d(x, y)$ the distance between x and y. Be careful because it is quite possible that there will be more than one metric on a set that is of interest. When this happens, we have to be very clear which metric is being used.

Examples:

1. Let $X = \mathbb{R}^n$ and $d = d_2$ as above. Clearly, $d(x, y) \geq 0$, $d(x, y) = d(y, x)$, and $d(x, y) = 0$ if and only if $x = y$. The last property is the proposition above. We call this the Euclidean metric on $\mathbb{R}^n$. Unless otherwise stated, this is the metric we use for $\mathbb{R}^n$.

2. Let $X = \mathbb{R}^n$ and define the taxi-cab metric

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

Again, it is easy to see that $d_1(x, y) = 0$ only when $x = y$ and that $d_1(x, y) = d_1(y, x)$. To see that the triangle inequality holds, let $x, y, z \in \mathbb{R}^n$ and add the inequalities $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

3. Let $X = \mathbb{R}^n$ and define the max metric by

$$d_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|$$

Again, the first two properties of a metric are easy to see.

For the triangle inequality, notice that for each $1 \le i \le n$, we have $|x_i - z_i| \le |x_i - y_i| + |y_i - z_i| \le d_\infty(x, y) + d_\infty(y, z)$. Hence, taking a maximum, $d_\infty(x, z) \le d_\infty(x, y) + d_\infty(y, z)$.

4. Let X be any set at all and define the discrete metric on X by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then d is a metric on X. The first three propoerties are clear. For the last, if $x \neq y$, we have $d(x, z) \le 1 = d(x, y) \le d(x, y) + d(y, z)$. On the other hand, if $x = y$, then $d(x, z) = d(y, z) \le d(x, y) + d(y, z)$.

5. Let $X = \mathbb{R}^2$ and define the radial metric d_r as follows: if x and y are on the same line through the origin of $\mathbb{R}^2$, let $d_r(x, y) = d_2(x, y)$, the usual Euclidean distance. Otherwise, if x and y are on different lines through the origin, define $d_r(x, y) = d_2(x, 0) + d_2(y, 0)$. It is an exercise to show that this is a metric on $\mathbb{R}^2$.

6. For any set X, let

$$\mathcal{B}(X) = \{f : X \to \mathbb{R} : \sup\{|f(x)| : x \in X\} < \infty\}$$

the set of bounded real-valued functions on X. Define

$$d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$

We must first show that this definition gives a finite answer for each f and g in $\mathcal{B}(X)$. In fact, since $|f(x) - g(x)| \le |f(x)| + |g(x)| \le \sup |f(x)| + \sup |g(x)| < \infty$, this metric is well defined (the supremum is not infinite). Again, all but the triangle inequality are trivial here. We can show the triangle inequality in a way similar to the last example.

7. Let $C([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$. Define the L_1 metric by setting
$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$$
In this case, it is a bit more subtle to show that $d_1(f,g) = 0$ only when $f = g$. In fact, a result from advanced calculus says that if $f \geq 0$ is continuous and $\int_0^1 f(x)\, dx = 0$, then $f = 0$. The triangle inequality can be obtained from integrating the pointwise version.

8. Let $X = \mathbb{Z}$ and suppose that $p \in \mathbb{Z}$ is prime. Define the p-adic metric $d_p(x,y)$ to be 0 if $x = y$. Otherwise, suppose that p^n is the largest power of p that divides $x - y$. Define $d_p(x,y) = p^{-n}$. This is called the p-adic metric on $\mathbb{Z}$. Again, it is an exercise to show this is, in fact, a metric on $\mathbb{Z}$.

The following proposition will turn out to be quite useful for us.

Proposition 2.7 *For $x, y \in \mathbb{R}^n$, we have the following inequalities:*

$$d_\infty(x,y) \leq d_2(x,y) \leq \sqrt{n} \cdot d_\infty(x,y)$$

$$d_\infty(x,y) \leq d_1(x,y) \leq n \cdot d_\infty(x,y)$$

$$d_2(x,y) \leq d_1(x,y) \leq \sqrt{n} \cdot d_2(x,y)$$

It is a general theme in mathematics that when a new structure is defined, we wish to consider both sub-structures and how to generate new structures from old ones. The following definitions do this.

Proposition 2.8 *Suppose that (X,d) is a metric space and $A \subseteq X$. Define, for $a, b \in A$, $d_A(a,b) = d(a,b)$. Then (A, d_A) is a metric space. We call d_A the subspace metric on A and (A, d_A) a subspace of (X,d).*

In terms of the defining functions, d_A is just the restriction of d to the set $A \times A$.

Examples:

1. Since $[0,1] \subseteq \mathbb{R}$, we get a subspace metric on $[0,1]$ induced by the usual metric on $\mathbb{R}$. Unless otherwise stated, this is the metric we use for $[0,1]$.

2. We have that $C([0,1]) \subseteq \mathcal{B}([0,1])$ since every continuous function on $[0,1]$ is bounded. So $C([0,1])$ becomes a metric space with the d_∞ metric. This is often called the uniform metric on $C([0,1])$.

3. Let $X = \mathbb{R}^2$ with the radial metric d_r. Let $A = \{(x,y) : x^2 + y^2 = \frac{1}{4}\}$. Then the subspace metric on A is the discrete metric on A.

Whenever we talk about metric properties of a subset, we use the subspace metric unless otherwise stated.

Definition 2.9 *Suppose that (X, d_x) and (Y, d_Y) are metric spaces. Define the product metric $d_{X\times Y}$ on $X \times Y$ by setting*

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}$$

Definition 2.10 *Let X be a metric space, $x_0 \in X$, and $r > 0$ a real number. Define*

a) the open ball $B_r(x_0) = B(x_0;r) = \{x \in X : d(x,x_0) < r\}$,

b) the closed ball $\overline{B}_r(x_0) = \overline{B}(x_0,r) = \{x \in X : d(x,x_0) \leq r\}$.

c) the sphere $S_r(x_0) = S(x_0;r) = \{x \in X : d(x,x_0) = r\}$

It is often helpful to consider the open and closed balls when attempting to understand a metric.

Examples:

a) Consider $\mathbb{R}^2$ with the Euclidean metric, d_2. The closed balls consist of closed, circular disks with center x_0 and or radius r.

b) In $\mathbb{R}^2$ with the Taxi-cab metric, d_1, the closed balls consist of closed squares with corners a distance r away from x_0 along each of the coordinate directions.

c) In $\mathbb{R}^2$ with the max metric, d_∞, the closed balls consist of closed squares of width $2r$ centered on x_0.

d) If X is a set and d is the discrete metric, then $B(x_0, r) = \{x_0\}$ if $r \leq 1$ and $B(x_0; r) = X$ if $r > 1$.

2.2 Continuity and Uniform Continuity

Definition 2.11 *A sequence in a set X is a function $x : \mathbb{N} \to X$. We usually write x_n instead of $x(n)$. We also write $\{x_n\}_{n=1}^{\infty}$ for the sequence x.*

Definition 2.12 *Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Suppose that $x \in X$. We say that $\{x_n\}$ converges to x and write $x_n \to x$ if for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $n \in \mathbb{N}$ with $n \geq N$, we have $d(x_n, x) < \varepsilon$. We say a sequence $\{x_n\}$ converges if there is an $x \in X$ such that $x_n \to x$. In this case, we also say that x is the limit of the sequence $\{x_n\}$.*

Examples

1. If X has the discrete metric, then the sequences $x_n \to x$ if and only if there is an $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$. We say that the sequence is eventually constant.

2. If X is a set, we say a sequence $f_n \in \mathcal{B}(X)$ converges uniformly to $f \in \mathcal{B}(X)$ if $f_n \to f$ in the supremum metric. This is should agree with the notion of uniform convergence you saw in advanced calculus.

3. Suppose that (X, d) is a metric space and $A \subseteq X$. If $\{a_n\}$ is a sequence in A and $a \in A$, then $a_n \to a$ in A if and only if $a_n \to a$ in X.

4. Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. Let $x \in \mathbb{R}$ be an irrational number. For each $n \in \mathbb{N}$, we can find a rational number q_n such that $|x - q_n| < \frac{1}{n}$. Then the sequence $\{q_n\}$ is a sequence in A that converges in X, but does not converge in A. The point is that the limit of the sequence is no longer in the subspace A.

Theorem 2.13 *Suppose that (X, d) is a metric space and $\{x_n\}$ is a sequence in X. Suppose also that $x, y \in X$ and $x_n \to x$ and $x_n \to y$. Then $x = y$. In other words, limits of sequences are unique.*

Proof: Suppose that $x \neq y$ and let $\varepsilon = d(x, y)/3 > 0$. Since $x_n \to x$, there is an $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $d(x_n, x) < \varepsilon$. Also, since $x_n \to y$, there is an $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies that $d(x_n, y) < \varepsilon$. Choose $n = \max\{N_1, N_2\}$ and find that $d(x, y) \leq d(x, x_n) + d(x_n, y) = d(x_n, x) + d(x_n, y) \leq 2d(x, y)/3$. Since $d(x, y) > 0$, this is a contradiction. ♦

Definition 2.14 *Let (X, d_X) and (Y, d_Y) be metric spaces, $x_0 \in X$ and $f : X \to Y$ a function.*

a) We say that f is continous at x_0 if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x \in X$ with $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \varepsilon$.

b) We say that f is continuous on X if it is continuous at every $x_0 \in X$.

c) We say that f is uniformly continuous on X if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x, x_0 \in X$ with $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \varepsilon$.

d) We say that f is Lipschitz if there is an $M < \infty$ such that $d_Y(f(x), f(x')) \leq M d_X(x, x')$ for all $x, x' \in X$.

e) We say that f is an isometry of X into Y if $d_Y(f(x), f(x')) = d_X(x, x')$ for all $x, x' \in X$.

Proposition 2.15 *We have the following:*

a) Every uniformly continuous function is continuous.

b) Every Lipschitz function is uniformly continuous.

c) Every isometry into is one-to-one.

Examples:

1. Suppose that X is given the discrete metric and $f : X \to Y$ is any function into a metric space Y. Then f is uniformly continuous: for any $\varepsilon > 0$, choose $\delta = 1$. Then $d(x, x_0) < \delta$ implies $x = x_0$, so $d(f(x), f(x_0)) = 0 < \varepsilon$.

2. Let $X = \mathbb{R}^n$ and let d, d' be any pair of the metrics d_1, d_2, d_∞. Then the identity map $i : (X, d) \to (X, d')$ is Lipschitz.

3. Let X be a metric space and $A \subseteq X$. Give A the subspace topology. Then the inclusion map $i_A : A \to X$ is an isometry of A into X.

4. Give the open unit interval $(0, 1)$ the subspace metric induced from the usual metric on $\mathbb{R}$. Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. Then f is continuous but is not uniformly continuous.

5. Let X and Y be metric spaces and define the projection $\pi_X : X \times Y \to X$ by $\pi_X(x, y) = x$. Then π_X is uniformly continuous. Similarly, $\pi_Y(x, y) = y$ defines a uniformly continuous $\pi_Y : X \times Y \to Y$.

6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x + y$. Then f is uniformly continuous.

7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = xy$. Then f is continuous, but is not uniformly continuous.

8. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then f is continuous, but is not uniformly continuous.

9. Let $X = Y = C([0, 1])$. Give X the uniform metric and Y the L_1 metric. Then the identity map $i : X \to Y$ is uniformly continuous. But the identity map $i : Y \to X$ is not continuous at any point.

Definition 2.16 *Let X be a metric space and $A \subseteq X$ a subset and $x \in X$. Define the distance from x to the set A by $d(x, A) = \inf\{d(x, a) : a \in A\}$.*

Note that $d(x, y) = d(x, \{y\})$. Notice also that if $x \in A$, then $d(x, A) = 0$. The converse fails, though.

Examples:

1. In $\mathbb{R}$, let $x = 0$ and $A = [1, 2]$. Then $d(x, A) = 1$.

2. Again in $\mathbb{R}$, let $x = 0$ and $A = (1, 2]$. Then $d(x, A) = 1$ but there is no point $a \in A$ with $d(x, a) = 1$.

3. If $x \in A$, then $d(x, A) = 0$.

4. Yet again in $\mathbb{R}$, let $x = 0$ and $A = (0, 11]$. Then $d(x, A) = 0$ but $x \notin A$.

Lemma 2.17 *Let X be a metric space, $A \subseteq X$ and $x, y \in X$. Then $|d(x, A) - d(y, A)| \leq d(x, y)$.*

Proof: For each $a \in A$, we have $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$. Hence, $d(x, A) - d(x, y) \leq d(y, a)$ for every $a \in A$. By definition of the infimum, $d(x, A) - d(x, y) \leq d(y, A)$. This gives $d(x, A) - d(y, A) \leq d(x, y)$. Now, reverse the roles of x and y to find $d(y, A) - d(x, A) \leq d(y, x) = d(x, y)$. The two inequalities now give the result. ♦

Proposition 2.18 *Let X be a metric space and $A \subseteq X$. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, A)$. Then f is Lipschitz.*

Proof: In fact, the above shows $|f(x) - f(y)| \leq d(x, y)$. ♦

This shows that there are plenty of uniformly continuous functions from any metric space X into $\mathbb{R}$.

Theorem 2.19 *Suppose that X and Y are metric spaces, $f : X \to Y$ and $x_0 \in X$. The following are equivalent:*

a) f is continuous at x_0

b) For each $\varepsilon > 0$, there is a $\delta > 0$ such that $f[B_\delta(x_0)] \subseteq B_\varepsilon(f(x_0))$

c) For each $\varepsilon > 0$, there is a $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}[B_\varepsilon(f(x_0))]$.

Theorem 2.20 *Let (X, d_X) and (Y, d_Y) be metric spaces and $x_0 \in X$. The following are equivalent:*

a) f is continuous at x_0,

b) for every sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \to x_0$, we have $f(x_n) \to f(x_0)$ in Y.

Proof: Suppose that f is continuous at x_0 and suppose that $\{x_n\}$ is a sequence in X such that $x_n \to x_0$. We need to show that $f(x_n) \to f(x_0)$. Let $\varepsilon > 0$. By continuity of f at x_0, there is a $\delta > 0$ such that whenever $x \in X$ and $d(x, x_0) < \delta$, we have $d(f(x), f(x_0)) < \varepsilon$. Now, since $x_n \to x_0$, there is a $N \in \mathbb{N}$ such that $n \geq N$ implies that $d(x_n, x_0) < \delta$. But then, we see that for $n \geq N$, $d(f(x_n), f(x_0)) < \varepsilon$ as required.

