

3. The Cantor Set

Once again Professor Flappenjaw's star students, Ernest Noddsoff and Hedda Earful, along with four other classmates, Nicole, Marieantoinette, Lauren, and Ashley, have come to him with questions about the real number line.

Hedda: I've been wondering, Professor, a point has zero width, right? So how come when you stick enough points together, you end up getting width?

Nicole: I've wondered about Hedda's question myself. Obviously if you start sticking points on the number line one at a time, their total length is zero. So where does the stuff that creates length come from? Does some kind of special glue hold points together?

Professor: I see that both of you have been thinking about a question that has perplexed mathematicians for centuries. Of course you realize, Hedda, from our study of countable sets, that you could not put all the real numbers between 0 and 1 onto the number line *one at a time*, for this would mean that the real numbers were a countable set, and we have seen that they are not.

Lauren: Are you saying that the reals have length because they are uncountable?

Nicole: And that points of an uncountable set must be packed in so tightly that they gotta take up some space?

Professor: Yes and no. A countable set of real numbers has zero length, just as you might think.

Hedda: Is this hard to prove, Professor?

Professor: No, but let's postpone the question for now.

Ernest: Then an uncountable set must always have some length to it?

Professor: Let me answer your question, Ernest, by beginning with the unit interval, $[0, 1]$, the set of points between 0 and 1, including the two endpoints. This interval has length 1, correct?

Ernest: Yep.

Professor: I want you to systematically squeeze the insides out of this interval. First, remove its middle third $(\frac{1}{3}, \frac{2}{3})$. Note that we leave the endpoint $\frac{1}{3}$ and $\frac{2}{3}$, but take out every point in between them.

Ashley: That leaves two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$.

Professor: Right, Ashley. Now what's the middle third of these two?

Ernest: $(\frac{1}{9}, \frac{2}{9})$ and—the other is harder—oh, I got it, $(\frac{7}{9}, \frac{8}{9})$.

Professor: Remove each middle third from the intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$.

Marieantoinette: Now we have four pieces: $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{3}{9}]$, $[\frac{6}{9}, \frac{7}{9}]$, and $[\frac{8}{9}, 1]$.

Professor: What are the middle thirds of these four intervals?

Ernest: Yikes, Professor, let me get my calculator.

Professor: You can do it in your head, Ernest. Just convert to denominator 27.

Ernest: Okay. The interval $[0, \frac{1}{9}] = [0, \frac{3}{27}]$. So the middle of it is $(\frac{1}{27}, \frac{2}{27})$. The middle of $[\frac{2}{9}, \frac{3}{9}] = [\frac{6}{27}, \frac{9}{27}]$ is $(\frac{7}{27}, \frac{8}{27})$. The middle of $[\frac{6}{9}, \frac{7}{9}] = [\frac{18}{27}, \frac{21}{27}]$ is $(\frac{19}{27}, \frac{20}{27})$. And the middle of $[\frac{8}{9}, 1] = [\frac{24}{27}, \frac{27}{27}]$ is $(\frac{25}{27}, \frac{26}{27})$.

Professor: See, I told you you could do it in your head. What do you get when you remove these four middle thirds?

Nicole: We have 8 intervals left and I suppose you want to take out their middle thirds.

Professor: Precisely. Although I think you get the picture without actually needing to work out the exact fractions.

Ernest: Whew!

Professor: Now if I continue this process forever, each time removing the middle thirds of the intervals which remain in an ever expanding collection of very tiny subintervals, what do I end up with?

Ernest: Gee, Professor. It sure looks like you have squeezed out all the insides so maybe you end up with the empty set.

Marieantoinette: That can't be right, Ernest. You will still have the endpoints such as $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{9}$, $\frac{2}{9}$.

Ernest: I forgot about the endpoints. Hedda must be right. When all the middle thirds are removed, all that will remain are the endpoints of the intervals you have removed.

Professor: You're right, Hedda, the endpoints remain, but surprisingly, so do lots and lots of other points.

Ernest: Who thought of this mess in the first place?

Professor: Georg Cantor.

Lauren: The same guy who invented Cantor's diagonalization method for showing the reals are uncountable?

Professor: The very same man. Perhaps Cantor's most famous aphorism is: *The essence of mathematics lies in its freedom*. Unfortunately, not all of Cantor's ground breaking ideas were accepted by other mathematicians of his time. The unacceptance of his ideas contributed in part to his mental breakdown.

Lauren: Bummer.

