

MODULE 4

A Foundation for Geometry

There is no royal road to geometry – Euclid

1. Points and Lines

We are ready (finally!) to talk about geometry. Our first task is to say something about *points* and *lines*. We will assume that we have a (nonempty) space X and that the **points** in our geometry are just elements of X . There are no restrictions (for now) on the set X . It could be a finite set, the real number line, the x - y plane, or even three dimensional space. We assume further that we have a set $\mathcal{L}$ of subsets of X . The elements of $\mathcal{L}$ are called the **lines** of our geometry.

As with metric spaces we need some axioms or rules about the possible points and lines of our geometry. The first three rules are called the **incidence axioms**. They are very basic.

Incidence Axioms

Axiom 1. There are at least two different lines in $\mathcal{L}$.

Axiom 2. Each line contains at least two different points.

Axiom 3. For any two distinct points A and B , there is exactly one line ℓ in $\mathcal{L}$ which contains them.

The third axiom is often stated as “two points determine a line” and dates back to Euclid. The unique line determined by A and B is written $\overleftrightarrow{AB}$. Points which lie on the same line are called **collinear**.

Our first theorem in geometry is an easy consequence of Axiom 3

Theorem 4.1. Distinct lines m and n intersect in at most one point

4.1 Exercises

1. Prove Theorem 4.1.
2. Prove that there must be at least three points in X .

2. Distance and Betweenness

We will assume that our space has a distance function d defined on it which obeys the four distance axioms of a metric space.

It is convenient to use a simplified notation for the distance between two points. Assuming that we are using the distance function d , then we shall denote the distance $d(A, B)$ between two points A and B by

$$AB \stackrel{\text{def}}{=} d(A, B).$$

We use the topological axioms of distance as a springboard into geometry. In terms of this simplified notation, the four distance axioms can be stated.

Distance Axioms

Axiom 1. $AB \geq 0$ [non-negativity]

Axiom 2. $AB = 0$ if and only if $A = B$ [zero property]

Axiom 3. $AB = BA$ [symmetry]

Axiom 4. $AC \leq AB + BC$ [triangle inequality]

Remark. Most authors do not include Axiom 4, the triangle inequality, since it is possible to prove the triangle inequality from the other axioms of geometry. Since we are not critically interested in developing a minimal set of axioms here, it suits our purpose to assume a priori that our distance function d satisfies the triangle inequality.

Given three distinct points A , B , or C on a line, we often need to know in which order they appear. In general we expect that one of them will lie *between* the other two. But the question is: which one is in the middle? The Greek geometers did not define the concept of “betweenness,” relying on diagrams instead.

We will use the notation $A - B - C$ to designate that A , B , and C are collinear points and that B is between A and C .

We can use distance to define **betweenness**, as follows:

Definition 4.1. $A - B - C$ if and only if A , B , C are three distinct collinear points and

$$AB + BC = AC.$$

The following theorem formalizes the intuitively obvious statement that if three points line up in order A, B, C , then going backwards they are in order C, B, A .

Theorem 4.2. $A - B - C$ implies $C - B - A$.

The definition of betweenness does not assert that a betweenness relation must hold between three collinear points. It merely defines the condition that must be met if a betweenness relation actually occurs. The existence of a betweenness relation requires another axiom.

Betweenness Axiom. Given three distinct collinear points. At least one betweenness relation $A - B - C$, $A - C - B$, or $B - A - C$ holds.

The next theorem says that only one of the points A , B , or C can lie in the middle. For this reason, we call this the “unique middle” theorem.

Theorem 4.3. [unique middle] If $A - B - C$ is true, then $A - C - B$ and $B - A - C$ are both false.

4.2 Exercises

1. Prove Theorem 4.2.
2. Prove Theorem 4.3.

3. Examples of Geometries

Example 4.1. [The Euclidean plane] Let the space X be the x - y plane $\{(x, y) : x \text{ and } y \text{ are real numbers}\}$. The lines in $\mathcal{L}$ are sets of the form

$$\{(x, y) : ax + by = c\},$$

where a , b , and c are real numbers, not all zero. The usual definition of distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane is

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 4.2 (The x - y plane with the taxicab metric). X and $\mathcal{L}$ are defined as in Example 4.1. The distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane is

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

Example 4.3 (Lines on a sphere). Let X be a sphere and suppose that we restrict travel to the surface of the sphere. Then the distance between two points P and Q on the sphere is measured along the great circle of the sphere which passes through the two points. A great circle is a circle on the sphere whose center is the center of the sphere. Note that because of a technicality, the third incidence axiom does not hold on a sphere. Specifically, if A is the North Pole and B is the South Pole, then any longitudinal line passes through these two points. It is possible to modify the incidence axioms to allow for this exception.