The converse is trickier. We show the contrapositive. So, suppose that f is *not* continous at x_0. Then, there is an $\varepsilon > 0$ such that whenever $\delta > 0$, there is an $x \in X$ with $d(x, x_0) < \delta$, but $d(f(x), f(x_0)) \geq \varepsilon$. Fix this ε and choose, for each $n \in \mathbb{N}$, $\delta = 1/n$. Then we obtain a $x_n \in X$ such that $d(x_n, x_0) < 1/n$ but $d(f(x_n), f(x_0)) \geq \varepsilon$. But then $\{x_n\}$ is a sequence in X such that $x_n \to x_0$, but $f(x_n) \not\to f(x_0)$. ♦

Corollary 2.21 *Suppose that X, Y, Z are metric spaces, $f : X \to Y$ is continous at $x_0 \in X$ and $g : Y \to Z$ is continous at $f(x_0) \in Y$. Then $g \circ f : X \to Z$ is continous at x_0.*

Proof: We use the last result. Suppose that $x_n \to x_0$ in X. By continuity of f at x_0, we have $f(x_n) \to f(x_0)$ in Y. By continuity of g at $f(x_0)$, this gives $g(f(x_n)) \to g(f(x_0))$, showing that $g \circ f(x_n) \to g \circ f(x_0)$. Applying the theorem again shows $g \circ f$ is continuous at x_0. ♦

Example Let X be $\mathbb{R}^n$ and d, d' any pair of metrics from d_1, d_2, d_∞. Then since the identity map is continuous, if $\{x_n\}$ is any sequence in X and $x \in X$, then $x_n \to x$ for d if and only if $x_n \to x$ for d'. In other words, the three metrics d_1, d_2 and d_∞ have exactly the same convergent sequences.

Furthermore, because of the above characterization of continuity using sequences, the metrics have exactly the same continuous functions.

Definition 2.22 *Let X be a set and d_1, d_2 metrics on X. We say that d_1 and d_2 are equivalent metrics if $i : (X, d_1) \to (X, d_2)$ and $i : (X, d_2) \to (X, d_1)$ are both continuous.*

So, if two metrics are equivalent, a sequence converges in one if and only if it converges in the other to the same point.

Also, if d_1 and d_2 are equivalent metrics on X and Y is any metric space, $f : X \to Y$. Then f is continous with the metric d_1 if and only if it is equivalent with the metric d_2. A similar statement holds if we have two equivalent metrics on Y.

Proposition 2.23 *Suppose that d_1 and d_2 are metrics on the space X. Suppose also that there are constants c and C such that $cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)$ for all $x, y \in X$. Then d_1 and d_2 are equivalent metrics.*

Proposition 2.24 *Let (X, d) be a metric space. Define $d'(x, y) = \min\{1, d(x, y)\}$ and $d''(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Then d' and d'' are bounded metrics equivalent to d.*

2.3 Cluster Points, Open sets, and Closed sets

Definition 2.25 *Let (X, d) be a metric space and $A \subseteq X$. We say that $x \in X$ is a limit point of A (also called a cluster point of A) if for every $\varepsilon > 0$ there is an $a \in A$ with $a \neq x$ such that $d(x, a) < \varepsilon$. Let A' denote the set of all limit points of A.*

Theorem 2.26 *Let (X, d) be a metric space and $A \subseteq X$. Then $x \in X$ is a limit point of A if and only if there is a sequence $\{a_n\}$ of distinct elements of A such that $a_n \to x$.*

Proof: First assume that such a sequence in A exists. Since the terms of the sequence are distinct, at most one of them is x (possibly none of them is). Let $\varepsilon > 0$. Choose N so that $n \geq N$ implies that $d(a_n, x) < \varepsilon$. We can certainly pick $a_n \neq x$, so x is a limit point of A by the above definition.

Conversely, suppose that x is a limit point of A. Let $\varepsilon_1 = 1$ to find $a_1 \in A$ so that $d(a_1, x) < 1$ and $a_1 \neq x$. Next, let $\varepsilon_2 = \min\{\frac{1}{2}, d(a_1, x)\}$. Note that this $\varepsilon > 0$. Again, since x is a limit point, there is an $a_2 \in A$ with $a_2 \neq x$ and $d(a_2, x) < \varepsilon_2$. Notice that $a_2 \neq a_1$ also.

Now, if $a_1, \cdots a_n \in A$ have been found, let $\varepsilon_{n+1} = \min\{\frac{1}{n+1}, d(a_1, x), \cdots, d(a_n, x)\}$. Then $\varepsilon_{n+1} > 0$, so there is an $a_{n+1} \in A$ with $a_{n+1} \neq x$ and $d(x, a_{n+1}) < \varepsilon_{n+1}$. By induction, we then get a sequence $\{a_n\}$. Since $d(x, a_n) < \frac{1}{n}$ for each n, we have $a_n \to x$ as desired. ♦

Corollary 2.27 *In a metric space, x is a limit point of the set A if and only if for each $\varepsilon > 0$, the set $A \cap B(x, \varepsilon)$ is infinite.*

Examples:

1. Let $\mathbb{R}$ have the usual (absolute value) metric. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $A' = \{0\}$.
2. Let $A = [0, 1] \subseteq \mathbb{R}$ with the usual metric. Then $A' = [0, 1]$.
3. Let $\mathbb{Q} \subseteq \mathbb{R}$, again with the usual metric. Then $A' = \mathbb{R}$.
4. Let X be any set with the discrete metric and $A \subseteq X$. Then $A' = \emptyset$.
5. Let $X = \mathbb{R}^2$ with the radial metric d_r. Let $A = \{(q, 1) : q \in \mathbb{Q}\}$. Then $A' = \emptyset$.
6. Let $X = \mathbb{R}^2$ with the radial metric and $A = \{(q, 0) : q \in \mathbb{Q}\}$. Then $A' = \{(x, 0) : x \in \mathbb{R}\}$.

Definition 2.28 *Let X be a metric space and $A \subseteq X$. We say that A is a bounded set if there is an $x \in X$ and an $R < \infty$ such that $A \subseteq B(x; R)$.*

Proposition 2.29 *Suppose that $\{x_n\}$ is a convergent sequence in a metric space X. Then $\{x_n : n \geq 1\}$ is a bounded set in X.*

The converse fails. We will see a particularly good example below.

Definition 2.30 *let $\{x_n\}$ be a sequence in a metric space X. A subsequence of $\{x_n\}$ is a sequence of the form $\{x_{n_k}\}_{k=1}^{\infty}$ where $\{n_k\}$ is an increasing sequence in $\mathbb{N}$, i.e. $n_{k+1} > n_k$.*

Proposition 2.31 *Suppose that $x_n \to x$ in a metric space X. If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then $x_{n_k} \to x$ also.*

Examples:

a) Let X be an infinite set with the discrete metric and let $\{x_n\}$ be any sequence of distinct points of X. Then $\{x_n\}$ is a bounded sequence with no convergent subsequence.

b) Give $\mathbb{R}^2$ the radial metric d_r and let $\{x_n\}$ be any sequence of distinct points in the unit circle. Then $\{x_n\}$ is a bounded sequence that has no convergent subsequences.

Definition 2.32 *Let X be a metric space and $A \subseteq X$. We say that*

a) A is open if for each $a \in A$, there is a $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq A$.

b) A is closed if $X \setminus A$ is open.

If we want to make the metric used explicit, we can say A is d-open or d-closed.

Examples:

a) Every set is both open and closed in the discrete topology.

b) Every open ball $B(x; r)$ is open.

c) Every closed ball $\overline{B}(x; r)$ is closed.

d) Let $X = [0, 1] \cup [2, 3]$ with the subspace topology from $\mathbb{R}$. Let $A = [0, 1]$. Then A is open since for all $a \in A$, $B(a, 1) \subseteq A$. Similarly, $X \setminus A = [2, 3]$ is open, so A is also closed.

Theorem 2.33 *Let X be a metric space. Then*

a) $\emptyset$ and X are open

b) If U and V are open, then $U \cap V$ is open

c) If $\{U_\alpha\}_{\alpha \in I}$ is a family of open sets, then $\bigcup_{\alpha \in I} U_\alpha$ is open.

Proof:

a) The emptyset is open trivially (there are no elements of the empty set) and X is open using any $r > 0$.

b) Now, suppose that U and V are open and that $x \in U \cap V$. Then, there are $r_1, r_2 > 0$ such that $B(x; r_1) \subseteq U$ and $B(x; r_2) \subseteq V$. let $r = \min\{r_1, r_2\}$. Then $B(x; r) \subseteq B(x; r_1) \cap B(x; r_2) \subseteq U \cap V$.

c) Now suppose that $\{U_\alpha\}$ is a collection of open sets. Let $x \in \bigcup U_\alpha$. Then, there is an α such that $x \in U_\alpha$. Since U_α is open, there is an $r > 0$ such that $B(x; r) \subseteq U_\alpha \subseteq \bigcup U_\alpha$.

♦

Theorem 2.34 *Let X be a metric space. Then*

a) $\emptyset$ and X are closed

b) If C and D are closed, then $C \cup D$ is closed

c) If $\{C_\alpha\}_{\alpha \in I}$ is a family of closed sets, then $\bigcap_{\alpha \in I} C_\alpha$ is closed.

Proof:

a) We have $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$, so both $\emptyset$ and X are closed.

b) Suppose that C and D are closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is the intersection of two open sets, so is open. Hence $C \cup D$ is closed.

c) We have that $X \setminus (\bigcap C_\alpha) = \bigcup (X \setminus C_\alpha)$ is open.

♦

Theorem 2.35 *Suppose that X and Y are metric spaces and $f : X \to Y$ is a function. The following are equivalent:*

a) f is continuous.

b) For each $V \subseteq Y$ which is open, we have $f^{-1}[V]$ is open in X.

c) For each $D \subseteq Y$ which is closed, we have $f^{-1}[D]$ is closed in X.

In other words, continuity only depends on the open sets, not on the specific metric used. This is more relevant when we consider the following:

Proposition 2.36 *Let d_1 and d_2 be metrics on the set X. Then d_1 and d_2 are equivalent precisely when the collection of all d_1-open sets is exactly the same as the collection of d_2-open sets.*

Definition 2.37 *Let X be a metric space and $A \subseteq X$. Define the closure of A by $\overline{A} = \{x \in X : d(x, A) = 0\}$.*

Theorem 2.38 *Let X be a metric space, $A \subseteq X$. Then*

a) $\overline{A}$ is a closed set and $A \subseteq \overline{A}$.

b) If C is any closed set in X with $A \subseteq C$, then $\overline{A} \subseteq C$. In other words, $\overline{A}$ is the smallest closed set containing A.

c) If $x \in X$, then $x \in \overline{A}$ if and only if for each $\varepsilon > 0$, there is an $a \in A$ such that $d(x, a) < \varepsilon$.

d) If $x \in X$, then $x \in \overline{A}$ if and only if there is a sequence $\{a_n\}$ in A such that $a_n \to x$.

e) A is closed if and only if $\overline{A} \subseteq A$.

f) $\overline{A} = A \cup A'$.

Proof:

a) If we let $f(x) = d(x, A)$, then f is continuous and $\overline{A} = f^{-1}[\{0\}]$ is the inverse image of a closed set, so is closed. Also, if $x \in A$, then $d(x, A) \leq d(x, x) = 0$, so $x \in \overline{A}$. Thus $A \subseteq \overline{A}$.

b) Now, suppose that $A \subseteq C$ where C is closed. To show that $\overline{A} \subseteq C$, we actually show that $X \setminus C \subseteq X \setminus \overline{A}$. So, suppose that $x \in X \setminus C$. Since $X \setminus C$ is open, there is a $\varepsilon > 0$ such that $B(x; \varepsilon) \subseteq X \setminus C \subseteq X \setminus A$. Hence, $A \subseteq X \setminus B(x; \varepsilon)$. But this says that if $a \in A$, then $d(x, a) \geq \varepsilon$. Hence, $d(x, A) \geq \varepsilon > 0$, so $x \notin \overline{A}$. This gives the result

c) We have $x \in \overline{A}$ if and only if $\inf\{d(x,a) : a \in A\} = 0$. The statement is equivalent to this.

d) Suppose $x \in \overline{A}$. Using the last item, for each $n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$ to find $a_n \in A$ such that $d(x, a_n) < \frac{1}{n}$. Then $a_n \to x$. Conversely, if $\{a_n\}$ is a sequence in A with $a_n \to x$, then $d(x, A) \leq d(x, a_n) \to 0$, so $d(x, A) = 0$, giving $x \in \overline{A}$.

e) In fact, if $\overline{A} \subseteq A$, then, since $A \subseteq \overline{A}$, $A = \overline{A}$ is closed. Conversely, if A is closed, we have from the above that $\overline{A} \subseteq A$.

♦

Definition 2.39 *Let X be a metric space and $A \subseteq X$. Define the interior of A by $A^\circ = int(A) = \{x \in A :$ there is $\varepsilon > 0$ such that $B(x;\varepsilon) \subseteq A\}$.*

Proposition 2.40 *Let X be a metric space and $A \subseteq X$. Then*

a) A° is open

b) If $U \subseteq A$ is open, then $U \subseteq A^\circ$. In other words, A° is the largest open subset of A.

c) $X \setminus A^\circ = \overline{X \setminus A}$

d) $X \setminus \overline{A} = (X \setminus A)^\circ$.

Proof:

a) Let $x \in A^\circ$. Then, there is a $r > 0$ such that $B(x;r) \subseteq A$. We show that $B(x;r) \subseteq A^\circ$. In fact, if $y \in B(x;r)$, then, since open balls are open, there is a $\varepsilon > 0$ such that $B(y;\varepsilon) \subseteq B(x;r) \subseteq A$. But this shows that $y \in A^\circ$.

b) Suppose that U is open and $U \subseteq A$. If $x \in U$, then there is a $\varepsilon > 0$ such that $B(x;\varepsilon) \subseteq U \subseteq A$, so $x \in A^\circ$.

c) Since $A^\circ \subseteq A$ is open, $X \setminus A \subseteq X \setminus A^\circ$ with the latter being closed. Hence $\overline{X \setminus A} \subseteq X \setminus A^\circ$. Conversely, $X \setminus A \subseteq \overline{X \setminus A}$ so, $X \setminus (\overline{X \setminus A}) \subseteq A$ with the first being open. Hence, $X \setminus (\overline{X \setminus A}) \subseteq A^\circ$, so $X \setminus A^\circ \subseteq \overline{X \setminus A}$.

d) Rewrite the previous as $A^\circ = X \setminus (\overline{X \setminus A})$ and replace A by $X \setminus A$ to get the result.