Professor: Eventually he did achieve the recognition he deserved. The set obtained by repeatedly removing middle thirds forever is called the Cantor set in his honor. The question we are trying to settle is whether the Cantor set consists of more than just the endpoints of the middle thirds that have been removed at some stage: 0 , 1 , $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, etc. In trying to answer this question, it is helpful to change to base 3 notation.

Ernest: I've just barely got used to base 2 notation and now we gotta use base 3. What's so good about base 3?

Professor: It perfectly suits this problem. In base 10, or decimal notation, we split the interval $[0, 1]$ into ten equally spaced subintervals, the tenths. A real number between 0 and 1 must lie in one of these 10 subintervals. We split that subinterval into ten smaller subintervals, the hundredths, and so on. In base 2, or binary notation, we divide each intervals into halves instead of tenths. But in base 3, or ternary notation as it is sometimes called, we subdivide by 3's.

Hedda: I think I know what you're driving at. In base 3 the fraction $\frac{1}{3} = .1$ and $\frac{2}{3} = .2$, and this eliminates the need for the awkward fractions. Assuming that we are writing all our numbers in base 3, rather than decimal, notation, the middle third of $[0, 1]$ is $(.1, .2)$.

Ashley: I get it! The remaining intervals are $[0, .1]$ and $[\frac{2}{3}, 1]$ and their middle thirds are $(.01, .02)$ and $(.21, .22)$. These ternary numbers sure make the problem easier.

Professor: Can you see how to characterize the points that remain in the Cantor set using ternary notation where we write each numbers as [He writes on the blackboard]

$$x = 0.t_1t_2t_3\ldots \text{ base } 3?$$

Ashley: Let's see. For the first subinterval we remove all real numbers where $t_1 = 1$. Next we remove all numbers where $t_2 = 1$.

Nicole: Wait a minute. We only remove $t_2 = 1$ when $t_1 = 0$ or $t_1 = 2$.

Ashley: But if $t_1 = 1$, we have already removed the entire subinterval $(.1, .2)$ the first time around. We don't have to worry about its middle third, since the entire interval is gone.

Nicole: Gotcha. The third time we remove those remaining numbers where $t_3 = 1$. You know, it looks like the Cantor set will consist of all real numbers which do not have a one in their base 3 representations.

Professor: Very good, Nicole, that's exactly the answer that I was looking for.

Nicole: No offense, but being a student is good training in trying to figure out what a professor wants to hear you say.

Professor: No offense taken. I'm sure there's something to what you're saying. Anyway, the Cantor set contains all those real numbers whose base 3 representation contains only 0's and 2's—no 1's are allowed. Here are three problems for you to go home and think about:

Question 1: Can you match the real numbers in the Cantor Set, one-for-one, with the real numbers between 0 and 1?

Question 2: What are the base 3 representations of the endpoints 0 , 1 , $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, etc.?

Question 3: Explain why the Cantor set contains more—actually, a lot more—than just the endpoints of Question 2.

1.3 Exercises

The first three exercises ask you to answer the Professor's three questions. If you get stuck, here are some hints. Do not look at the hints until after you have given the problems an honest attempt.

1. Answer Question 1.

Hint for Question 1: Find a correspondence between the ternary representation of points in the Cantor set and the binary representation of points in the interval $[0, 1]$.

2. Answer Question 2.

Hint for Question 2: The 999 Question in base 3 becomes

The 222 Question: Does $0.222222\cdots = 1$ (base 3)?

Like the 999 Question, the answer to the 222 Question is “yes,” giving us two different representations for the endpoints used in constructing the Cantor set. For example, in base 3, $\frac{1}{9} = .01 = .0022222\cdots$ and $\frac{2}{9} = .02 = .0122222\cdots$. Since we are not allowed to use the ternary digit 1 in representing numbers in the Cantor set, we should use the representation $.002222\cdots$ for $1/9$ and the representation $.02$ for $2/9$. Formulate a general rule based on whether the endpoint of the “middle third” is a left endpoint or a right endpoint.

3. Answer Question 3.

Hint for Question 3: Do the ternary representations you found in answering Question 2 comprise all possible ways of writing a base 3 number without using the ternary digit 1? For example, could the ternary number $.02020202\cdots$ be an endpoint?

4. What would the Cantor Set look like in two dimensions? Start with an equilateral triangle of side length 1. Connect the endpoints of the midpoints of each side. This gives an equilateral triangle inside the original triangle. Show that the side length of this smaller triangle is $\frac{1}{2}$. Show that area of the smaller triangle is one fourth the area of the outer triangle. Removing this middle triangle leaves three outer triangles. Remove the middle fourth triangle from each of these. Draw the next two iterations of this process. What points remain if we continue removing the middle fourth of each remaining triangles forever?