

MATH 229 LECTURE NOTES: CHAPTER 3
APPLICATIONS OF DERIVATIVES

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1. FINITE INTERVALS

There are four kinds of finite intervals:

Notation	Condition	Name
$[a, b]$	$a \leq x \leq b$	closed
(a, b)	$a < x < b$	open
$[a, b)$	$a \leq x < b$	half-open
$(a, b]$	$a < x \leq b$	half-open

2. INFINITE INTERVALS

There are five kinds of infinite intervals:

Notation	Condition	Name
$[a, \infty)$	$x \geq a$	closed ray
(a, ∞)	$x > a$	open ray
$(-\infty, b]$	$x \leq b$	closed ray
$(-\infty, b)$	$x < b$	open ray
$(-\infty, \infty)$	any x	real number line

3. INCREASING VERSUS DECREASING

If I is any one of these nine intervals (finite or infinite)
and $f(x)$ is a function defined on I ,

- We say f is increasing on I

if for any two points x_1 and x_2 in I ,

if $x_1 < x_2$ then $f(x_1) < f(x_2)$

In other words, as x values increase, the y values on the graph increase.

- We say f is decreasing on I

if for any two points x_1 and x_2 in I ,

if $x_1 < x_2$ then $f(x_1) > f(x_2)$

In other words, as x values increase, the y values on the graph decrease.

4. RELATION OF DERIV TO GRAPH

function	derivative
increasing	positive
decreasing	negative
horizontal	zero
straight line	constant
steep rising	large positive
gradual rising	small positive
steep falling	large negative
gradual falling	small negative

5. MAJOR THEOREM

Let f be a function which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) .

If $f'(x) > 0$ for every x in (a, b) , then $f(x)$ is increasing on $[a, b]$.

If $f'(x) < 0$ for every x in (a, b) , then $f(x)$ is decreasing on $[a, b]$.

If $f'(x) = 0$ for every x in (a, b) , then $f(x)$ is constant on $[a, b]$.

6. DERIVATIVES

$$y = f(x) = x^3 - 6x^2 + 9x + 5$$

$$\begin{aligned} y' = f'(x) &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x - 1)(x - 3) \end{aligned}$$

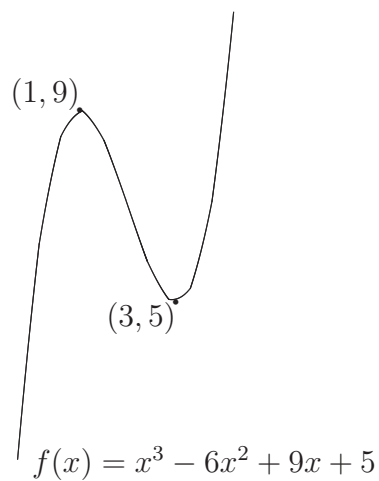
7. DETERMINING SIGN OF f'

$$f'(x) = 3(x - 1)(x - 3)$$

To determine the sign of $f'(x)$ use the following chart

Interval	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
$x - 1$	neg	pos	pos
$x - 3$	neg	neg	pos
$(x - 1)(x - 3)$	pos	neg	pos
Interpretation	increasing	decreasing	increasing

8. PICTURE OF GRAPH



9. SUBSET OF THE DOMAIN

Let f be a real valued function defined on a set S of real numbers.

The set S is not necessarily the entire domain of f ,

although S must, of course, lie inside the domain of f .

For example, if $f(x) = \sqrt{x}$, then the domain of f is the set of real numbers ≥ 0 .

We could take S to be the closed interval $[1, 2]$.

10. ABSOLUTE MAXIMUM&MINIMUM

The function f has an **absolute maximum** on the set S

if there is at least one point c in S such that

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } S.$$

The number $f(c)$ is called the **absolute maximum value** of f on S .

We say f has an **absolute minimum** on the set S if there is at least one point c in S such that

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } S.$$

The number $f(c)$ is called the **absolute minimum value** of f on S .

11. LOCAL MAXIMUM&MINIMUM

Let f be a real value function defined on a set S of real numbers.

We say the function f has a **local maximum** at a point c in S

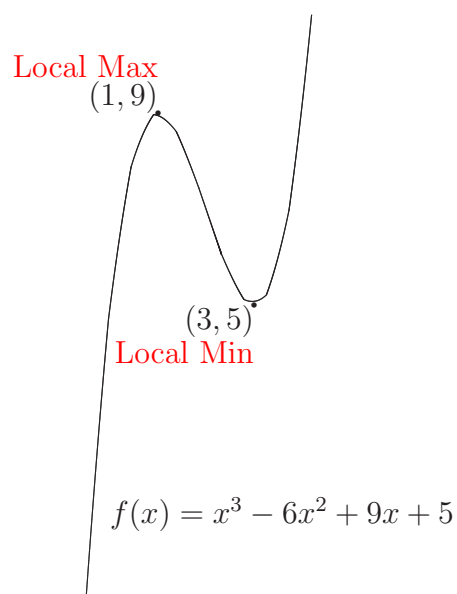
if there is an open interval I containing c such that

$$f(x) \leq f(c) \quad \text{for all } x \text{ lying in both } I \text{ and } S.$$

The concept of **local minimum** is similarly defined by requiring $f(x) \geq f(c)$ instead of $f(x) \leq f(c)$.