♦

Example: Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then there are no non-empty open subsets of $X \setminus A$, so $(X \setminus A)^\circ$ is empty. Thus $\overline{A} = X$.

Definition 2.41 *We say a set $A \subseteq X$ is dense if $\overline{A} = X$.*

Proposition 2.42 *Let X be a metric space and $A \subseteq X$. The following are equivalent:*

a) A is dense in X.

b) For every non-empty open set U, $U \cap A \neq \emptyset$

c) For every $x \in X$, there is a sequence $\{a_n\}$ from A such that $a_n \to x$.

Exercises:

1. Show that the radial metric d_r is, in fact, a metric on $\mathbb{R}^2$.

2. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that $d_{X\times Y}$ is a metric on $X \times Y$.

3. Let (X, d_X) and (Y, d_Y) be metric spaces. Define d_1 on $X \times Y$ by $d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. Show that d_1 is a metric on $X \times Y$.

4. Suppose that d_1 and d_2 are metrics on the set X. Define $d_3(x, y) = d_1(x, y) + d_2(x, y)$. Show that d_3 is a metric on X.

5. Suppose that d_1 and d_2 are metrics on the set X. Define $d_3(x, y) = \max\{d_1(x, y), d_2(x, y)\}$. Show that d_3 is a metric on X and for $x \in X$, $r > 0$ that $B_3(x, r) = B_1(x, r) \cap B_2(x, r)$. Here, B_1, B_2, B_3 are the balls for the metrics d_1, d_2, d_3 resp.

6. Describe the open balls $B(x; r)$ for $\mathbb{R}^2$ with the radial metric d_r. Be sure to consider when $r < d_2(x, 0)$ from $d_2(x, 0) > r$. What happens if $d_2(x, 0) = r$?

7. Let (X, d) be a metric space and $A \subseteq X$. Let d_A be the subspace metric on A. Let $a_0 \in A$ and $r > 0$. Show that $B_A(a_0; r) = B_X(a_0; r) \cap A$.

8. Let (X, d_X) and (Y, d_Y) be metric spaces. Give $X \times Y$ the product metric $d_{X\times Y}$. Let $(x_0, y_0) \in X \times Y$. Show that $B_{X\times Y}((x_0, y_0); r) = B_X(x_0; r) \times B_Y(y_0; r)$.

9. Let d_2 and d_∞ be the Euclidean and max metrics on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $r > 0$. Show that $B_\infty(x; \frac{r}{\sqrt{n}}) \subseteq B_2(x; r) \subseteq B_\infty(x; r)$.

10. Let $X = \mathbb{R}^2$ and define $d((x_1, y_1), (x_2, y_2)) = |x_1| + |y_1| + |x_2| + |y_2|$. Is d a metric on X?

11. Suppose that d is a metric on X and define $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Show that d' is a metric on X.

12. Give $\mathbb{R}^2$ the Euclidean metric d_2 and $\mathbb{R}$ the usual (absolute value) metric. Give $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ the d_1 metric from problem 3. Describe the open balls for this metric.

13. Let X be a metric space and $y \in B(x; r_1)$. Show that $B(y; r_2) \subseteq B(x; r_1 + r_2)$.

14. Let X be a metric space and suppose that $r_1 + r_2 < d(x, y)$. Show that $B(x; r_1) \cap B(y; r_2) = \emptyset$.

15. Show that if X is a metric space and $A \subseteq X$ is a bounded subset, then for every $x \in X$ there is an $R < \infty$ such that $A \subseteq B(x; R)$. In other words, every $x \in X$ can be used in the definition of boundedness.

16. Let X be a metric space and $A \subseteq X$ a finite set. Show that $A' = \emptyset$.

17. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\{(x_n, y_n)\}$ be a sequence in $X \times Y$ and $(x, y) \in X \times Y$. Give $X \times Y$ the product metric. Show that $(x_n, y_n) \to (x, y)$ if and only if $x_n \to x$ and $y_n \to y$.

18. Let X be a metric space and $A \subseteq X$. Suppose that $x \in A'$. Show that $d(x, A) = 0$. What about the converse?

19. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is an $M < \infty$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous. Hint: Mean Value Theorem.

20. Suppose that X,Y and Z are metric spaces and $f : Z \to X$ and $g : Z \to Y$ are functions. Define $h : Z \to X \times Y$ by $h(z) = (f(z), g(z))$. Show that h is continuous if and only if both f and g are continuous.

21. Suppose that $f, g : X \to \mathbb{R}$ are contiunuous functions. Show that $f + g : X \to \mathbb{R}$ defined by $(f + g)(x) = f(x) + g(x)$ is continuous.

22. Prove Proposition 2.23

23. Let X and Y be metric spaces. Show that the usual product metric $d_{X\times Y}$ and the metric given by $d_1(x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ are equivalent metrics on $X \times Y$.

24. Suppose that (X, d) is a metric space and define $d'(x, y) = \min\{1, d(x, y)\}$. Show that d' is a metric on X that is equivalent to d.

25. Let (X, d) be a metric space. Show that $|d(x, y) - d(w, z)| \leq d(x, w) + d(y, z)$ for all $x, y, w, z \in X$.

26. Suppose that (X, d) is a metric space. Give $X \times X$ the product metric and consider d to be a function $d : X \times X \to \mathbb{R}$. Show that d is continuous.

27. Suppose that (X, d) is ametric space and $x_n \to x$, $y_n \to y$ are convergent sequences in X. Show that $d(x_n, y_n) \to d(x, y)$ as a sequence in $\mathbb{R}$.

28. Suppose that d_1 and d_2 are metrics on a space X. Let $d_3(x, y) = \max\{d_1(x, y), d_2(x, y)\}$. Show that a subset $U \subseteq X$ is d_3-open if and only if it is both d_1-open and d_2-open.

29. Suppose that X and Y are metric spaces and that $U \subseteq X$ and $V \subseteq Y$ are open. Show that $U \times V \subseteq X \times Y$ is open in the product metric on $X \times Y$.

30. Suppose that X and Y are metric spaces and that $C \subseteq X$ and $D \subseteq Y$ are closed. Show that $C \times D \subseteq X \times Y$ is closed in the product metric.

31. Suppose that A is a subspace of the metric space X. Show that $V \subseteq A$ is open in A if and only if there is an open set $U \subseteq X$ such that $V = A \cap U$.

32. Suppose that A is a subspace of the metric space X. Show that $C \subseteq A$ is closed in A if and only if there is an closed set $D \subseteq X$ such that $C = A \cap D$.

33. Suppose that $f, g : X \to \mathbb{R}$ are continuous functions. Show that $\{x \in X : f(x) = g(x)\}$ is closed in X.

34. Suppose that $f, g : X \to \mathbb{R}$ are continuous functions and $A \subseteq X$ is dense in X. Suppose that $f(a) = g(a)$ for all $a \in A$. Show that $f = g$.

35. Let X and Y be metric spaces and $A \subseteq X$, $B \subseteq Y$. Show that $\overline{A \times B} = \overline{A} \times \overline{B}$ where the first closure is taken in the product space.

36. Let X be a metric space and $A \subseteq X$. Suppose that $B \subseteq A$. Show that $\mathrm{cl}_A(B) = A \cap \mathrm{cl}_X(B)$.

37. Show that the analogous result to the last exercise for interiors is false.

Chapter 3

Cauchy Sequences and Completeness in Metric Spaces

3.1 Basic Properties

Definition 3.1 *Let X be a metric space and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. We say that $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $n, m \geq N$ we have $d(x_n, x_m) < \varepsilon$.*

Proposition 3.2 *Every convergent sequence is Cauchy.*

Proof: Suppose that $x_n \to x$ in X. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < \varepsilon/2$. Then, if $n, m \geq N$, we have $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. ♦

The converse is not always true. For example, any sequence in $\mathbb{Q}$ converging to an irrational element of $\mathbb{R}$ will be Cauchy in $\mathbb{Q}$ but not convergent in $\mathbb{Q}$.

A more interesting example is given next.

Example Let $X = C[0,1]$ with the d_1 metric,

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$$

Let f_n be a function that is 0 on the interval $[0, 1/2-1/n]$; is 1 on $[1/2+1/n, 1]$ and is linear on the interval $[1/2 - 1/n, 1/2 + 1/n]$.

It is easy to see that $d_1(f_n, f_m) = |1/m - 1/n|$, so $\{f_n\}$ is a Cauchy sequence in X.

Claim: There is no $f \in X$ with $f_n \to f$. Suppose that $f \in C[0,1]$.

Claim: If $1/2 < x \leq 1$ and $f(x) \neq 1$, then $f_n \not\to f$.

Let $\varepsilon = |f(x) - 1| > 0$. By the assumed continuity of f at x, there is a $\delta > 0$ such that whenever $|t - x| < \delta$, we have $|f(t) - f(x)| < \varepsilon/2$. For such t, we then have $|f(t) - 1| \geq \varepsilon/2$. Also, we may choose $\delta < |x - 1/2|$. But now, for any n where $1/n < \delta$, we have

$$\begin{aligned} d_1(f, f_n) &= \int_0^1 |f(t) - f_n(t)|\, dt \\ &\geq \int_{x-\delta}^{x+\delta} |1 - f_n(t)|\, dt \\ &\geq \delta \cdot \varepsilon/2 > 0 \end{aligned}$$

This is enough to show that $f_n \not\to f$.

Similarly, we have, for $x \in [0, 1/2)$, if $f(x) \neq 0$, then $f_n \not\to f$.

In other words, if we have $f_n \to f \in C[0,1]$, we must have $f(x) = 0$ for $0 \leq x < 1/2$ and $f(x) = 1$ for $1/2 < x \leq 1$. Clearly, this violates the continuity of f. ♦

Proposition 3.3 *Suppose that $\{x_n\}$ is a Cauchy sequence in the metric space X. Then $\{x_n : n \in \mathbb{N}\}$ is a bounded set.*

Proof: Let $\varepsilon = 1$ to find $N \in \mathbb{N}$ such that $n, m \geq N$ imply that $d(x_n, x_m) < 1$. let $x = x_N$ and $R = 1 + \max\{1, d(x_1, x_N), \cdots, d(x_{N-1}, x_N)\}$. Then, for all $n \geq 1$, we have $d(x_n, x) < R$. ♦

Proposition 3.4 *Suppose that $f : X \to Y$ is a uniformly continous function between metric spaces. Suppose that $\{x_n\}$ is a Cauchy sequence in X. Then $\{f(x_n)\}$ is a Cauchy sequence in Y.*

This turns out to be a very good way to show that certain functions are NOT uniformly continuous.

Example: let $X = (0,1)$ and $Y = \mathbb{R}$, both with the usual metrics. Let $f : X \to Y$ be defined by $f(x) = 1/x$. Then f is continuous but the sequence $\{1/n\}$ in X is Cauchy with $\{f(1/n) = n\}$ not Cauchy in Y. Hence f is continuous but not uniformly continous on X.

Definition 3.5 *A metric space X is said to be complete if every Cauchy sequence in X converges in X.*

We assume it to be known that $\mathbb{R}$ is complete in the usual metric.

Proposition 3.6 *Suppose that (X, d) is a complete metric space and $A \subseteq X$ is given the subspace metric, d_A. Then (A, d_A) is complete if and only if A is closed in X.*

Proof: First suppose that A is complete. We show that $\overline{A} \subseteq A$. Let $x \in \overline{A}$. Then there is a sequence $a_n \to x$ with $a_n \in A$ for all $n \in \mathbb{N}$. Since $\{a_n\}$ converges in X, it is Cauchy in X. Hence, it is Cauchy in A. Since A is complete, $\{a_n\}$ must converge in A. By uniqueness of limits, this requires that $x \in A$. Hence A is closed.

Conversely, suppose that A is closed in X and suppose that $\{a_n\}$ is a Cauchy sequence in A. Then $\{a_n\}$ is a Cauchy sequence in X. Since X is complete, there is an $x \in X$ such that $a_n \to x$. But now, since A is closed, we must have $x \in A$, so $\{a_n\}$ converges in A. Hence A is complete. ♦

Proposition 3.7 *Let $\{x_n\}$ be a Cauchy sequence in a metric space X. If some subsequence of $\{x_n\}$ converges, then $\{x_n\}$ converges.*

3.2 Advanced Properties of Complete Metric Spaces

Many important existence theorems in mathematics rely on completeness of some metric space. Often these results reduce to one of the following theorems.

Theorem 3.8 *Suppose X is a complete metric space $f : X \to X$ and $\alpha < 1$. Suppose that f satisfies $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$ (we say that f is a contractive map). Then f has a unique fixed point: there is a unique $a \in X$ such that $f(a) = a$.*

Proof: We first show uniqueness. If $f(a) = a$ and $f(a') = a'$, then $d(a, a') = d(f(a), f(a')) \leq \alpha d(a, a')$. Since $\alpha < 1$, this implies that $d(a, a') = 0$, so $a = a'$.

Let $x_1 \in X$ be any point. Define $x_2 = f(x_1), x_3 = f(x_2), \cdots$ be successive images of x_1 under f. let $D = d(x_1, x_2)$. Then $d(x_2, x_3) = d(f(x_1), f(x_2)) \leq \alpha d(x_1, x_2) = \alpha D$. Similarly, $d(x_3, x_4) \leq \alpha^2 D$. Inductively, we find $d(x_n, x_{n+1}) \leq \alpha^{n-1} D$.

Now, if $m < n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \alpha^{m-1} D + \alpha^m D + \cdots + \alpha^{n-2} D \\ &\leq \alpha^{m-1} D + \alpha^m D + \cdots \\ &= \frac{\alpha^m D}{1 - \alpha} \end{aligned}$$

Since $\alpha < 1$, $\alpha^m \to 0$, so $\{x_n\}$ is a Cauchy sequence in X. Let $x_n \to a$. Then $x_{n+1} \to a$ also, but $x_{n+1} = f(x_n) \to f(a)$. By uniqueness of limits, $a = f(a)$. ♦

As a result of the application of this result, we prove the following basic theoem concerning differential equations:

Theorem 3.9 *Suppose that $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Also assume that there is an $\alpha < 1$ such that*

$$|f(x, y_1) - f(x, y_2)| \leq \alpha |y_1 - y_2|$$

for all x, y_1, y_2.

