

2. Hotel Infinity

2.1. The Original Problem: Ignore the No Vacancy Sign. You are Captain Smirk of the Starfleet Galaxy (made by Ford) and you have landed in a new and unexplored region of the universe in which time worms have warped the fabric of space-time, allowing the existence of an infinite hotel. Hotel ∞ , as it's called, has infinitely many rooms, numbered (what else?) 1, 2, 3, 4, 5, etc. Being tired from your long journey—traveling at Super Warp Speed and fighting Cling-ons [*Insert joke about Uranus*—you go to the front desk of Hotel ∞ and ask for a room.

Manager: I'm sorry, all our rooms are occupied.

You: What do you mean “occupied?” You have infinitely many rooms!

Manager: I know. It's the busy season. The maids aren't happy about the situation either.

Question 1. What do you do? It's the only hotel in town and you need a room. There is a perfect solution, which gets you a room, yet doesn't kick any one else out of the hotel.

Go think about it for a while.

Are you back? The solution is: you go back to the manager and say: “Ask the guest in Room 1 to move to Room 2, ask the guest in Room 2 to move to Room 3, and so on down the line. Room 1 will be empty and I'll take it.”

At first the solution seems a little fishy. After all, how can you fit one more guest into a hotel that is completely filled? On the other hand, although each guest is slightly inconvenienced, no one is put out on the street.

Question 2. How do you get rooms for all 273 members of your crew?

2.2. Bus Line ∞ . Bus Line ∞ shows up at Hotel ∞ with infinitely many passengers. They all want rooms and once again the hotel is full.

Question 3. Now what do you do? How can you find rooms for infinitely many passengers in the already full Hotel ∞ ?

Question 3 is harder than Question 2. You cannot just shift every one by the same amount. Moving the occupant of Room n to, say, Room number $n + 1000000$ will free up a million rooms. But you need more than that. The solution is: (*Stop reading if you want*

more time to think about it!) Tell the guest in Room 1 to move to Room 2, the guest in Room 2 to move to Room 4, the guest in Room 3 to move to Room 6, and in general, the guest in Room number n to move to Room $2n$. This will free up all the odd-numbered rooms. Since there are infinitely many odd numbers, all the passengers of Bus Line ∞ can be accommodated.

Question 4. Two buses from Bus Line ∞ shows up at Hotel ∞ , each loaded with infinitely many passengers and the hotel is full. What should the manager do?

The solution to Question 4 is much like the solution to Question 3. Divide the positive integers into three classes: (1) numbers which are one more than a multiple of 3, (2) numbers which are two more than a multiple of 3, and (3) the multiple of 3 themselves:

Class 1:	1	4	7	10	13	16	19	...
Class 2:	2	5	8	11	14	17	20	...
Class 3:	3	6	9	12	15	18	21	...

Ask the current guests to move to the room whose number is triple their current room number, then unload Bus 1 into Class 1 rooms and Bus 2 into Class 2 rooms.

2.3. The Infinite Number of Buses Problem. Infinitely many buses from Bus Line ∞ , numbered Bus 1, Bus 2, Bus 3, ..., show up at Hotel ∞ , each loaded with infinitely many passengers.

Question 5: Assuming (for once) that Hotel ∞ is completely empty, how can the hotel manager find rooms for all the passengers?

Question 5 is much harder than any question we have seen so far—not to mention the traffic jam that an infinity of buses can create. A key to solving Question 5 is to identify each passenger with a pair of numbers (i, j) . The first number i identifies the bus number and the second number identifies the seat number on that particular bus. Thus the seventeenth passenger on bus number 62 would be given the ID numbers $(62, 17)$ and will be referred to as *Passenger* $(62, 17)$. We can place all of the passengers on an infinite grid of rows:

Bus 1:	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	...
Bus 2:	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	...
Bus 3:	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	...
Bus 4:	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	...
Bus 5:	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	...
⋮			⋮			⋮

Passenger(1,1), the first passenger on Bus 1, is assigned Room number 1. Who's next? The trick is to go down the diagonals, starting at the top. Rooms 2 and 3 are assigned to *Passengers*(1,2) and (2,1), in that order. The third diagonal consists of *Passengers*(1,3), (2,2), and (3,1), who are assigned Rooms 4, 5, and 6. To decide which lucky passenger gets Room number 7, we go to the next diagonal at the top—the winner is *Passenger*(1,4), to be followed by *Passengers*(2,3), (3,2), and (4,1). Proceeding in this way, one diagonal at a time, every passenger on every bus will eventually be placed in a room. This method works because the wait is always finite. Only a finite number of other passengers lie *ahead* of any particular passenger. If, instead, all passengers on Bus 1 were assigned rooms *before* the passengers on Bus 2, then *Passenger*(2,1) would wait forever. Incidentally, if you're wondering why we use buses instead of planes, the engineers tried building a plane that would hold infinitely many passengers, but it never got off the ground.

2.4. Counting Infinite Sets. The mathematics behind the infinite hotel problem can be phrased in terms of sets: Given two sets A and B , how can we tell whether they have the same number of elements? One answer, of course, is to just “count the number of elements” of both sets and see if they are the same.

