

Anti-Derivatives

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1. ANTI-DERIVATIVES

We say that $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

Notice that the letters are case-sensitive.

Essentially, you worked out the derivative of a function $F(x)$, got the answer $f(x)$.

Can you figure out what was the original function just from knowing its derivative?

The answer is “sometimes”

Suppose you know

$$F'(x) = 3x^2 - 14x + 11$$

What's $F(x)$?

$$F(x) = x^3 - 7x^2 + 11x \quad \text{works}$$

$$\text{But so does } F(x) = x^3 - 7x^2 + 11x + 17$$

2. THE TROUBLE WITH CONSTANTS

Since the derivative of any constant is zero, you cannot know the value of that constant just from the derivative.

The following fact shows that, except for an undetermined constant, anti-derivatives are essentially unique:

If two functions $f(x)$ and $g(x)$ have the same derivative, then $f(x) = g(x) + C$, for some constant C .

When asked to find the antiderivative of

$$F'(x) = 3x^2 - 14x + 11$$

the most general answer is to write

$$F(x) = x^3 - 7x^2 + 11x + C$$

where C is a constant.

3. FINDING THE CONSTANT

If we know one data point $(x_0, F(x_0))$ on the curve $y = F(x)$ we can determine the value of the constant.

For example, the anti-derivative of $F'(x) = 3x^2 - 14x + 11$

$$\text{is } y = F(x) = x^3 - 7x^2 + 11x + C,$$

where C is a constant.

Suppose we know $F(1) = 9$.

Then plugging in $x = 1$, $y = 9$ gives the equation

$$9 = F(1) = 1^3 - 7 \cdot 1^2 + 11 \cdot 1x + C = 5 + C$$

Thus, $C = 9 - 5 = 4$ and $F(x) = x^3 - 7x^2 + 11x + 4$.

4. INTEGRAL NOTATION

Mathematicians write the symbol

$$\int f(x) dx$$

to mean the anti-derivative of $f(x)$

The symbol $\int$ is called the “integral sign” and the anti-derivative of $f(x)$ is also called the **integral** of $f(x)$.

Thus,

$$\int 3x^2 - 14x + 11 dx = x^3 - 7x^2 + 11x + C$$

5. THE POWER RULE

For any value of n except -1 ,

$$\boxed{\int x^n dx = \frac{1}{n+1} x^{n+1} + C}$$

You can easily check this formula by differentiation:

$$\begin{aligned} & \frac{d}{dx} \frac{1}{n+1} x^{n+1} + C \\ &= \frac{1}{n+1} (n+1) x^{(n+1)-1} \\ &= x^n \end{aligned}$$

More good news: this formula works for rational numbers, negative numbers, and real numbers, as well as integers.

6. EXAMPLES

$$\int x^{10} dx = \frac{1}{11} x^{11} + C$$

$$\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{1}{-2}x^{-2} + C = -\frac{1}{2}x^{-2} + C$$

$$\int \sqrt{x} dx = \int x^{1/2} dx = \frac{1}{3/2}x^{3/2} + C = \frac{2}{3}x^{3/2} + C$$

7. TWO RULES

We can integrate sums such as $x^3 + x$ by integrating each term separately:

$$\int x^3 + x dx = \int x^3 dx + \int x dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$$

Why don't we write $C + C$?

The rule that justifies this is

$$\boxed{\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx}$$

There is also a rule for “pulling out constants”:

$$\boxed{\int a \cdot f(x) dx = a \int f(x) dx}$$

where a is a constant

8. MORE PRACTICE

$$\int 15x^2 + 2 dx = \frac{15}{3}x^3 + 2x + C$$

$$\int 6 - \frac{4}{x^2} dx = \int 6 - 4x^{-2} dx = 6x - \frac{4}{-1}x^{-1} + C = 6x + \frac{4}{x} + C$$

$$\int 6\sqrt[3]{x} dx = \int 6x^{1/3} dx = \frac{6}{4/3}x^{4/3} + C = \frac{9}{2}x^{4/3} + C$$

$$\begin{aligned} \int 10x + \frac{2}{\sqrt{x}} + 1 dx &= \int 10x + 2x^{-1/2} + 1 dx = \frac{10}{2}x^2 + \frac{2}{1/2}x^{1/2} + x + C \\ &= 5x^2 + 4x^{1/2} + x + C \end{aligned}$$

9. A QUOTIENT

Integrate $\int \frac{x^2 - x}{x} dx$

We first simplify the the fraction

$$\int \frac{x^2 - x}{x} dx = \int \frac{x(x - 1)}{x} dx = \int x - 1 dx = \frac{1}{2}x^2 - x + C$$

Question: Could we have integrated this quotient by some kind of “quotient rule” for integrals?

10. THE BAD NEWS ABOUT INTEGRATION

There is no **product rule**, **quotient rule**, or **chain rule** for integrals.

Integrate $\int (x^3 - 1)^2 dx$

There is no “chain rule” for integrals, so we have to square out the binomial:

$$\int (x^3 - 1)^2 dx = \int x^6 - 2x^3 + 1 dx = \frac{1}{7}x^7 - \frac{2}{4}x^4 + x + C$$

11. INTEGRALS YOU CAN'T DO

$$\int \frac{1}{x} dx$$

Why can't you write $\frac{1}{x} = x^{-1}$ and use the power rule?

$$\int \frac{x}{\sqrt{x+1}} dx$$

Wait till section 4.5

$$\int x \cos x dx$$

Wait till Calc II

12. MORE INTEGRALS YOU CAN'T DO

$$\int \sin x \cos x \, dx$$

What trig identity does this remind you of?

$$\int \tan x \, dx$$

Why can't you write $\tan x = \frac{\sin x}{\cos x}$?

Wait till Calc II

$$\int \sec^2 x \, dx$$

Actually you can do this one. The answer is $\tan x$

$$\int \sec^3 x \, dx$$

A real toughie, covered in Calc II

$$\int \sin(x^2) \, dx$$

No one can do this one in a finite number of symbols.