Let $y_0 \in \mathbb{R}$. Then there is a unique solution $y : [0, 1] \to \mathbb{R}$ of the differential equation

$$y' = f(x, y)$$

such that $y(0) = y_0$.

Proof: From the Fundamental Theorem of Calculus, y is a solution as required if and only if y satisfies the equation

$$y(x) = y_0 + \int_0^x f(t, y(t))\ dt$$

for all $x \in [0,1]$.

Now, we define a map $T : C[0,1] \to C[0,1]$ as follows: if $z \in C[0,1]$, let

$$Tz(x) = y_0 + \int_0^x f(t, z(t))\ dt$$

Notice that Tz is, in fact, a continuous function by properties of integration. We claim that T is a contractive map, which will prove the result since a solution of the differential equation is simply a fixed point for T.

So, suppose that $z_1, z_2 \in C[0,1]$. Then, for $x \in [0,1]$

$$\begin{aligned} |Tz_1(x) - Tz_2(x)| &= |\int_0^x f(t, z_1(t)) - f(t, z_2(t))\ dt| \\ &\leq \int_0^x |f(t, z_1(t)) - f(t, z_2(t))|\ dt \\ &\leq \int_0^x \alpha |z_1(t) - z_2(t)|\ dt \\ &\leq \alpha |x - 0| d_\infty(z_1, z_2) \\ &\leq \alpha d_\infty(z_1, z_2) \end{aligned}$$

Since this is true for all $x \in [0,1]$, we have $d_\infty(Tz_1, Tz_2) \leq \alpha d_\infty(z_1, z_2)$. Since $\alpha < 1$, the previous result gives the required existence and uniqueness. ♦

Recall that a set A is dense in a metric space X if $\overline{A} = X$. Alternatively, $A \cap U \neq \emptyset$ whenever $\emptyset \neq U \subseteq X$ is open.

Theorem 3.10 *Suppose that X is a complete metric space and $\{U_n\}$ is a sequence of dense open sets in X. Then $\cap U_n$ is dense in X.*

Proof: Let $U \neq \emptyset$ be open in X. We need to show that $U \cap (\cap U_n) \neq \emptyset$.

Since U_1 is dense, $U \cap U_1 \neq \emptyset$. So pick $x_1 \in U \cap U_1$. Since both U and U_1 are open there is a $\varepsilon_1 > 0$ such that $\overline{B}(x_1, \varepsilon_1) \subseteq U \cap U_1$ and $\varepsilon_1 < 1$. Let $V_1 = B(x_1, \varepsilon_1)$.

We proceed via induction. Suppose that $x_1, \cdots, x_n$ and $\varepsilon_1, \cdots, \varepsilon_n$ such that $\varepsilon_k < 1/k$, and $\overline{B}(x_k, \varepsilon_k) \subseteq V_{k-1} \cap U_k$ where $V_k = B(x_k, \varepsilon_k)$.

Then $V_n \neq \emptyset$ is open and U_{n+1} is dense, so $V_n \cap U_{n+1} \neq \emptyset$. Pick $x_{n+1} \in V_n \cap U_{n+1}$ and $\varepsilon < 1/(n+1)$ such that $\overline{B}(x_{n+1}, \varepsilon_{n+1}) \subseteq V_n \cap U_{n+1}$ and let $V_{n+1} = B(x_{n+1}, \varepsilon_{n+1})$. the induction continues.

Notice that for $n \geq N$, we have $x_n \in V_n \subseteq V_N$, so $n, m \geq N$ gives $d(x_n, x_m) < 2/N$, so $\{x_n\}$ is a Cauchy sequence. Since X is complete, this sequence converges, say to x. We claim $x \in U \cap (\cap U_n)$.

In fact, for each n, we have $m \geq n$ implies that $x_m \in \overline{B}(x_n, \varepsilon_n)$, which is a closed set in X. Hence the limit $x \in B(x_n, \varepsilon_n) \subseteq U_n$. Note also, $x \in \overline{B}(x_1 \varepsilon_1) \subseteq U$. This finishes the proof. ♦

As an example, suppose that $C = \{x_n\}$ is a countable subset of $\mathbb{R}$. For each $x_n \in C$, the set $U_n = \mathbb{R} \setminus \{x_n\}$ is dense in $\mathbb{R}$. Hence, $\mathbb{R} \setminus C = \cap U_n$ is dense in $\mathbb{R}$. We extend this example as follows.

Definition 3.11 *We say that $A \subseteq X$ is nowhere dense in X if $\overline{A}^\circ = \emptyset$, in other words, the closure of A contains no non-empty open sets.*

We say that A is meager (old terminology, first category) if $A = \cup_n A_n$ is a countable union where each A_n is nowhere dense.

Theorem 3.12 *let X be a complete metric space and $U \neq \emptyset$. Then U is not meager.*

3.3 Completions of Metric Spaces

We now want to embed any given metric space into a complete metric space. Such an embedding where the image is dense turns out to be special.

Definition 3.13 *Let (X, d) be a metric space. Suppose that (Y, ρ) is a complete metric space and $\Phi : X \to Y$ is an isometry of X into Y such that $\Phi[X]$ is dense in Y. We then say that (Y, ρ, Φ) is a completion of the metric space X.*

Since $\Phi[X]$ is isometric to X, we think of X being inside of Y as a dense subset.

Example: Let $X = (0, 1)$ with the usual metric and let $Y = [0, 1]$ with the usual metric. Let $\Phi : X \to Y$ be the inclusion map. Since Y is a closed subset of the complete space $\mathbb{R}$, Y is complete. Furthermore, Φ (being a restriction map) is an isometry from X into Y and the image, X is dense in Y. Hence $[0, 1]$ is a completion of the space $(0, 1)$.

Example Suppose that $\Phi : X \to Y$ is an isometry of X into a complete metric space Y. Then $(\overline{f[X]}, \rho, f)$ is a completion of X.

Theorem 3.14 *Suppose that (Y, ρ, Φ) is a completion of the metric space (X, d). Suppose that $f : X \to Z$ is a uniformly continuous function from X into a complete metric space Z. Then there is a unique $F : Y \to Z$ such that $F \circ \Phi = f$. Furthermore, F is uniformly continuous.*

In other words, f has a unique extension to a function originating from Y.

Proof: We first show uniqueness. Suppose that $F_1, F_2 : Y \to Z$ are two continuous functions with $F_1 \circ \Phi = f = F_2 \circ \Phi$. Then $F_1(y) = F_2(y)$ for all $y \in \Phi[X]$. Since both F_1 and F_2 are continous and since $\Phi[X]$ is dense in Y, we obtain that $F_1 = F_2$.

To prove existence, we have to define $F(y)$ for all $y \in Y$. To this end, suppose that $y \in Y$. Since $\Phi[X]$ is dense in Y, there is a sequence $\Phi(x_n) \to y$. The sequence $\Phi(x_n)$ is then a Cauchy sequence. Since $\rho(\phi(x_n), \Phi(x_m)) = d(x_n, x_m)$, we see that $\{x_n\}$ is a Cauchy sequence in X. Now, since f is uniformly continous, the sequence $\{f(x_n)\}$ is Cauchy in Z. Since we assumed that Z is complete, there is a $z \in Z$ such that $f(x_n) \to z$. We wish to define $F(y) = z$.

The problem with doing so is that there are other Cauchy sequences from $\Phi[X]$ that converge to y. Do we get the same $z \in Z$ if we use a different sequence $\{x_n\}$ from X? In other words, is F well defined using the procedure above?

To this end, suppose that $\Phi(x'_n) \to y$ also. As before $\{x'_n\}$ is a Cauchy sequence in X, so $\{f(x'_n)\}$ is a Cauchy sequence in Z. Suppose that $f(x'_n) \to z'$. We need to show $z = z'$.

But, since $\Phi(x_n) \to y$ and $\Phi(x'_n) \to y$, $\rho(\Phi(x_n), \Phi(x'_n)) \to 0$. Since Φ is an isometry, this means that $d(x_n, x'_n) \to 0$. But then uniform continuity of f gives

that $d_Z(f(x_n), f(x'_n)) \to 0$. But, we kow that $d_Z(f(x_n), f(x'_n)) \to d_Z(z, z')$. Thus $d_Z(z, z') = 0$ showing that $z = z'$ and so F is well defined.

Now, suppose that $x \in X$. To determine $F \circ \Phi(x) = F(\Phi(x))$, we need to find a sequence, $\{x_n\}$, in X such that $\Phi(x_n) \to \Phi(x)$. An example of such a sequence is the constant sequence, $x_n = x$! But then $f(x_n) = f(x)$ for all n, so the corresponding $z = f(z)$. In other words, $F(\Phi(x)) = f(x)$, so $F \circ \Phi = f$.

Finally, we need to show that F is uniformly continous. To this end, let $\varepsilon > 0$ be given. Since f is assumed to be uniformly continuous, there is a $\delta > 0$ such that $x, x' \in X$ with $d(x, x') < \delta$ implies that $d_Z(f(x), f(x')) < \varepsilon/2$.

Now suppose that $y, y' \in Z$ with $\rho(y, y') < \delta$. Choose sequences $\{x_n\}$ and $\{x'_n\}$ such that $\Phi(x_n) \to y$ and $\Phi(x'_n) \to y'$. Then by our definition, $f(x_n) \to F(y)$ and $f(x'_n) \to F(y')$. Since $d(x_n, x'_n) = \rho(\Phi(x_n), \Phi(x'_n)) \to \rho(y, y') < \delta$, there is an N such that $n \geq N$ implies that $d(x_n, x'_n) < \delta$. But then $d_Z(f(x_n), f(x'_n)) < \varepsilon/2$, so $d_Z(F(y), F(y')) = \lim d_Z(f(x_n), f(x'_n)) \leq \varepsilon/2 < \varepsilon$. This completes the proof. ♦

Theorem 3.15 *Suppose that (Y, ρ, Φ) and (Y', ρ', Φ') are two completions of the metric space X. Then there is an isometry $J : Y \to Y'$ from Y onto Y' such that $J \circ \Phi = \Phi'$*

In other words, completions of X are unique up to isometry.

Now that we have shown that completions of a space X are essentially unique we turn our attention to finding such a completion. By our second exmaple, it is enough to find an isometry of X into a complete metric space. For this, it is good to have a collection of complete metric spaces.

Theorem 3.16 *Let X be a set. Recall that $\mathcal{B}(X) = \{f : X \to \mathbb{R} | \sup\{|f(x)| : x \in X\} < \infty\}$ with the metric $d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Then $(\mathcal{B}(X), d_\infty)$ is a complete metric space.*

Proof: Suppose that $\{f_n\}$ is a Cauchy sequence in $\mathcal{B}(X)$. We need to show that $\{f_n\}$ converges in $\mathcal{B}(X)$. We will show this in several steps. We first find a candidate function, f, for the limit. Then we show this candidate is actually in $\mathcal{B}(X)$. Then we show that $f_n \to f$ in $\mathcal{B}(X)$.

Step I: We need to define a function $f : X \to \mathbb{R}$.

Let $x \in X$. Then $|f_n(x) - f_m(x)| \leq d_\infty(f_n, f_m)$, showing that the sequence of real numbers $\{f_m(x)\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, this sequence converges. Define $f(x)$ by letting $f_n(x) \to f(x)$.

Step II: We show that $f \in \mathcal{B}(X)$.

Since $\{f_n\}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that whenever $n, m \geq N$, we have $d_\infty(f_n, f_m) < 1$. As before, we find $R > 0$ so that $d_\infty(f_n, f_N) < R$ for all $n \in \mathbb{N}$. Now, let $M = \sup\{|f_N(x)| : x \in X\}$. Then, for each $x \in X$ and each $n \in \mathbb{N}$, we have $|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq d_\infty(f_n, f_N) + M \leq R + M$. Taking a limit as $n \to \infty$ gives that $|f(x)| \leq R + M$. Thus, $\sup\{|f(x)| : x \in X\} \leq R + M$, showing that $f \in \mathcal{B}(X)$.

Step III: We show $d_\infty(f_n, f) \to 0$. Hence $\{f_n\}$ converges in $\mathcal{B}(X)$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that whenever $n, m \geq N$ we have $d_\infty(f_n, f_m) < \varepsilon/2$. We now show that $d_\infty(f_n, f) < \varepsilon$ whenever $n \geq N$. To do this, let $x \in X$ and $n \geq N$. Then, for every $m \geq N$, we have $|f_n(x) - f_m(x)| \leq d_\infty(f_n, f_m) < \varepsilon/2$. Let $m \to \infty$ to find that $|f_n(x) - f(x)| \leq \varepsilon/2$. This holds for all $x \in X$, so $d_\infty(f_n, f) \leq \varepsilon/2 < \varepsilon$, as desired. ♦

Theorem 3.17 *Let X be a metric space. Then there is an isometry of X into $\mathcal{B}(X)$. Hence, X has a completion.*

Proof: Fix any $x_0 \in X$. For any $y \in X$, define a function $f_y : X \to \mathbb{R}$ by setting $f_y(x) = d(y, x) - d(x_0, x)$. Then $|f_y(x)| \leq d(y, x_0)$ so $f_y \in \mathcal{B}(X)$. Define $\Phi : X \to \mathcal{B}(X)$ by setting $\Phi(y) = f_y$ for all $y \in X$.

For any $x \in X$, we have

$$|f_y(x) - f_z(x)| = |(d(y, x) - d(x_0, x)) - (d(z, x) - d(x_0, x))| = |d(y, x) - d(z, x)| \leq d(y, z)$$

which shows that $d_\infty(f_y, f_z) \leq d(y, z)$.

On the other hand, putting $x = z$ into this formula gives $|f_y(z) - f_z(z)| = d(y, z)$, so $d_\infty(f_y, f_z) \geq d(y, z)$. Hence $d_\infty(\Phi(y), \Phi(z)) = d(y, z)$ showing that Φ is an isometry of X into $\mathcal{B}(X)$. ♦

Theorem 3.18 *The space $(C([0, 1]), d_\infty)$ is a complete metric space.*

Exercises:

1. Show that a set with the discrete metric is complete.

2. Suppose that (X, d_X) and (Y, d_Y) are complete metric spaces. Show that $(X \times Y, d_{X\times Y})$ is a complete metric space.

