

CALC I REVIEW

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1. WHAT IS A DERIVATIVE?

Multiple Choice Question:

The derivative is

- (a) a slope
- (b) a limit
- (c) a rate of change
- (d) all of the above

2. WHAT IS AN INTEGRAL?

Multiple Choice Question:

The integral $\int_a^b f(x) dx$ is

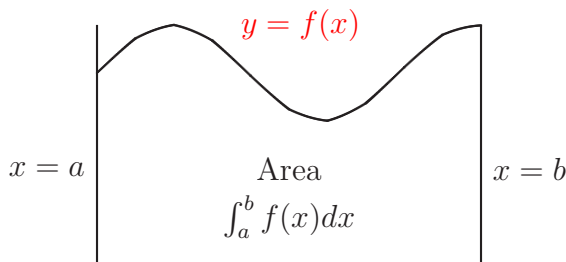
- (a) an area
- (b) a limit
- (c) an antiderivative
- (d) all of the above

3. THE AREA PROBLEM

Given a curve $y = f(x)$

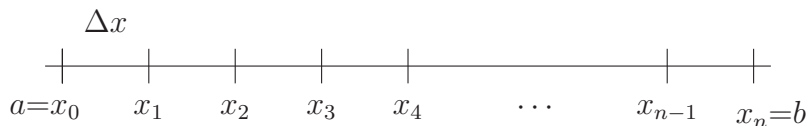
For now we will assume that $f(x)$ is ≥ 0

How do we find the area under the curve and above the x -axis, between two vertical sides $x = a$ and $x = b$?



4. SUB-DIVIDING $[a, b]$

Set-up: divide the interval $[a, b]$ into n equally spaced segments, whose division points are $x_0, x_1, \dots, x_n$.



The width of each segment is

$$\Delta x = \frac{b - a}{n}$$

The division points x_k are $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $x_3 = a + 3\Delta x$ and, in general,

$$x_k = a + k\Delta x$$

5. RIEMANN METHOD

In the Riemann method, we use n rectangles to approximate the area under the curve $y = f(x)$

Each rectangle has as its base the subinterval $[x_{k-1}, x_k]$ of width Δx

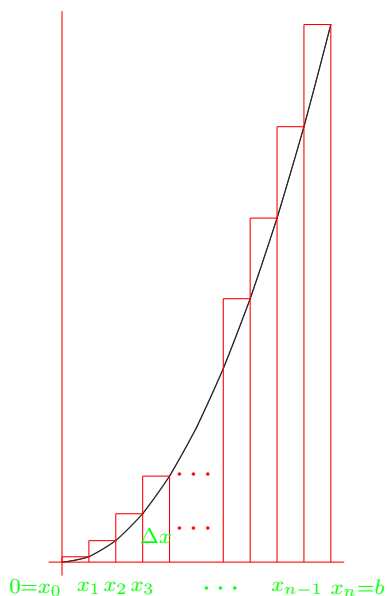
The height of each rectangle is $f(x_k)$

Since the area of a rectangle is base $\times$ height, the k -th rectangle has area

$$\Delta x \cdot f(x_k)$$

Our approximation, called a **Riemann Sum** is the sum of the areas of these n rectangles

6. RIEMANN METHOD PICTURE



7. EXAMPLE $y = x^2$

$$y = f(x) = x^2$$

n = number of rectangles

$$[a, b] = [0, 3]$$

$$\Delta x = \frac{3}{n}$$

$$x_k = \Delta x \cdot k$$

$$A(n) = \Delta x \sum_{k=1}^n f(x_k)$$

8. COMPUTING THE EXACT AREA

The following table gives the values of $A(n)$ for various values of n :

n	$A(n)$
3	14
6	11.375
12	10.15625
24	9.5703125
100	9.13545
1000	9.0135045
10000	9.001350045
1000000	9.0000135

It looks like the exact area under the curve $y = x^2$ between 0 and 3 is 9

9. LIMIT DEFINITION OF INTEGRAL

We determine the exact area A under the curve $y = f(x)$ from a to b by calculating the **limit** of the Riemann Sums $A(n)$ as the number of rectangles n goes to infinity.

That is, the definite integral of $f(x)$ from a to b is the following **limit**

$$\int_a^b f(x) dx \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} A(n) = \lim_{n \rightarrow \infty} \Delta x \sum_{k=1}^n f(x_k)$$

For many simple functions, such as $f(x) = x^n$, the limit that gives the area under the curve has been worked out.