A local maximum/minimum is sometimes called a relative maximum and minimum.

12. RECENT EXAMPLE



13. STRAIGHT LINE EXAMPLE

Consider $f(x) = x$, $S = [0, 2)$.

This function has absolute minimum on S when $x = 0$.

No absolute maximum exists.

This function fails to have an absolute maximum

because $x = 2$ is not in the domain set S .

14. RECIPROCAL EXAMPLE

Consider $f(x) = \frac{1}{x}$, $S = (0, 2]$.

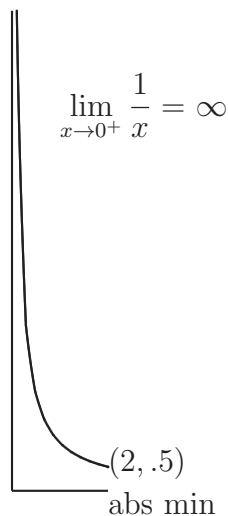
The reciprocal function has absolute minimum on S when $x = 2$.

No absolute maximum exists.

This function fails to have an absolute maximum

because it is not continuous at $x = 0$.

that is, because $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$

15. GRAPH OF $1/x$ 

16. VANISHING DERIVATIVE THEOREM

Assume $f(x)$ is a continuous function defined on an open interval I .

Assume that $f(x)$ has a local maximum or minimum at a point c inside I .

If $f'(c)$ exists, then $f'(c) = 0$.

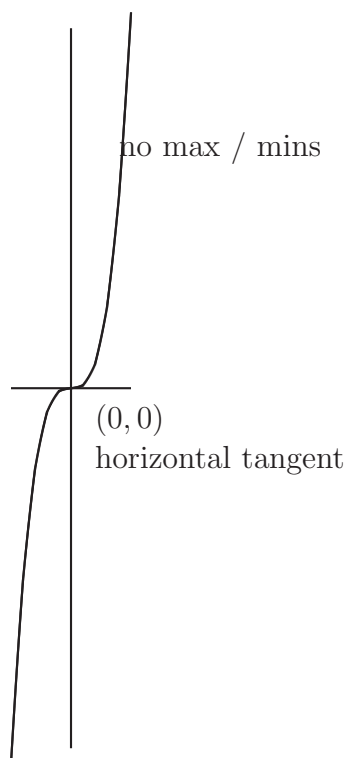
17. FIRST WARNING

The Vanishing Derivative Theorem does **not** say that if $f'(c) = 0$, then f has a maximum or a minimum at $x = c$.

Example. Consider $f(x) = x^3$.

Here $f'(x) = 3x^2$ and so clearly $f'(0) = 0$,

yet $x = 0$ is neither a local maximum nor a local minimum.

18. GRAPH OF x^3 

19. SECOND WARNING

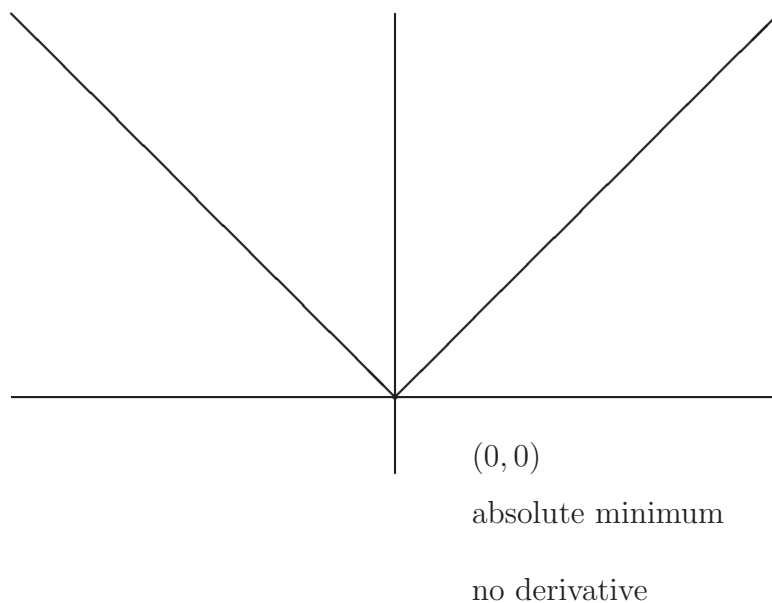
The Vanishing Derivative Theorem does **not** say that if f has a maximum or a minimum at $x = c$, then $f'(c) = 0$.

Example Consider $f(x) = |x|$.