3. Let X and Y be metric spaces and $f : X \to Y$ is uniformly continuous. Suppose that $\{x_n\}$ is a Cauchy sequence in X. Show that $\{f(x_n)\}$ is a Cauchy sequence in Y.

4. Suppose that $f : X \to Y$ is a uniformly continous function between metric spaces. Suppose that $\{x_n\}$ and $\{x'_n\}$ are two sequences in X such that $d_X(x_n, x'_n) \to 0$ in $\mathbb{R}$. Show that $d_Y(f(x_n), f(x'_n)) \to 0$.

5. Show the converse of the last exercise: Suppose that $f : X \to Y$ is a function such that whenever $\{x_n\}$ and $\{x'_n\}$ are sequences in X such that $d_X(x_n, x'_n) \to 0$, we obtain $d_Y(f(x_n, x'_n) \to 0$. Show that f is uniformly continuous.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(x^2)$. Use the last exercise to show that f is not uniformly continuous.

7. Suppose that (Y, ρ, f) is a completion of the metric space X and (Y', ρ', f') a completion of the metric space X'. Show that $(Y \times Y', \rho'', f \times f')$ is a completion of the product space $X \times X'$ where ρ'' is the product metric for ρ and ρ' and $(f \times f')(x, x') = (f(x), f'(x'))$.

8. Suppose that $f : X \times Y \to Z$ is a uniformly continuous map. Let X', Y', Z' be completions of X, Y, Z resp. Show that f can be extended uniquely to a uniformly continous map from $X' \times Y' \to Z'$.

9. Suppose that d_1 and d_2 are two metrics on a set X. Suppose also that there are constants $c, C > 0$ such that $c \cdot d_1(x, y) \leq d_2(x, y) \leq C \cdot d_1(x, y)$. Show that (X, d_1) is complete if and only if (X, d_2) is complete.

10. Consider the metric on $\mathbb{R}$ given by $d(x, y) = |\tan^{-1} x - \tan^{-1} y|$. Show that d is equivalent to the usual metric on $\mathbb{R}$ but that $(\mathbb{R}, d)$ is not complete.

Chapter 4

Topology

4.1 Basic Definitions

Definition 4.1 *A topological space is a set X together with a collection $\tau \subseteq \mathcal{P}(X)$ of subsets of X with the following properties:*

a) $\emptyset, X \in \tau$

b) If $U, V \in \tau$, then $U \cap V \in \tau$

c) If $\{U_\alpha\}_{\alpha \in I}$ is a collection of sets with $U_\alpha \in \tau$ for each $\alpha \in I$, then $\bigcup_{\alpha \in I} U_\alpha \in \tau$.

A collection τ satisfying these properties is said to be a topology on X. A subset A of X is said to be τ-open (or simply open if τ is understood) if $A \in \tau$. We also say that A is closed if $X \setminus A \in \tau$. Notice that, by definition, $\emptyset$ and X are both open and closed.

Examples:

(i) The discrete topology has $\tau = \mathcal{P}(X)$. Clearly this gives a topology on X in which every subset of X is open.

(ii) Let $\tau = \{\emptyset, X\}$. Then τ is a topology on X called the indiscrete topology.

(iii) Let $X = \{0, 1, 2\}$ and $\tau = \{\emptyset, \{0\}, X\}$. Then τ is a topology on X.

(iv) Let $X = \{0, 1, 2\}$ and $\tau = \{\emptyset, \{0, 1\}, X\}$. Then τ is a topology on X.

(v) Let $\tau = \{\emptyset\} \cup \{A \subseteq X : \text{card}\{X \setminus A\} < \infty\}$. This is called the co-finite topology on X. Notice that if X is finite, this is the same as the discrete topology.

(vi) Let (X, d) be a metric space and let τ_d be the collection of d-open sets. Then τ_d is a topology on X. We say that τ_d is the topology induced by the metric d. Just to be explicit, we have that $U \in \tau_d$ if and only if for each $x \in U$, there is an $\varepsilon > 0$ such that $B(x; \varepsilon) \subseteq U$. Whenever we talk about the toplogy of a metric space, this is the topology we are using.

(vii) We say that a topological space (X, τ) is metrizable if there is a metric d on X such that $\tau = \tau_d$.

Notice that if d_1 and d_2 are metrics on a set X, and if τ_1, τ_2 are the induced topologies, then d_1 and d_2 are equivalent metrics precidely when $\tau_1 = \tau_2$.

Definition 4.2 *If (X, τ) is a topological space and $A \subseteq X$, we say that A is closed (or τ-closed) in X if $X \setminus A$ is open.*

Notice that for metric spaces, this agrees with our previous definition.

Using DeMorgan's laws, we see have the following proposition.

Proposition 4.3 *Let (X, τ) be a topological space. Then we have*

a) $\emptyset$ and X are closed.

b) If C and D are closed, so is $C \cup D$.

c) If $\{C_\alpha\}$ is any collection of closed sets, then $\cap_\alpha C_\alpha$ is closed.

Definition 4.4 *Let X be a topological space and $A \subseteq X$. Define*

a) the interior of A, $A^\circ = \text{int}(A) = \text{int}_X(A) = \cup\{U : U \subseteq A, U \text{ open}\}$,

b) the closure of A, $\overline{A} = \text{cl}(A) = \text{cl}_X(A) = \cap\{C : A \subseteq C, C \text{ closed}\}$,

c) the boundary of A, $\partial A = \text{bdy}(A) = \text{bdy}_X(A) = \overline{A} \cap \overline{X \setminus A}$

Also, we say that the set A is dense if $\overline{A} = X$.

Proposition 4.5 *Let X be a topological space and $A \subseteq X$. Then*

a) A° is the largest open subset of A

b) $\overline{A}$ is the smallest closed subset of X containing A

c) A is open if and only if $A = A^\circ$

d) A is closed if and only if $A = \overline{A}$

e) $X \setminus A^\circ = \overline{X \setminus A}$

f) $X \setminus \overline{A} = (X \setminus A)^\circ$

g) $\partial A = \overline{A} \setminus A^\circ$.

h) A is dense if and only if $A \cap U \neq \emptyset$ for every $U \neq \emptyset$ open.

Proposition 4.6 *Let (X, τ) be a topological space and let $A \subseteq X$. Let $\tau_A = \{U \cap A : U \in \tau\} = \{V \subseteq A : \text{there exists } U \in \tau, V = U \cap A\}$. Then τ_A is a topology on A. We call this topology the subspace topology of A. We say elements of τ_A are open* in *A.*

Proposition 4.7 *Let (X, τ) be a topology and $A \subseteq X$. Then*

a) $K \subseteq A$ is closed in A if and only if $K = C \cap A$ for some C closed in X.

b) If A is open in X and V is open in A, then V is open in X.

c) If A is closed in X and K is closed in A, then K is closed in X.

d) if $B \subseteq A$, then $\mathrm{cl}_A(B) = A \cap \mathrm{cl}_X(B)$.

Proposition 4.8 *Let (X, d) be a metric space and $A \subseteq X$. Let τ_d be the metric topology on X, τ_A the subspace topology induced from τ_d, d_A the restricted metric on A, and τ'_A the metric topology on A induced by d_A. Then $\tau_A = \tau'_A$. In other words, the two topologies on subspaces of metric spaces agree.*

4.2 Continuity

Definition 4.9 *Let X and Y be two topological spaces and let $f : X \to Y$ be a function. We say that f is continuous if $f^{-1}[V]$ is open in X whenever $V \subseteq Y$ is open in Y.*

Notice that this definition agrees with our previous one if X and Y are metric spaces.

Definition 4.10 *We say that a function $f : X \to Y$ between topological spaces is a homeomorphism if f is one-to-one, onto, continuous and $f^{-1} : Y \to X$ is continuous. In this case, $U \subseteq X$ is open in X if and only if $f[U] \subseteq Y$ is open in Y.*

If an homeomorphism between X and Y exists, we say that X and Y are homeomorphic, or alternatively, topologically equivalent.

Example:

a) If X has the discrete topology, every function $f : X \to Y$ to any topological space is continuous.

b) If Y has the indiscrete topology, then every function $f : X \to Y$ from any topological space is continuous.

c) If $\mathbb{R}_d$ is the set $\mathbb{R}$ given the discrete topology and $i : \mathbb{R}_d \to \mathbb{R}$ is the identity map to $\mathbb{R}$ with the usual topology, then i is one-to-one, onto, and continuous, but is NOT a homeomorphism.

d) The spaces $\mathbb{R}$ and $(0, 1)$ with the usual topologies are homeomorphic.

e) if $A \subseteq X$ is given the subspace topology, then the inclusion map $i_A : A \to X$ is continuous. This follows since $i_A^{-1}[V] = V \cap A$.

Proposition 4.11 *Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps between topological spaces. Then $g \circ f : X \to Z$ is continuous.*

Proposition 4.12 *Homeomorphism is an equivalence relation on the class of topological spaces.*

Theorem 4.13 *Let X and Y be topological spaces and $f : X \to Y$. The following are equivalent:*

a) f is continuous

b) Whenever $C \subseteq Y$ is closed in Y, then $f^{-1}[C]$ is closed in X.

c) Whenever $A \subseteq X$, we have $f[\overline{A}] \subseteq \overline{f[A]}$.

d) Whenever $x \in X$, V open in Y such that $f(x) \in V$, there exists U open in X such that $x \in U$ and $f[U] \subseteq V$.

Notice the similarity of this last condition to the $\varepsilon - \delta$ definition of continuity.

4.3 Bases

One big problem we face is that it is quite difficult in most situations to define interesting topologies directly from the definition. For similar reasons, it can be difficult to check the definition of continuity for a function. One way of alleviating these problems is by considering a smaller collection of sets that 'generates' the topology. This leads us to the concept of a base for a topology.

Definition 4.14 *Let (X, τ) be a topological space. We say that $\mathcal{B} \subseteq \mathcal{P}(X)$ is a base for τ if $\mathcal{B} \subseteq \tau$ and whenever $U \in \tau$, there is a collection $\{B_\alpha\} \subseteq \mathcal{B}$ such that $U = \cup B_\alpha$.*

Proposition 4.15 *Let $\mathcal{B}$ be a base for the topology of X. Then $U \subseteq X$ is open if and only if for each $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

The following is how we often recognize a base for a topology:

Proposition 4.16 *let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a base for the topology of X if and only if both the following are true:*

a) Every $B \in \mathcal{B}$ is open in X

b) Whenever $x \in U$ and U is open in X, there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

It is easy to confuse the purpose of the last two results. The first one already has a base and we want to recognize open sets, while the second wants to recognize when a collection of sets forms a base for a given topology.

Example: If (X, d) is a metric space, then $\mathcal{B} = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base for the topology of X. We can find a somewhat smaller base by considering $\{B(x, 1/n) : x \in X, n > 0\}$.

Proposition 4.17 *Let $\mathcal{B}$ be a base for the topology of X. Let $A \subseteq X$. Then $A^\circ = \cup[\{B : x \in B \subseteq A, B \in \mathcal{B}\}$*

Proposition 4.18 *Let $\mathcal{B}$ be a base for the topology of X and $A \subseteq X$. For $x \in X$, the following are equivalent:*

a) $x \in \overline{A}$

b) For each $B \in \mathcal{B}$ with $x \in B$, we have $B \cap A \neq \emptyset$

Theorem 4.19 *Let X and Y be topological spaces with bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ resp. Let $f : X \to Y$ be a function. The following are equivalent:*

a) f is continuous

b) For each $B \in \mathcal{B}_Y$, $f^{-1}[B]$ is open in X

c) Suppose that $B \in \mathcal{B}_Y$ and $f(x) \in B$. Then there is a $B' \in \mathcal{B}_X$ such that $x \in B'$ and $f[B'] \subseteq B$.

In other words, we can recognize continuity of a function by looking at basic open sets.

In the previous results, we have a topology and want to either recognize a base or use a base to answer topological questions. We next turn to a result that allows us to define new topologies.

Theorem 4.20 *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$ a collection of subsets of X. Suppose that*

a) $\cup\{B : B \in \mathcal{B}\} = X$

b) Whenever $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then there is a unique topology τ on X for which $\mathcal{B}$ forms a base.

It should be noted that the first condition can always be achieved by considering $\mathcal{B} \cup \{X\}$ instead of X is required. Also, in practice it is quite common for $\mathcal{B}$ to be closed under intersections, which makes the second condition trivial (just let $B_3 = B_1 \cap B_2$).

Example: Let $X = \mathbb{R}$ and let $\mathcal{B} = \{[a, b) : a < b\}$ be the collection of half-open intervals. Then the above shows that $\mathcal{B}$ is a base for a topology on X (notice that $\mathcal{B}$ is closed under non-empty intersections). When we give $\mathbb{R}$ this topology, we call it the Sorgenfry line and designate it by $\mathcal{L}$. This topology is also known as the half-open interval topology.

Example: Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Let $\mathcal{B}$ consist of Euclidean open balls contained in $\{(x, y) : y > 0\}$ together with sets of the form $\{(x, 0)\} \cup B((x, R); R)$. Then $\mathcal{B}$ is a base for a topology of X. We call the resulting space the Moore plane. This is also known as the Niemitzki plane.

Example: Let X be a set and consider $\mathcal{B}(X)$, the collection of bounded real-valued functions on X. Previously, we gave $\mathcal{B}(X)$ a metric topology corresponding to uniform convergence of functions. We now give a different topology corresponding to pointwise convergence of functions.

Suppose $x_1, \cdots x_n \in X$, $f_0 \in \mathcal{B}(X)$ and $\varepsilon > 0$. Define

$$B(f_0; x_1, \cdots, x_n; \varepsilon) = \{f \in \mathcal{B}(X) : |f(x_j) - f_0(x_j)| < \varepsilon \text{ for all } 1 \leq j \leq n\}$$

Let $\mathcal{U}$ be the collection of all sets of this form in $\mathcal{B}(X)$.

We claim that $\mathcal{U}$ is a a base for a topology on $\mathcal{B}(X)$.

We also claim that for each $x \in X$, the evaluation function $e_x : \mathcal{B}(X) \to \mathbb{R}$ defined by $e_x(f) = f(x)$ is continuous.