Question 6. Do you always have to count the number of elements?

No! To tell whether the number of fingers on your left hand is the same as the number of fingers on your right hand, put the fingers of your two hands together. If no finger is left unpaired, then the numbers are the same. (Of course, it is not the world's most difficult problem to count to five.) If at a dance, everyone is dancing with a partner of the opposite sex, then there are as many girls as boys at the dance.

Definition 1.1. If we can pair the elements of set A with the elements of set B in such a way that each element of set A is paired with a unique element of set B and no element of B is omitted, then the two sets have the same number of elements.

There are two great things about this definition. The first is that it doesn't require us to count the number of elements in either set. But the second, more important advantage is that our definition applies to infinite sets.

Question 7. Does the set of positive integers have the same number of elements as the set of negative integers?

Yes, the pairing is obvious:

$$\begin{array}{ccc} 1 & \leftrightarrow & -1 \\ 2 & \leftrightarrow & -2 \\ 3 & \leftrightarrow & -3 \\ \vdots & & \vdots \\ n & \leftrightarrow & -n \\ \vdots & & \vdots \end{array}$$

No surprise here.

Question 8. Does the set of positive integers have the same number of elements as the set of perfect squares 1, 4, 9, 16, etc.?

Yes, since they can be paired as follows:

$$\begin{array}{ccc} 1 & \leftrightarrow & 1 \\ 2 & \leftrightarrow & 4 \\ 3 & \leftrightarrow & 9 \\ \vdots & & \vdots \\ n & \leftrightarrow & n^2 \\ \vdots & & \vdots \end{array}$$

There is something a little unsettling about this example. It certainly seems like there are a lot more integers than perfect squares. After all, loads of numbers are not perfect squares: 2, 3, 5, 6, 7, etc. Up to one million, we have a million numbers and only one thousand squares. Our discomfort is reminiscent of the uneasy feeling about fitting more guests into Hotel ∞ even though it is completely full. Obviously, strange things happen when we deal with infinity.

Question 9. Can we match the set of positive fractions with the set of positive integers?

Write the fraction $\frac{m}{n}$ as *Fraction*(m, n). Notice the similarity between *Fraction*(m, n) in this problem and *Passenger*(m, n) in the Infinite Bus problem. Using the diagonal method, we can give a number to *Fraction*(m, n) in exactly the same way that we assigned a room number to *Passenger*(m, n), with one slight, but important, difference. *Do you see what it is?* *Fraction*(1, 2) and *Fraction*(2, 4) correspond to the same fraction, one half, since $\frac{2}{4}$ simplifies to $\frac{1}{2}$. (Many people say $\frac{2}{4}$ reduces to $\frac{1}{2}$, implying some kind of weight loss.) Indeed, there is an infinite chain of pairs equivalent to $\frac{1}{2}$: *Fraction*(1, 2) = *Fraction*(2, 4) = *Fraction*(3, 6) = *Fraction*(4, 8) = $\dots$. The solution to the infinite bus problem would assign separate rooms to each of these pairs, and yet, as fractions, they are all the same number, namely $\frac{1}{2}$. *How do you suggest we fix this problem?*

2.5. Making Room for the Reals. By now you probably believe that every infinite set has the same number of elements as any other infinite set and that all this fuss boils down to the equation $\infty = \infty$. Things are not this simple, as the next example illustrates.

Question 10. All of the points between 0 and 1 on the number line wish to obtain rooms in Hotel ∞ . (This makes the maids happy, because points never mess up the rooms.) Is there room in Hotel ∞ for all these points?

Since the points on the number line are just the real numbers, we can restate Question 10 in a more mathematical way: Can you match the positive integers with the real numbers x in the interval $0 < x < 1$?

Very often mathematicians play devil's advocate. To show something can't be done, assume, to the contrary, that it can be done, and then try to follow your assumption to an incorrect conclusion. My father used to say "you cannot shave a wasp with a monkey wrench." If you assume otherwise, then get yourself a wasp, a wrench, and some shaving cream, and try it. (Of course, to say something is impossible does not mean that you personally cannot do it. If you cannot dunk a basketball, this does not mean that dunking a basketball is impossible. On the other hand, dunking a basketball through a soda straw is impossible for everyone.)

Returning to the problem of finding rooms for the points in the interval between 0 and 1, we wish to show that no matter how you pair the positive integers with the real numbers between 0 and 1, you always end up leaving out some of the points. Suppose you have a way to pair the positive integers and the points between 0 and 1. The number 1 is paired with some real number—call it x_1 ; the number 2 is paired with another real number x_2 , etc. We get an infinite list of the real numbers: $x_1, x_2, x_3, \dots$. Our goal is to find a number between 0 and 1 which is missing from this list. The secret is to look at the decimal expansion of each of the numbers x_i . Write $Digit(i, j)$ for the j th digit to the right of the decimal point of the i th number x_i in the list. (Compare the idea of $Digit(i, j)$ to the use of $Passenger(i, j)$ in the infinite bus problem.) As an example, suppose the first three numbers in the list have the following decimal expansions:

$$x_1 = 0.19776\,73176\dots$$

$$x_2 = 0.34182\,18910\dots$$

$$x_3 = 0.21724\,52839\dots$$

Then $Digit(2, 7) = 8$ and $Digit(3, 5) = 4$. Our goal is to construct a number y whose decimal expansion $y = 0.e_1e_2e_3\dots$ differs from every real number x_i on the list. The way to do this

is easy: choose the first digit e_1 of y to be the digit that is one more than $Digit(1, 1)$, choose the second digit e_2 to be the digit that is one more than $Digit(2, 2)$, etc. Here we understand that the digit that is one more than 9 is 0 (not 10). For the beginning decimal expansions listed above for x_1 , x_2 , and x_3 , we see that $y = 0.258 \dots$. (Make sure you understand where the digits 2, 5, and 8 are coming from.) The number y , so constructed, differs from the i th number x_i in the i th digit. In making the list, we left out the real number y .

This argument is called Cantor's diagonalization argument. It shows conclusively that no matter how hard you try, you can never match all the real numbers with the positive integers. This reminds me of a candy store that had a giant fishbowl full of bubblegum. For a quarter you can reach your hand into the fishbowl and pull out as many pieces of bubblegum as you could hold in your hand. It didn't take long to realize that you couldn't pull your hand out of the small hole at the top of the fishbowl without dropping most of the bubblegum. In the same way, any attempt to grab all the real numbers between 0 and 1 and put them in a list just results in the bulk of them slipping back onto the number line.

If A is an infinite set and the elements can be paired with the positive integers, then we can list the elements of A as $a_1, a_2, a_3, \dots$. Such sets are called **countable**. An infinite set whose elements cannot be paired with the positive integers is called **uncountable**. We have just seen that the set of real numbers in the interval between 0 and 1 is an uncountable set.

1.2 Exercises

1. Exhibit a matching between the set of positive even numbers $\{2, 4, 6, 8, \dots\}$ and the set of odd numbers $\{1, 3, 5, \dots\}$.
2. Four buses, filled with infinitely many passengers, show up at Hotel ∞ , which is also full. How do you find everyone a room?
3. Find a formula for the sum $S = 1 + 2 + 3 + 4 + \dots + (n - 1) + n$ by experimenting with small values of n .
4. This problem relates to the Infinite Bus Problem. If you are $Passenger(7, 4)$ what is your room number? $Passenger(30, 20)$? Can you come up with a formula so that $Passenger(i, j)$ can figure out her Room number in the hotel? Hint: once you determine the diagonal that (i, j) lies on, Exercise 3 comes in handy.
5. How do you solve the problem of equivalent fractions in Question 9?

6. Show that there are as many rational numbers in the interval $(0, 1)$ as there are rational numbers in the interval $(1, \infty)$.

7. The interval $\{x; a < x < b\}$ of all points on the number line strictly between a and b is denoted (a, b) . Show that there are as many points in the interval $(0, 1)$ as there are points in the interval $(7, 8)$. Show that there are as many points in $(0, 1)$ as there are in the in $(0, 2)$, even though the second interval is twice as long.

8. Assume the following statement is true about Hotel ∞ :

() For every positive integer n , there is an occupied room whose number exceeds n .*

Answer the following two questions and explain your answer:

- (a) Does $(*)$ imply that there are infinitely many occupied rooms in Hotel ∞ ?
- (b) Does $(*)$ imply that all rooms of Hotel ∞ are occupied?

9. Find a one-to-one correspondence between the points in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the entire real number line. [Hint: a trig function will do the job.]

10. Find a one-to-one correspondence between the positive integers and the set of all triples (i, j, k) of positive integers.

11. Are there are as many points in the unit square $\{(x, y) : 0 < x < 1 \text{ and } 0 < y < 1\}$ as there are in the the unit interval $\{x : 0 < x < 1\}$?

12. In Question 10 in the text, we left out lots and lots of numbers, since there are so many different ways to make y differ from all entries in the list $x_1, x_2, x_3, \dots$. Describe some other ways to obtain a number y not on the list.

13. In our solution to Question 10, we constructed a decimal number y whose digits differed from the digits of every element in the list $x_1, x_2, x_3, \dots$. Unfortunately, there is a subtle error in this argument, due to the mischevious nature of the 999 problem. Suppose, for example, that $x_1 = 0.19999\dots$ and that the i th digit of the other numbers x_i in the list is always 9. By construction, $y = 0.20000\dots$. While it is true that the digits of y differ from the digits of every x_i in the list, it is not the case that y is a new number. Our solution to the 999 problem shows that $0.2000\dots = 0.19999\dots$ and therefore, $y = x_1$. Ending in 9's is the only way that a number can have two different decimal representations. Explain how we can fix this “bug” in the construction of y .

14. It is known that students do not always believe what their math teachers tell them about infinity. What do you believe? Do you really believe there are as many square as positive integers? Do you believe the infinity of reals exceeds the infinity of integers? Do you believe it is possible to live forever?