It is clear from the graph that the absolute value function has a local (in fact, absolute) minimum at $x = 0$,

yet $f'(0)$ is not even defined.

20. GRAPH OF ABSOLUTE VALUE FUNCTION



21. MAX-MIN THEOREM

MAX-MIN THEOREM FOR CONTINUOUS FUNCTIONS

Assume $f(x)$ is a continuous function defined on a set S and that $f(x)$ has a local maximum or minimum at a point c in $[a, b]$.

Then we have three possibilities:

(i) $f'(c) = 0$

(ii) $f'(c)$ is undefined

or (iii) c is a boundary point of S

Usually (iii) means that S is an endpoint of an interval.

Points of type (i) or (ii) are called **critical points**.

22. EXAMPLE 1

Find all local and absolute max/min points of $f(x) = x^3 - 12x$ on the interval $[0, 4]$.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2)$$

The critical points are 2 and -2 . Since -2 does not lie in the interval $[0, 4]$, we may ignore it.

The only points that can possibly be local max / mins are the critical point $x = 2$ and the endpoints 0 and 4.

23. DETERMINING MAX / MINS

$f(x) = x^3 - 12x$ on the interval $[0, 4]$.

$$f'(x) = 3(x - 2)(x + 2)$$

x	$y = x^3 - 12x$	Result
0	0	?
2	$8 - 24 = -16$	Abs Min
4	$64 - 48 = 16$	Abs Max

24. EXAMPLE 2

Find all local and absolute max/min points of $f(x) = 3x^{2/3} - 2x$ on the interval $[-1, 2]$.

$$f'(x) = 3 \cdot \frac{2}{3}x^{-1/3} - 2 = \frac{2}{x^{1/3}} - 2$$

$$f'(x) = 0 \iff \frac{2}{x^{1/3}} - 2 = 0$$

$$\iff \frac{2}{x^{1/3}} = 2$$

$$\iff \frac{1}{x^{1/3}} = 1$$

$$\iff x^{1/3} = 1$$

$$\iff x = 1$$

Note that the derivative is undefined when $x = 0$

Why?

The critical points are 0 and 1 and the endpoints are -1 and 2 .

25. DETERMINING MAX / MINS

$f(x) = 3x^{2/3} - 2x$ on the interval $[-1, 2]$.

$$f'(x) = \frac{2}{x^{1/3}} - 2$$

x	$y = 3x^{2/3} - 2x$	Result
-1	$3+2 = 5$	Abs Max
0	0	Abs Min
1	$3 - 2 = 1$	?
2	.7662	?

26. EXAMPLE 1 REVISITED

Find all local max/min points of $f(x) = x^3 - 12x$ on the interval $[0, 4]$.

$$f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$$

The critical points are 2 and -2 . We ignore -2 since it is not in the interval $[0, 4]$. The endpoints 0 and 4.

We can use the sign of the derivative to determine when the function is going up and going down, thereby isolating local maximum and minimums.

27. FINDING LOCAL EXTREMA

Use the chart for $f'(x) = 3(x - 2)(x + 2)$

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$x - 2$	neg	neg	pos
$x + 2$	neg	pos	pos
$(x - 2)(x + 2)$	pos	neg	pos
Interpretation	increasing	decreasing	increasing

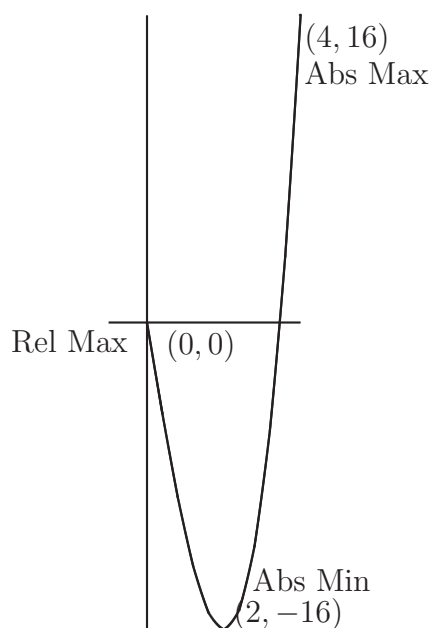
Conclusions:

0 is a local max since the graph is decreasing to the right of $x = 0$

2 is a local min since the graph is decreasing to the left of 2 and increasing to the right of 2

4 is a local max since the graph is increasing to the left of $x = 4$

28. GRAPH OF $x^3 - 12x$



29. FIRST DERIVATIVE TEST

Suppose $f(x)$ is continuous on a closed interval $[a, b]$ and c is a critical point of $f(x)$ in the open interval (a, b)

Interval	(a, c)	(c, b)	Conclusion
Case 1	$y' < 0$	$y' > 0$	MIN
Case 2	$y' > 0$	$y' < 0$	MAX
Case 3	$y' > 0$	$y' > 0$	INCR
Case 4	$y' < 0$	$y' < 0$	DECR