Definition 4.21 *Let X be a topological space and $x \in X$. We say that $A \subseteq X$ is a neighborhood of x if $x \in A^\circ$.*

Definition 4.22 *Let X be a topological space and $x \in X$. A family of sets $\mathcal{N}$ is said to be a neighborhood base at x if every $N \in \mathcal{N}$ is a neighborhood of x and whenever U is open with $x \in U$, there is an $N \in \mathcal{N}$ such that $x \in N \subseteq U$.*

Example: Let $\mathbb{R}$ have the usual topology and let $x \in \mathbb{R}$. Then $\mathcal{N} = \{[x - \varepsilon, x + \varepsilon] : \varepsilon > 0\}$ is a neighborhood base at x. Notice that none of these neighborhoods is open in $\mathbb{R}$.

Proposition 4.23 *If X is a topological space and $\mathcal{B}$ is a collection of open sets, then $\mathcal{B}$ is a base for X if and only if for every $x \in X$, $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is a neighborhood base at x.*

4.4 Product Spaces

Proposition 4.24 *Let X and Y be topological spaces. Let $\mathcal{B} = \{U \times V : U \text{ is open in } X, V \text{ is open in } Y\}$. Then $\mathcal{B}$ is a base for a topology on $X \times Y$. We call this topology the product topology.*

Theorem 4.25 *Let X and Y be topological spaces. Give $X \times Y$ the product topology and let π_X, π_Y be the projection maps. Then π_X and π_Y are continuous. Furthermore, suppose Z is a topological space and $f : Z \to X \times Y$. Then f is continuous if and only if both $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.*

Notice that if we write $g = \pi_X \circ f$ and $h = \pi_Y \circ f$, then $f(z) = (g(z), h(z))$ for all $z \in Z$. So the continuity claim is simply that f is continuous if and only if both g and h are continuous.

Example: Let (X, d_X) and (Y, d_Y) be metric spaces. Give $X \times Y$ the metric $d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$. then the metric topology on $X \times Y$ is the same as the product of the metric topologies on X and Y.

In particular, the Euclidean topology on $\mathbb{R}^n$ is simply the product of the usual topology on $\mathbb{R}$.

Definition 4.26 *Suppose that X and Y are topological spaces and $f : X \to Y$ is a function. We say that f is an open map if $f[U]$ is open in Y for each U*

open in X. We say that f is a closed map if $f[C]$ is closed in Y whenever C is closed in X.

Proposition 4.27 *Suppose that $\mathcal{B}_X$ is a base for the topology of X. A map $f : X \to Y$ is an open map if and only if $f[B]$ is open in Y for each $B \in \mathcal{B}_X$.*

Corollary 4.28 *The projection maps π_X and π_Y on the product space $X \times Y$ are open maps.*

Example: It need not be the case that projections are closed maps. In fact, let $X = Y = \mathbb{R}$. Let $C = \{(x, y) : xy = 1\}$. Then C is closed in $X \times Y$, but $\pi_X[C] = \mathbb{R} \setminus \{0\}$ is not closed in X.

We will find cases later where projections are closed maps, though.

4.5 Quotient Spaces

Now suppose that X is a topological space and $\sim$ is an equivalence relation on X. We wish to put a topology on the collection of equivalence classes $X/\sim$.

Definition 4.29 *If $\sim$ is an equivalence relation on X and $q : X \to X/\sim$ is defined by $q(x) = [x]$, we define the quotient topology on $X/\sim$ by setting $U \subseteq X/\sim$ to be open if and only if $q^{-1}[U]$ is open in X.*

Proposition 4.30 *This gives a topology on $X/\sim$ which makes the map q a continous function. Furthermore, a function $f : X/\sim \to Y$ is continuous if and only if $f \circ q : X \to Y$ is continuous.*

Examples

a) Let $X = [0, 1]$ and define $x \sim y$ if and only if $x = y$ or $\{x, y\} = \{0, 1\}$. In other words, we identify the points 0 and 1 to a single point. We claim that $X/\sim$ is homeomorphic to the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

b) Let $X = [0, 1]^2 = [0, 1] \times [0, 1]$. Define $(x, y) \sim (w, z)$ if and only if $(x, y) = (w, z)$ or $\{x, w\} = \{0, 1\}$ and $y = z$. Then $X/\sim$ is homeomorphic to the cylinder $S^1 \times [0, 1]$.

c) Let $X = [0,1]^2 = [0,1] \times [0,1]$. Define $(x,y) \sim (w,z)$ if and only if $(x,y) = (w,z)$ or $\{x,w\} = \{0,1\}$ and $y = 1 - z$. Then $X/\sim$ is called the Möbius strip.

d) Let X be any topological space and $A \subseteq X$. Define $x \sim y$ if and only if $x = y$ or $x, y \in A$. We often write X/A instead of $X/\sim$ for this equivalence relation. We say that A is identified to a point.

e)

Exercises:

1. Suppose that $\mathcal{C} \subseteq \mathcal{P}(X)$ has the following properties:

   a) $\emptyset, X \in \mathcal{C}$

   b) If $C, D \in \mathcal{C}$, then $C \cup D \in \mathcal{C}$

   c) If $\{C_\alpha\} \subseteq \mathcal{C}$, then $\cap C_\alpha \in \mathcal{C}$.

   Show that there is a unique topology, τ on X such that $\mathcal{C}$ is the collection of τ-closed subsets of X.

2. Suppose that τ_1 and τ_2 are topologies on the set X. Show that $\tau_1 \cap \tau_2$ is a topology on X.

3. Suppose that $\{\tau_\alpha\}$ is a collection of topologies on X. Show that $\cap \tau_\alpha$ is a topology on X.

4. Give an example of two topologies τ_1 and τ_2 on a set X such that $\tau_1 \cup \tau_2$ is NOT a topology on X.

5. Give an example of $B \subseteq A \subseteq \mathbb{R}$ such that $\text{int}_A(B) \neq \text{int}_X(B) \cap A$.

6. Suppose that $\mathcal{B}$ is a base for the topology of X and $A \subseteq X$. Show that $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$ is a base for the subspace topology on A.

7. Show that open intervals are open in the Sorgenfry line, $\mathcal{L}$.

8. Show that the identity map $i : \mathcal{L} \to \mathbb{R}$ is continuous, but the identity map $i : \mathbb{R} \to \mathcal{L}$ is not continuous.

9. Let τ_1 and τ_2 be topologies on a set X. Let $\mathcal{B} = \{U \cap V : U \in \tau_1, V \in \tau_2\}$. Show that $\mathcal{B}$ is a base for a topology τ_3 on X. Show that $\tau_1 \cup \tau_2 \subseteq \tau_3$.

10. Let τ_1 and τ_2 are topologies on X. Show that the identity map $i : X \to X$ is continuous from (X, τ_1) to (X, τ_2) if and only if $\tau_2 \subseteq \tau_1$. We say that τ_2 is weaker than τ_1 in this case.

11. Suppose that (X, τ) is a topological space and $f : X \to Y$ is a function. Define $\tau_Y = \{V \subseteq Y : f^{-1}[V] \in \tau\}$. Show that (Y, τ_Y) is a topological space making f continuous.

12. Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are two collections of subsets of X satisfying the hypotheses of Theorem 4.20 and let τ_1 and τ_2 be the topologies generated by $\mathcal{B}_1$ and $\mathcal{B}_2$ resp. Suppose that whenever $x \in B \in \mathcal{B}_2$ there is a $B' \in \mathcal{B}_1$ such that $x \in B' \subseteq B$. Show that $\tau_2 \subseteq \tau_1$.

13. Suppose that $f : X \to Y$ is a function between topological spaces and that $\mathcal{B}$ is a base for the topology of X. Suppose that $f[B]$ is open in Y for each $B \in \mathcal{B}$. Show that f is an open map.

14. Suppose that $\mathcal{B}_X$ is a base for the topology of X and $\mathcal{B}_Y$ is a base for the topology of Y. Let $\mathcal{B} = \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$. Show that $\mathcal{B}$ is a base for the product topology of $X \times Y$.

15. Suppose that X and Y are topological spaces. Suppose that $A \subseteq X$ and $B \subseteq Y$. Show that the subspace topology on $A \times B$ induced from $X \times Y$ is the same as the product of the subspace topologies on A and B resp.

16. Suppose that $A \subseteq X$ is closed in X and $B \subseteq Y$ is closed in Y. Show that $A \times B$ is closed in $X \times Y$.

17. Suppose that X and Y are topological spaces, $A \subseteq X$, $B \subseteq Y$. Show that in the product space $\overline{A \times B} = \overline{A} \times \overline{B}$

18. Show that a one-to-one, onto $f : X \to Y$ is open if and only if $f^{-1} : Y \to X$ is continuous. Do the same for closed maps.

19. Let X be a topological space and let $\Delta = \{(x, x) : x \in X\}$ be the diagonal in $X \times X$. Show that X is homeomorphic to Δ via the function $f(x) = (x, x)$.

20. Let τ_1 and τ_2 be topologies on X and let τ_3 be the topology from exercise 9. Give $X \times X$ the product topology $(X, \tau_1) \times (X, \tau_2)$. Show that the subset $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ is homeomorphic to (X, τ_3).

21. Let X and Y be topological spaces and $y_0 \in Y$. Show that $X \times \{y_0\} \subseteq X \times Y$ is homeomorphic to X via the map that sends x to (x, y_0). This is often a convenient way to embed a space into a product.

22. Let $\sim$ be an equivalence relation on X. Show that $A \subseteq X/\sim$ is closed if and only if $q^{-1}[A]$ is closed in X.

23. Let X and Y be topological spaces. Define an equivalence relation on $X \times Y$ by setting $(x_1, y_1) \sim (x_2, y_2)$ if and only if $y_1 = y_2$. Show that $(X \times Y)/\sim$ is homeomorphic to X via the map that sends $[(x, y)]$ to x.

Chapter 5

Separation and Countability Properties

5.1 Countability Properties

Definition 5.1 *We say that a topological space X is*

a) Separable if there is a countable subset $D \subseteq X$ which is dense.

b) First countable if every point of X has a countable neighborhood base.

c) Second countable if X has a base with countably many elements.

Example:

a) Every metric space is first countable: $\{B(x, 1/n) : n \geq 1\}$ is a countable base of neighborhoods at x

b) A discrete space is separable if and only if it is countable.

c) The Sorgenfry line, $\mathcal{L}$ is separable and first countable. It is not second countable.

d) The Moore plane is separable and first countable.

Proposition 5.2 *If X is second countable, then X is separable and first countable.*

Proof: Suppose that $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ is a base for the topology of X.

If $x \in X$, then $\{B_n : x \in B_n\}$ is a neighborhood base at x.

Without loss of generality, we may assume that every $B_n \neq \emptyset$. Pick $x_n \in B_n$ and set $D = \{x_n : n \in \mathbb{N}\}$. The D is a countable dense subset of X. ♦

Proposition 5.3 *If X is a separable metric space, then X is second countable.*

Proof: Let $D = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of the metric space X. Define

$$\mathcal{B} = \{B(x_n; \frac{1}{k}) : n, k \in \mathbb{N}\}$$

Then $\mathcal{B}$ is a countable collection of open subsets of X. We claim that $\mathcal{B}$ is a base for the topology of X.

So, suppose that $x \in U$ and U is open in X. Then there is a $\varepsilon > 0$ such that $B(x; \varepsilon) \subseteq U$. Choose $k \in \mathbb{N}$ so that $\frac{2}{k} < \varepsilon$.

Then $B(x; \frac{1}{k})$ is a non-empty open subset of X, so the density of D guarantees an $x_n \in B(x; \frac{1}{k})$. This means that $d(x, x_n) < \frac{1}{k}$, so

$$x \in B(x_n; \frac{1}{k}) \subseteq B(x; \frac{2}{k}) \subseteq U$$

. ♦

Theorem 5.4 *Subspaces and products of first countable spaces are first countable.*

Theorem 5.5 *Subspaces and products of second countable spaces are second countbale.*

Theorem 5.6 *Open subspaces and products of separable spaces are separable.*

Example: The Moore plane is separable, but the subspace $\{(x, 0 : x \in \mathbb{R}\}$ is not. In fact, that subspace is discrete and uncountable. From the above, the Moore plane is NOT second countbale.

Example: The Sorgenfry line $\mathcal{L}$ is separable and first countable, so the product $\mathcal{L} \times \mathcal{L}$ is also separable and first countbale. The subspace $A = \{(x, -x) : x \in \mathcal{L}\}$ is uncountable and discrete so it is not second countable. Hence, $\mathcal{L} \times \mathcal{L}$ is not second countbale. But this shows again that $\mathcal{L}$ is not second countable.

In particular, neither the Moore plane nor the Sorgenfry line are metrizable.

5.2 Convergence of Sequences

Definition 5.7 *Let X be a topological space, $\{x_n\}$ a sequence in X and $x \in X$. We say that x_n converges to x and write $x_n \to x$ if for every open set U with $x \in U$, there is a N such that $n \geq N$ implies $x_n \in U$.*

We notice that for metric spaces this definition is equivalent to the $\varepsilon - N$ definition.

Example: Let X be any set with the indiscrete topology. If $\{x_n\}$ is a sequence and $x \in X$, then $x_n \to x$. In other words, every sequence converges to every point of X! This shows dramatically that limits of sequences do not have to be unique in general topological spaces.

Proposition 5.8 *Let $\mathcal{B}$ be a base for the topology on X, $\{x_n\}$ a sequence in X and $x \in X$. The following are equivalent:*

a) $x_n \to x$

b) For each $B \in \mathcal{B}$ with $x \in B$, there is an N such that $n \geq N$ implies $x_n \in B$.

Next, we have the result that continuous functions preserve convergence of sequences:

Proposition 5.9 *Let X be a topological space $A \subseteq X$ and $\{a_n\}$ a sequence in A such that $a_n \to x \in X$. Then $x \in \overline{A}$.*

Proof: In fact, if U is any open set with $x \in U$, there is a N such that $a_N \in A \cap U$, so $A \cap U \neq \emptyset$. ♦

In the case of metric spaces, we know this has a converse: if $x \in \overline{A}$, then there is a sequence $\{a_n\}$ from A with $a_n \to x$. We show this is false for general topological spaces.