We will apply the limit definition to the case when $n = 2$ in complete (and painful) detail.

10. EXAMPLE $y = x^2$

For example, suppose $f(x) = x^2$ and we wish to compute the area from $a = 0$ to b .

We begin by computing the area under the n rectangles, using right end points.

Here

$$\Delta x = \frac{b}{n} \text{ and } x_k = k \Delta x$$

$$\begin{aligned} A(n) &= \sum_{k=1}^n \Delta x \cdot f(x_k) \\ &= \Delta x \sum_{k=1}^n f(x_k) \quad [\text{factor out } \Delta x] \end{aligned}$$

11. EXAMPLE PART 2

$$\begin{aligned} A(n) &= \Delta x \sum_{k=1}^n f(x_k) = \Delta x \sum_{k=1}^n x_k^2 \quad [f(x) = x^2] \\ &= \Delta x \sum_{k=1}^n (k \Delta x)^2 \quad [\text{because } x_k = k \Delta x] \\ &= \Delta x \sum_{k=1}^n k^2 \Delta x^2 \quad [\text{algebra}] \\ &= (\Delta x)^3 \sum_{k=1}^n k^2 \quad [\text{factor out } (\Delta x)^2] \\ &= \frac{b^3}{n^3} \sum_{k=1}^n k^2 \quad [\text{because } \Delta x = \frac{b}{n}] \end{aligned}$$

12. EXAMPLE PART 3

At this point we need a formula for $\sum_{k=1}^n k^2$.

Page 309, formula $\boxed{6}$: $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$

You can verify this formula $\boxed{6}$ for $n = 1, 2$, and 3 .

$$\begin{aligned}\text{Thus, } A(n) &= \frac{b^3}{n^3} \sum_{k=1}^n k^2 \\ &= \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= \frac{b^3}{6} \left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right). \quad [\text{Note } n \text{ cancels.}]\end{aligned}$$

13. EXAMPLE CONCLUSION

The exact area A under the curve is the limit of the rectangular approximations $A(n)$ as $n \rightarrow \infty$:

$$\begin{aligned}A &= \lim_{n \rightarrow \infty} A(n) \\ &= \lim_{n \rightarrow \infty} \frac{b^3}{6} \left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{b^3}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \\ &= \frac{b^3}{6} (1+0)(2+0) = \frac{b^3}{3}\end{aligned}$$

When $b = 3$, $A = \frac{3^3}{3} = 9$, confirming our previous guess.

14. THE FUNDAMENTAL THEOREM OF CALCULUS

$$\boxed{\int_a^b f(x) dx = F(b) - F(a)}$$

where $F(x)$ is an anti-derivative of $f(x)$

Calculus was born when Sir Issac Newton and Gottfried Wilhelm Leibniz independently discovered this essential connection between integrals (areas) and derivatives (rates of change).

The Fundamental Theorem of Calculus says that in order to find the area A under the curve $y = f(x)$ from a to b ,

first find an anti-derivative of $f(x)$, that is, a function $F(x)$ whose derivative is $f(x)$

then the area is $A = F(b) - F(a)$

15. THE EASY WAY

Compute the definite integral $\int_0^3 x^2 dx$

An anti-derivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$

So by the Fundamental Theorem of Calculus

$$\int_0^3 x^2 dx = F(3) - F(0) = \frac{1}{3}3^3 - \frac{1}{3}0^3 = 9$$

The area under the curve $y = 2x$ for 0 to 3 is 9,

a fact that we laboriously worked out earlier.

16. GOOD NEWS FOR ANTIDERIVATIVES

Like derivatives, there is a sum rule

$$\int (f + g) = \int f + \int g,$$

a constant rule

$$\int (cf) = c \int f,$$

and the power rule

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C.$$

What happens if $n = -1$?

Joke

Together these three rules allow us to integrate all polynomials.

17. BAD NEWS FOR ANTIDERIVATIVES

The bad news is that

there is no product rule,

there is no quotient rule,

there is no chain rule

for integration.

18. EXAMPLE

Since the derivative of $\sin x$ is $\cos x$, we know

$$\int \cos x \, dx = \sin x + C$$

What is $\int \frac{1}{\cos x} \, dx$?

What is $\int (\cos x)^2 \, dx$?

What is $\int \cos(x^2) \, dx$?