

4. Fractions and Fermat's Method of Descent

The real number line is composed of two types of numbers: the rationals and the irrationals. A **rational**, or fraction, can always be written as a repeating decimal expansion. The set of rational numbers is commonly written as $\mathbb{Q}$, where the 'Q' stands for "quotients." A real number which cannot be written as a fraction is called **irrational**. To write an irrational all you have to do is make sure its decimal expansion does not repeat. Thus, the number

$$x = .101001000100001 \dots$$

is irrational. It gets harder, however, to determine if a known number has a repeating decimal pattern or not. Take, for example, the square root of 2. The first 50 digits of $\sqrt{2}$ are

$$1.4142135623730950488016887242096980785696718753769$$

No pattern so far, but how do we know that the pattern isn't very, very long—maybe billions of digits long? You can't tell from just the first fifty, or million, or billion digits whether $\sqrt{2}$ is rational or not. It turns out that $\sqrt{2}$ actually is irrational, but in order to prove this, we need to use our brains, not some computer's brawn. The proof we shall give is based on a Fermat's method of infinite descent.

Pierre de Fermat (1601-1665) is often called the "father of modern number theory." In many areas of mathematics, Fermat was a player, although he was unusual in two ways: he was an amateur and he published very few of his results. His day job was royal councillor at the Parliament of Toulouse. Had he done nothing besides giving legal advice to the court, he would not be remembered today. Happily, he spent much of his leisure time—his day job paid well—absorbed by mathematical problems. As early as 1629 he began to make significant contributions in analytic geometry. His interest in number theory began with the publication of the Greek treatise *Arithmetica of Disphantus* in Latin by de Bechet in 1621. Diophantus was a Greek mathematician who came after Euclid and began his own cult dedicated to the study of integer solutions to mathematical problems. (The problem of finding integers x and y such that $5x + 7y = 1$ is a linear Diophantine equation. Can you figure out two solutions?) Anyway, Fermat read de Bechet's translation carefully and began his own investigations into some fundamental questions in number theory. The problem was: Fermat published virtually nothing during his lifetime. He was content to send his great discoveries to his friend Marin Mersenne (1588-1648). Fermat's letters contained statements of his results, often without proof, as challenge problems.

One of Fermat's favorite techniques for proving statements in number theory was his method of infinite descent. We shall illustrate Fermat's descent method in the proof of the following well known fact.

Theorem 1.1. The square root of 2 is irrational.

PROOF. Suppose $\sqrt{2}$ is, contrary to what you might have learned in high school, a rational number. Then there are positive integers a and b such that $\frac{a}{b} = \sqrt{2}$. Multiplying by b and squaring both sides gives

$$a^2 = 2b^2.$$

The method of our proof is to show that no matter how small the positive integers a and b are, we can always find another solution

$$A^2 = 2B^2,$$

where A and B are positive integers and A is smaller than a . Observe that

$$b^2 < 2b^2 < 4b^2.$$

(One apple is less than 2 apples and 2 apples are less than 4 apples.) We may substitute a^2 for $2b^2$ to get

$$b^2 < a^2 < (2b)^2.$$

Since b , a , and $2b$ are all positive integers, we are justified in taking the square root of this inequality, to get the following important result:

Important Inequality: $b < a < 2b$.

Take

$$A = 2b - a \quad \text{and} \quad B = a - b.$$

We wish to show the following four facts:

Fact 1. $A > 0$. Important Inequality $\implies 2b > a \implies 2b - a > 0 \implies A > 0$.

Fact 2. $B > 0$. Important Inequality $\implies a > b \implies a - b > 0 \implies B > 0$.

Fact 3. $A < a$. Important Inequality $\implies b < a \implies 2b < 2a \implies 2b - a < a \implies A < a$.

Fact 4. $A^2 = 2B^2$. Plugging $A = 2b - a$ and $B = a - b$ into $A^2 - 2B^2$ gives

$$\begin{aligned} A^2 - 2B^2 &= (2b - a)^2 - 2(a - b)^2 \\ &= (4b^2 - 4ab + a^2) - 2(a^2 - 2ab + b^2) \\ &= 4b^2 - 4ab + a^2 - 2a^2 + 4ab - 2b^2 \\ &= 2b^2 - a^2 = 0. \end{aligned}$$

We did it! Both A and B are positive integers, $A^2 = 2B^2$, and A is smaller than a . If we repeat this procedure, this time with (A, B) taking the place of (a, b) , then we will find yet another pair—call it (A_2, B_2) —where A_2 is a positive integer less than A and satisfying $A_2^2 = 2B_2^2$. The problem is that we cannot continue to find smaller and smaller positive integers which are all less than a . Otherwise there would be infinitely many positive integers less than a and clearly, *that ain't right!* The whole mess started from our initial assumption that $\sqrt{2}$ is a rational number. This assumption must be flat out wrong and $\sqrt{2}$ must be an irrational number. (Was there ever a doubt?) $\square$

It is interesting to compare this proof with the standard proof usually seen (if not remembered) in high school or standard algebra textbooks. After defining the irrational numbers, most books like to give an example, something they can point to and say “Look, readers, behold an irrational number.” To most authors the first thing that comes to their mind when searching for an example of an irrational number is the square root of 2. It is a familiar number and it is not too difficult to demonstrate its irrationality. The “standard proof” that $\sqrt{2}$ is irrational is to factor both sides of the integer equation

$$a^2 = 2b^2$$

into primes. The exponent of the prime 2 of the number a^2 will be even, while the exponent of the prime 2 of the number $2b^2$ will be odd, an obvious contradiction. This argument is shorter than the descent argument. On the other hand, it requires that we know about factorization into primes. The descent argument does not rely on a knowledge of primes and is in this regard a simpler proof.

1.4 Exercises

1. Use the descent method to show that $\sqrt{3}$ is irrational. The idea is to follow the proof for $\sqrt{2}$: Assume $a^2 = 3b^2$, where a and b are positive integers, and then define $A = 3b - a$, $B = a - b$.

2. What happens if we try to use the method in Exercise 1 above to show that $\sqrt{4}$ is irrational?
3. One of Fermat's discoveries was to find a test for deciding whether a prime can be written as the sum of two squares. For example, $5 = 1^2 + 2^2$, but 7 is not the sum of two squares. Make a list of the primes up to 100. There should be 25 primes in your list, starting with 2 and ending with 97. Determine which of these 25 primes are the sum of two squares. Do you see the pattern that Fermat saw?
4. Fermat's habit of omitting proofs ultimately caused quite a great stir for centuries to come. Look up "Fermat's Last Theorem" in an encyclopedia or on the internet. What was the problem? Who solved it and when?