Proposition 5.10 *Let X be a set and given $\mathcal{B}(X)$ the topology of pointwise convergence 4.3. Then $f_n \to f$ if and only if $f_n(x) \to f(x)$ for all $x \in X$ (justifying the name of the topology).*

Proof: Suppose that $f(x) \in V$ with V open in Y. Then $x \in f^{-1}[V]$, which is open in X. So, there is a N such that $n \geq N$ implies that $x_n \in f^{-1}[V]$. But then $n \geq N$ implies that $f(x_n) \in V$. Hence $f(x_n) \to f(x)$. ♦

Example: Consider $\mathcal{B}([0,1])$ with the topology of pointwise convergence. Let A be the set of all functions f where $f(x) \in \{0,1\}$ for all $x \in [0,1]$ and $\{x : f(x) = 1\}$ is finite. So elements of A are functions which only take the values 0 or 1 and only take the value 1 at finitely many points. Let 1 be the constant function taking the value 1.

Claim: $1 \in \overline{A}$

Claim: There is no sequence f_n in A with $f_n \to 1$.

In particular, $\mathcal{B}([0,1])$ with the topology of pointwise convergence is NOT metrizable.

The situation changes for first countable spaces. For them, the closure operation is determined by convergence of sequences.

Theorem 5.11 *Let X be a first countable space and $A \subseteq X$. For $x \in X$, the following are equivalent:*

a) $x \in \overline{A}$

b) There is a sequence $\{a_n\}$ from A with $a_n \to x$

In particular, the space $\mathcal{B}([0,1])$ with the topology of pointwise convergence is *not* first countable. Hence, it is not metrizable.

Proposition 5.12 *Let $f : X \to Y$ be a continuous function between topological spaces. Let $x_n \to x$ in X. Then $f(x_n) \to f(x)$ in Y.*

Once again, the converse is false for general topological spaces. But for first countable spaces, we have

Theorem 5.13 *Let X be a first countable space and Y a topological space. For a function $f : X \to Y$, the following are equivalent:*

a) f is continuous

b) Whenever $\{x_n\}$ is a sequence in X with $x_n \to x \in X$, then $f(x_n) \to f(x)$ in Y.

For spaces that are not first countable, sequences are no longer sufficient to establish continuity of functions or whether a subset is closed. There is a more general notion of *nets* that can allow these determinations for general topological spaces. We will not cover nets in this course.

It is natural to ask whether there is a connection between first countability of a space and separability. Since an uncountable discrete space is first countable, but not separable, one implication is clearly false. The other direction follows from the following rather surprising example.

Theorem 5.14 *The space $\mathcal{B}([0,1])$ with the topology of pointwise convergence is separable.*

5.3 The Separation Axioms

Definition 5.15 *Let X be a topological space. We say that X is*

a) T_1 if whenever $x \neq y$ in X, there is an open set U with $x \in U$ and $y \notin U$.

b) T_2 if whenever $x \neq y$ in X, there are open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$

c) regular if whenever C is closed and $x \notin C$, there are open sets U, V such that $x \in U$, $C \subseteq V$ and $U \cap V = \emptyset$.

d) normal if whenever C, D are closed sets with $C \cap D = \emptyset$, then there are open U, V such that $C \subseteq U$, $D \subseteq V$ and $U \cap V = \emptyset$.

e) T_3 separation axiom if X is both T_1 and regular.

f) T_4 is X is both T_1 and normal.

Spaces that T_2 are also said to be Hausdorff spaces.

Proposition 5.16 *For a topological space X, the following are equivalent:*

a) X is T_1

b) Sets of the form $\{x\}$ with $x \in X$ (i.e, singletons) are closed in X.

For this reason, it is very uncommon to work with topological spaces that do not satisfy at least the T_1 separation axiom. In fact, it is typical to assume at least the T_2 axiom.

This also shows that $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$.

Theorem 5.17 *Metric spaces are T_4.*

Example: The Sorgenfry line, $\mathcal{L}$ is T_4.

Proposition 5.18 *Let X be a topological space and let $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ be the diagonal. The following are equivalent:*

a) X satisfies the T_2 separation axiom

b) Δ is closed in $X \times X$.

Proposition 5.19 *Let $f : X \to Y$ be a continuous function where Y is a T_2 space. Then*

$$f = \{(x, f(x)) : x \in X\}$$

is closed in $X \times Y$.

Proposition 5.20 *Let X be a T_2 space, $\{x_n\}$ a sequence in X and suppose that $x_n \to x$ and $x_n \to y$. Then $x = y$.*

Theorem 5.21 *Products and subspaces of T_1 spaces are T_1.*

Theorem 5.22 *Products and subspaces of T_2 spaces are T_2.*

Theorem 5.23 *The following are equivalent for a T_1 topological space X:*

a) X is T_3.

b) Whenever $x \in U$ with U open, there is a closed set C such that $x \in C^\circ \subseteq C \subseteq U$.

c) Whenever $x \in U$ with U open, there is an open set V such that $x \in V \subseteq \overline{V} \subseteq U$.

d) For every $x \in X$, the closed neighborhoods of x form a neighborhood base at x.

Theorem 5.24 *Products and subspaces of T_3 spaces are T_3.*

Theorem 5.25 *The following are equivalent for a T_1 topological space:*

a) X is T_4.

b) Whenever C is closed, U is open and $C \subseteq U$, there is a V open with $C \subseteq V \subseteq \overline{V} \subseteq U$.

Of the separation properties we discuss here, the T_4 property is the most important. This is because of the next result which guarantees the existence of many real-valued bounded continuous functions.

Theorem 5.26 (Urysohn's Lemma) *Let X be a T_4 space and C, D disjoint closed subsets of X. Then there is a continuous $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in D$.*

Notice that any other closed interval in $\mathbb{R}$ can be used in place of $[0, 1]$.

Another hugely important result for T_4 spaces is the following extension theorem.

Theorem 5.27 (Tietze's Extension Theorem) *Let X be a T_4 space, $C \subseteq X$ a closed subspace and $f : C \to [0, 1]$ a continuous function. Then there is a continuous $F : X \to [0, 1]$ such that $F|_C = f$.*

Theorem 5.28 *Suppose that X is a separable space and $A \subseteq X$ is a closed, discrete subspace that has the same cardinality as $\mathbb{R}$. Then X is not T_4.*

Example: The space $\mathcal{L} \times \mathcal{L}$ has $A = \{(x, -x) : x \in \mathcal{L}\}$ as a closed discrete subspace. Hence, $\mathcal{L} \times \mathcal{L}$ is not T_4. Since $\mathcal{L}$ is T_4, we find that products of T_4 spaces need not be T_4.

It is also possible for a subspace of a T_4 space to fail to be T_4, but it is much more difficult to give an example.

Exercises:

1. Show that subspaces of separable metric spaces are separable. Give an example of a non-separable subspace of a separable space.

2. Show that open subspaces of separable spaces are separable.

3. Suppose that X is separable and $f : X \to Y$ is continuous and onto. Show that Y is separable.

4. Suppose that X is first countable and $x \in X$. Show that there is a neighborhood base $\{U_n : n \in \mathbb{N}\}$ such that $U_{n+1} \subseteq U_n$ for each $n \geq 1$.

5. Show that products of second countable spaces are second countable.

6. Show that subspaces of first countable spaces are first countable.

7. Show that a sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in a product space $X \times Y$ converges to (x, y) if and only if $x_n \to x$ and $y_n \to y$.

8. Suppose that X is second countable and $f : X \to Y$ is both continuous, open, and onto. Show that Y is second countable.

9. Show that products of T_2 spaces are T_2.

10. Suppose that $f, g : X \to Y$ are two continuous functions and Y is T_2. Show that $\{x \in X : f(x) = g(x)\}$ is closed in X.

11. Show that products of T_3 spaces are T_3.

12. Suppose that X is a first countable T_3 space. Show that every point of X has a countable neighborhood base consisting of closed sets.

13. Suppose that X is T_3 and $A \subseteq X$ is closed. Show that X/A is T_2.

14. Suppose that X is T_4, $A, B \subseteq X$ are disjoint closed sets. Define $x \sim y$ if and only if $x = y$ or $x, y \in A$ or $x, y \in B$. Show that $X/\sim$ is T_2.

15. Suppose that X is T_3 and $A \subseteq X$ is an infinite subset. Shot that there are open sets $\{U_n : n \in \mathbb{N}\}$ such that $\overline{U_n} \cap \overline{U_m} = \emptyset$ if $n \neq m$ and $U_n \cap A \neq \emptyset$ for all $n \geq 1$.

16. Suppose that X is T_4 and C, D, E are three mutually disjoint subsets of X. Let $c, d, e \in \mathbb{R}$. Show that there is a continuous $f : X \to \mathbb{R}$ such that $f = c$ on C, $f = d$ and D and $f = e$ on E.

17. Suppose that X is T_4, C is closed in X and $f : C \to [0, 1]^n$ is a continuous function ($n \geq 1$). Show that there is an $F : X \to [0, 1]^n$ such that $F(x) = f(x)$ for all $x \in C$.

18. Suppose that X is T_4 and $C \subseteq X$ is closed. Define $\Phi : \mathcal{B}(X) \to \mathcal{B}(C)$ by setting $\Phi(f) = f|_C$. Show that Φ is onto.

19. We say a subset A of a topological space X is a retract of X if there is a continuous $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Suppose that X is T_4 and $A \subseteq X$ is closed and homeomorphic to $[0, 1]$. Show that A is a retract of X.

Chapter 6

Covering Properties

Compactness

6.1 Basis Properties

Definition 6.1 *Let X be a topological space. An open cover of X is a collection $\{U_\alpha\}_{\alpha\in I}$ of open sets such that $X = \cup_{\alpha\in I} U_\alpha$. A subcover of $\{U_\alpha\}_{\alpha\in I}$ is a collection $\{U_\alpha\}_{\alpha\in J}$ where $J \subseteq I$ that is still a cover. A finite subcover is a subcover where the set J is a finite subset of I. Similarly, a countable subcover is a subcover where the set J is a countable subset of I.*

Examples

a) For $X = \mathbb{R}$, let $U_n = (-n, n)$. Then $\{U_n\}_{n\geq 1}$ is an open cover of X. For any N, we have a subcover $\{U_n\}_{n\geq N}$. This cover does not have a finite subcover.

b) If X is any metric space, and $x_0 \in X$, let $U_n = B(x_0; n)$. Then $\{U_n\}_{n\geq 1}$ is an open cover of X. This cover has a finite subcover precisely when X is bounded.

c) For $X = \mathbb{R}$, let $U_n = (n-1, n+1)$ for $n \in \mathbb{Z}$. Then $\{U_n\}_{n\in\mathbb{Z}}$ is an open cover of X. The only subcover of this cover is itself.

Definition 6.2 *We say that a topological space is Lindelöf if every open cover of X has a countable subcover.*

We say that a topological space X is compact if every open cover of X has a finite subcover.

Notice that infinite discrete spaces are not compact and uncountable discrete spaces are not Lindelöf.

Theorem 6.3 *The closed interval $[0, 1]$ is compact.*

Definition 6.4 *We say that a collection of sets $\{A_i\}_{i\in I}$ has the finite intersection property if for every finite $F \subseteq I$, we have $\cap_{i\in F} A_i \neq \emptyset$. We can similarly define the countable intersection property.*

Theorem 6.5 *A space X is compact if and only if every collection of closed sets, $\{C_i\}_{i\in I}$ with the finite intersection property has $\cap_{i\in I} C_i \neq \emptyset$*

The following result shows that we need only consider covers where the open sets come from a base for the topology of X.

Theorem 6.6 *Let $\mathcal{B}$ be a base for the topology of X. Then*

a) X is Lindelöf if and only if every cover by elements of $\mathcal{B}$ has a countable subcover.

b) X is compact if and only if every cover by elements of $\mathcal{B}$ has a finite subcover.

In particular, the following is now trivial.

Theorem 6.7 *Every second countable space is Lindelöf.*

Corollary 6.8 *Every separable metric space is Lindelöf.*

Example: The Sorgenfry Line, $\mathcal{L}$ is Lindelöf.

Theorem 6.9 *Closed subspaces of Lindelöf spaces are Lindelöf.*

Closed subspaces of compact spaces are compact.

For example, the subspace $A = \{(x, -x) : x \in \mathcal{L}\}$ is uncountable, discrete, and closed in $\mathcal{L} \times \mathcal{L}$. Hence $\mathcal{L} \times \mathcal{L}$ is NOT Lindelöf. So products of Lindelöf spaces need not be Lindelöf.

Theorem 6.10 *A compact subspace of a T_2 space is closed.*

Theorem 6.11 *A compact T_2 space is T_4.*

Because of this, Urysohn's Lemma shows that there are many continuous functions $f : X \to \mathbb{R}$ when X is compact T_2. Partly for this reason, much of abstract analysis is done in compact Hausdorff spaces or spaces that are closely related.

Theorem 6.12 *Suppose that $f : X \to Y$ is a continuous onto function.*

a) If X is Lindelöf, then Y is also.

b) If X is compact, Y is also.

Theorem 6.13 *Products of compact spaces are compact.*

Theorem 6.14 *Let X be a metric space and $A \subseteq X$ a compact subset. Then A is closed and bounded.*

The converse of this theorem is *FALSE* in general metric spaces. It is, however, true for subsets of $\mathbb{R}^n$.

Theorem 6.15 *A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example: Consider the space $C[0, 1]$ with the d_∞ metric (the metric of uniform convergence). Let $A = \overline{B}(0; 1)$ the closed unit ball around the origin. The A is clearly closed and bounded. We claim A is not compact.

For $n \geq 1$, let $f_n(x) = 0$ for $0 \leq x \leq \frac{1}{2^{n+1}}$ and $\frac{1}{2^n} \leq x \leq 1$, let $f_n(\frac{3}{2^{n+2}}) = 1$ and let f_n be linear on the two other intervals. Then the graph of f_n looks like a witches hat.

Now, $d_\infty(f_n, f_m) = 1$ for $n \neq m$, so $B = \{f_n : n \geq 1\} \subseteq A$ is closed, infinite, and discrete. Hence B is NOT compact, so A cannot be.

Theorem 6.16 *A Lindelöf T_3 space is T_4*

Theorem 6.17 (Urysohn's Metrization Theorem) *For a topological space X, the following are equivalent:*

a) X is separable and metrizable

b) X is second countable and T_3.

6.2 Variants of Compactness

We start with a few definitions.

Definition 6.18 *Let X be a topological space, $A \subseteq X$ and $x \in X$. We say that x is an accumulation point of A if whenever U is open with $x \in U$, then $U \cap A$ is an infinite set.*

In particular, if x is an accumulation point of A, then $x \in \overline{A}$. The converse need not hold (any isolated $x \in A$).

Definition 6.19 *Let $\{x_n\}$ be a sequence in a space X and $x \in X$. We say that x is a cluster point of the sequence if for each open set U with $x \in U$ and $N \in \mathbb{N}$, there is an $n \geq N$ with $x_n \in U$.*

Proposition 6.20 *For a sequence $\{x_n\}$ and a point $x \in X$, the following are equivalent:*

a) x is a cluster point of $\{x_n\}$

b) $x \in \cap_{N \geq 1} \overline{\{x_n : n \geq N\}}$

Definition 6.21 *We say that a topological space X*

a) is countably compact if every countable cover has a finite subcover.

b) is sequentially compact if every sequence in X has a convergent subsequence.

We note that countable compactness is equivalent to every *countable* collection of closed sets with the finite intersection property has non-empty intersection.

The connection between countable compactness and ordinary compactness is quite clear.

Corollary 6.22 *A space is compact if and only if it is both Lindelöf and countable compact.*

Theorem 6.23 *The following are equivalent:*

a) X is countably compact

b) Every sequence in X has a cluster point.

c) Every infinite subset of X has an accumulation point.

Proposition 6.24 *Every sequentially compact space is countably compact.*

Proposition 6.25 *Every first countable countably compact space is sequentially compact.*

6.3 Compact Metric Spaces

Definition 6.26 *We say that a metric space X is totally bounded if for each $\varepsilon > 0$, there are finitely many points $x_1, \cdots, x_n \in X$ such that $X = \cup_i B(x_i; \varepsilon)$.*

For $A \subseteq X$, we use the subspace metric on A to determine whether A is totally bounded. But since $B_A(a; \varepsilon) = A \cap B_X(a; \varepsilon)$, we find that A is totally bounded if and only if for each $\varepsilon > 0$, there are points $a_1, \cdots, a_n \in A$ such that $A \subseteq \cup_i B_X(a_i; \varepsilon)$.

Theorem 6.27 *Suppose that A is a totally bounded subset of the metric space X. Then $\overline{A}$ is totally bounded. Also, if $B \subseteq A$, then B is totally bounded.*

Proposition 6.28 *Let X be a totally bounded metric space. Then X is second countable.*

Proposition 6.29 *Let X be a metric space. Then X is totally bounded if and only if every sequence in X has a Cauchy subsequence.*

Theorem 6.30 *let X be a metric space. The following are equivalent:*

a) X is compact

b) X is countably compact

c) X is sequentially compact

d) Every sequence in X has a cluster point

e) Every infinite subset of X has an accumulation point

f) X is complete and totally bounded

Proof: Since compact spaces are countably compact, the first condition implies the next four, which are equivalent in the context of first countable spaces (hence in metric spaces).

So, now, assume that a metric space is sequentially compact. Then every sequence has a Cauchy subsequence, so X is totally bounded. Also, every Cauchy sequence has a convergent subsequence, so the entire Cauchy sequence converges. Hence X is complete.

Finally, suppose that X is complete and totally bounded. Then X is second countable, so it is Lindelöf. Also, every sequence has a Cauchy subsequence, which thereby converges by completeness. Hence, X is sequentially compact, so is countably compact. But Lindelöf and countably compact together give compactness. ♦

Finally, we consider compactness is the space $C[0,1]$.

Definition 6.31 *We say a collection of functions $\mathcal{F} \subseteq C[0,1]$ is equi-continuous if for each $x_0 \in X$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - x_0| < \varepsilon$ implies that $|f(x) - f(x_0)| < \varepsilon$ for every $f \in \mathcal{F}$.*

We say that $\mathcal{F}$ is pointwise bounded if for each $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is bounded in $\mathbb{R}$.

Theorem 6.32 (Arzela-Ascoli Theorem) *For a subset $\mathcal{F} \subseteq C[0,1]$, the following are equivalent:*

a) $\mathcal{F}$ is totally bounded in the d_∞ metric

b) $\mathcal{F}$ is equicontinuous and pointwise bounded.

Exercises:

1. Suppose that $K_1, \cdots, K_n$ are compact subsets of a space X. Show that $\cup_1^n K_i$ is compact. What is a similar statement for Lindelöf subsets?

2. let X be a compact space, $\{C_i\}_{i\in I}$ a collection of closed sets and U an open set such that $\cap_{i\in I} C_i \subseteq U$. Show that there is a finite $F \subseteq I$ such that $\cap_{i\in F} C_i \subseteq U$.

3. Suppose that X is compact, Y is T_2 and $f : X \to Y$ is continuous, one-to-one and onto. Show that f is a homeomorphism.

4. Let X and Y be topological spaces, $x \in X$, $K \subseteq Y$ a compact subset. Suppose that $W \subseteq X \times Y$ is an open set with $\{x\} \times K \subseteq W$. Show there are open sets $x \in U \subseteq X$ and $K \subseteq V \subseteq Y$ such that $U \times V \subseteq W$.

5. Let X and Y be topological spaces with $K \subseteq X$ and $L \subseteq Y$ compact subsets. Suppose that $W \subseteq X \times Y$ is open with $K \times L \subseteq W$. Show that there are open sets $K \subseteq U \subseteq X$ and $L \subseteq V \subseteq Y$ such that $U \times V \subseteq W$.

6. Suppose that X is a topological space and Y is compact. Show that the projection $\pi_X : X \times Y \to X$ is a closed map.

7. Show directly that a product of totally bounded metric spaces is totally bounded.

8. Show that if $A \subseteq X$ is totally bounded, then $\overline{A}$ is also totally bounded.

9. Suppose that X is a complete metric space and $A \subseteq X$. Show that A is totally bounded if and only if $\overline{A}$ is compact. For this reason, total boundedness is often called pre-compactness.

10. Let $\mathcal{F} = \{f \in C[0,1] :$f is differentiable and $|f(x)|, |f'(x)| \leq 1, x \in [0,1]\}$. Show that $\mathcal{F}$ is totally bounded in $C[0,1]$.

11. Show that a compact T_2 space is metrizable if and only if it is second countable.

12. We say a map $f : X \to Y$ between topological spaces is perfect if $f^{-1}[\{y\}]$ is compact for each $y \in Y$. Suppose that $f : X \to Y$ is a closed, perfect, onto continuous function and X is second countable. Show that Y is second countable.

13. Use the last result to show that if X is a compact metric space, Y is T_2 and $f : X \to Y$ is continuous and onto, then Y is metrizable.

14. Suppose that X is T_3, $K \subseteq U$ with K compact and U open. Show that there is a V open with $K \subseteq V \subseteq \overline{V} \subseteq U$.

15. Suppose that X is a metric space and $K \subseteq X$ is compact, and $C \subseteq X$ is closed with $C \cap K = \emptyset$. Show that $\inf\{d(x,y) : x \in K, y \in C\} > 0$.

Chapter 7

Connectedness

7.1 Basics

Definition 7.1 *We say that a topological space X is disconnected if there is a subset $\emptyset \neq A \subsetneq X$ which is both open and closed.*

We say that X is connected if it is not disconnected. In other words, X is connected if the only subsets that are both open and closed in X are $\emptyset$ and X.

Proposition 7.2 *For a topological space X, the following are equivalent:*

a) X is disconnected

b) There exist disjoint non-empty open sets U and V such that $X = U \cup V$

c) There exist disjoint non-empty closed sets C and D such that $X = C \cup D$

d) There is a non-constant continuous function $f : X \to \{0, 1\}$, to the two point discrete space.

Proposition 7.3 *For a subspace $A \subseteq X$ of a topological space, the following are equivalent:*

a) A is connected

b) Whenever U and V are open in X with $A \subseteq U \cup V$ and $A \cap U \cap V = \emptyset$, then either $A \subseteq U$ or $A \subseteq V$.

c) Whenever C and D are closed in X with $A \subseteq C \cup D$ and $A \cap C \cap D = \emptyset$, then either $A \subseteq C$ or $A \subseteq D$.

Theorem 7.4 *Suppose that X is connected and $f : X \to Y$ is a continuous map. Then $f[X]$ is connected.*

Theorem 7.5 (0,1) *is connected.*

Theorem 7.6 *Suppose that $\{A_\alpha\}$ is a collection of connected subsets of X and $\cap_\alpha A_\alpha \neq \emptyset$. Then $\cup_\alpha A_\alpha$ is connected.*

Theorem 7.7 *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Theorem 7.8 (Intermediate Value Theorem) *Suppose that X is a connected space, $f : X \to \mathbb{R}$ is continuous, $x, y \in X$ and $c \in \mathbb{R}$ such that $f(x) < c < f(y)$. Then there is a $z \in X$ such that $f(z) = c$.*

Theorem 7.9 *Suppose that $A \subseteq X$ is connected and $A \subseteq B \subseteq \overline{A}$. Then B is connected.*

Theorem 7.10 *Products of connected spaces are connected.*

Theorem 7.11 *Let X be a topological space. Define a relation on X by setting $x \simeq y$ if and only if there is a connected subset $A \subseteq X$ with $x, y \in X$. Then $\simeq$ is an equivalence relation on X and the equivalence classes are maximal connected subsets of X. Furthermore, each equivalence class is closed in X.*

Definition 7.12 *The equivalence classes of the above relation are called the components of X. In particular, X is a disjoint union of its components and every component is a closed maximally connected subset of X.*

Notice that a space is connected if and only if it has exactly one component. It need not be the case that components of X are open.

Theorem 7.13 *Let X be a compact T_2 space and $x \in X$. Then the component of x is the intersection of all subsets containing x that are both open and closed.*

Proof: Let C be the component of x and suppose that $x \in U$ and U is both open and closed. Then $U \cap C$ is both open and closed in C and is non-empty (since $x \in U \cap C$), so $U \cap C = C$, i.e, $C \subseteq U$. Hence, if we set K to the the intersection of all open and closed subsets of X containing x, then $C \subseteq K$.

Next, we show that K is connected, which would show $K \subseteq C$. Note that K is closed in X (it need not be open!).

Let $\mathcal{A} = \{U : x \in U, U \text{ open and closed}\}$, so $K = \cap\{U : U \in \mathcal{A}\}$.

Suppose that $K = K_1 \cup K_2$ where K_1, K_2 are closed in X and $K_1 \cap K_2 = \emptyset$. Since X is T_4, there are disjoint open sets V_1, V_2 with $K_1 \subseteq V_1$ and $K_2 \subseteq V_2$.

Then $\cap\{U : U \in \mathcal{A}\} = K \subseteq V_1 \cup V_2$. The first is an intersetion of closed sets, so there is a finite collection $U_1, \cdots, U_n \in \mathcal{A}$ such that

$$K \subseteq U = U_1 \cap \cdots \cap U_n \subseteq V_1 \cup V_2$$

Now, suppose that $x \in V_1$. Then $x \in U \cap V_1$ and $U \cap V_1$ is open in X. But $U \cap V_1 = U \setminus V_2$, so $U \cap V_1$ is also closed in X. But this means that $K \subseteq U \cap V_1$, so $K_2 \subseteq K \setminus U_1 = \emptyset$.

This shows K is connected, finishing the proof. ♦

This result can fail for non-compact spaces. In that case, the set K in the proof is called the quasi-component of x.

Definition 7.14 *Let X be a topological space and $x, y \in X$. A path from x to y is a continuous $p : [0,1] \to X$ such that $x = p(0)$ and $y = p(1)$.*

We say that X is path connected if there is a path between any pair of points of X.

Proposition 7.15 *Every path connected space is connected.*

The converse fails.

Example: Define the Topologist's Sine Curve

$$X = (\{0\} \times [-1,1]) \cup \{(x, \sin(1/x)) : 0 < x < 1\}$$

Then X is connected but not path connected.

Theorem 7.16 *Let X be a topological space and define a relation by setting $x \simeq y$ if there is a path in X from x to y. Then $\simeq$ is an equivalence relation on X. The equivalence classes are called path components of X.*

7.2 Local Connectivity

Definition 7.17 *We say a space X is locally connected if every point has a neighborhood base consisting of connected sets.*

Theorem 7.18 *A space X is locally connected if and only if every component of every open set is open.*

Theorem 7.19 *Every quotient of a locally connected space is locally connected.*

Proof: Let $q : X \to X/\sim = Y$ be a quotient map and $U \subseteq Y$ be open. Let A be a component of U. We need to show A is open in Y. We do this by showing $q^{-1}[A]$ is open. Let $x \in q^{-1}[A]$.

Then, $x \in q^{-1}[U]$ and $q^{-1}[U]$ is open in X. Let C be the component of $q^{-1}[U]$ containing x. Then C is connected and open in X. Hence $[x] \in q[C]$ is connected, so $q[C] \subseteq A$. But this means that $x \in C \subseteq q^{-1}[A]$. Since C is open in X, this shows that $q^{-1}[A]$ is open in X, as required. ♦

Example: [Comb Space] Let

$$X = ((\{0\} \cup \{1/n : n \geq 1\}) \times [0,1]) \cup ([0,1] \times \{0\})$$

. Then X is connected, but not locally connected.

Definition 7.20 *We say a space X is locally path connected if every point has a neighborhood base consisting of path connected sets.*

Theorem 7.21 *Every connected, locally path connected space is path connected.*

Exercises:

1. Suppose that $\{A_n\}_1^\infty$ is a collection of connected subsets of a space X. Suppose also that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \geq 1$. Show that $\cup_1^\infty A_n$ is connected.

2. Suppose that $f : [0,1] \to [0,1]$ are continuous. Show that f has a fixed point.

3. Let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and suppose that $f : S^1 \to \mathbb{R}$ is continuous. Show that there is an $(x,y) \in S^1$ such that $f(x,y) = f(-x,-y)$.

4. Show that the product of path connected spaces is path connected.

5. Show that the continuous image of a path connected space is path connected.

6. Show that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected.

7. Show that a connected T_4 space with at least two points is uncountable.

8. Suppose that X is a connected, locally connected space and $C \subseteq X$ is a closed, connected subset. Let U is a component of $X \setminus C$. Show that $X \setminus U$ is connected.

9. Suppose that $C_{n+1} \subseteq C_n$ are compact, connected subsets of a space X. Show that $\cap_n C_n$ is connected.

10. Suppose that X is a space with finitely many components. Show that each component is open.