Example 4.4 (Discrete space). Let X be *any* set with at least three elements. The set $\mathcal{L}$ consists of all possible two element sets $\{a, b\}$ where a and b lie in X . The following function is a distance function:

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b. \end{cases}$$

Note that it is impossible to find three distinct points on any line in this model, so no betweenness relations hold.

Example 4.5 (The Fano Plane). The Fano plane $\mathcal{F}$ is a collection of 7 points

$$X = \{A, B, C, D, E, F, G\}$$

and 7 lines, determined by the drawing in Figure 1.

The seven lines are

- (1) A, B, D
- (2) A, C, E
- (3) B, C, F
- (4) A, F, G
- (5) B, E, G
- (6) C, D, G
- (7) D, E, F

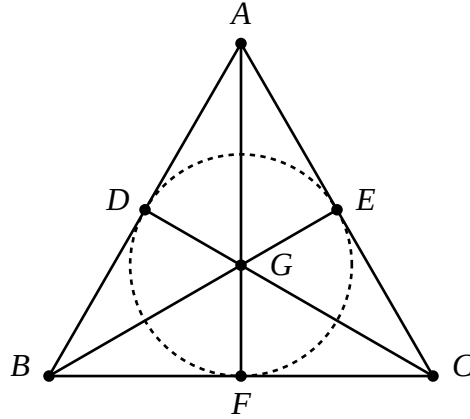


FIGURE 1. The Fano plane

The solid and dotted lines in the figure are only to aid visualization of the lines. There are only seven points and seven lines in the Fano plane. The Fano plane is also called the **projective plane** of order 2.

Distances are assigned so that the distance function satisfies the zero property and symmetry, that is, the distance from a point to itself is 0 and the distance from point x to point y is the same as the distance from point y to point x . Six distances are defined to be 2:

$$AB = AC = AF = BE = BC = CD = 2$$

The remaining distances $d(x, y)$, when $x \neq y$, have the value 1.

4.3 Exercises

1. On a sphere of radius 13, what is the distance between $A = (0, 0, 13)$ and $(3, 4, 12)$?
2. We have defined lines in the x - y plane with the standard distance function to be the same as lines in the x - y plane using the taxicab metric. Show that if A , B , and C are three distinct points on a line in the x - y plane, then the betweenness relation $A - B - C$ holds using the standard distance function if and only if $A - B - C$ holds using the taxicab metric.
3. What are the betweenness relations for the discrete space?

4. Ignoring symmetry, there are six betweenness relations in the Fano Plane. Find them.
5. Show that the Fano plane $\mathcal{F}$ has the following interesting property:

Property (*): Any line passing through any two points in $\mathcal{F}$ passes through a third point in $\mathcal{F}$.

You will never find a finite collection of *non-collinear* points in the Euclidean x - y plane that satisfy Property (*).

6. The Fano plane has a remarkable coloring property. To warm up on an easier example, consider the following three pairs of letters: (1) $\{A, B\}$, (2) $\{A, C\}$, and (3) $\{B, C\}$. Now suppose you color each letter, using exactly two colors, red and blue. Each letter must be colored the same in every pair it appears. For example, if you color A in the first pair red, then you must color A in the second pair red also. Explain why, no matter how the three letters are colored, at least one of the three pairs will have the same color.

Suppose we switch from pairs to triples. Can we construct an example similar to the three pairs displayed above? The key is to use the Fano plane.

Show that no matter how you color the seven points in the Fano plane $\mathcal{F}$ using two colors (red and blue), it will always be the case that at least one of the seven lines will contain three points all of the same color.

4. Segments and Rays

We can use the notion of betweenness to define segments and rays in our geometry. Suppose X is a set of points, d is a distance function on X , and $\mathcal{L}$ is the set of lines.

Definition 4.2 (line segment $\overline{AB}$). If A and B are distinct points in X , then

$$\overline{AB} \stackrel{\text{def}}{=} \{A, B\} \cup \{P : A - P - B\}.$$

The points A and B are called the **endpoints** of the segment $\overline{AB}$. A segment consists of its endpoints and the points in between them. The points strictly between A and B are called the **interior** points of $\overline{AB}$.

Definition 4.3 (ray $\overrightarrow{AB}$). If A and B are distinct points in X , then

$$\overrightarrow{AB} \stackrel{\text{def}}{=} \overline{AB} \cup \{P : A - B - P\}.$$

The starting point A is called the **endpoint** of ray $\overrightarrow{AB}$. The points on $\overrightarrow{AB}$ other than A are called the **interior** points of $\overrightarrow{AB}$.

The following three theorems, which are intuitively obvious if you draw diagrams, can be proven readily using our definitions of line segment and ray.

Theorem 4.4. $\overline{AB} = \overline{BA}$

Theorem 4.5. $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$

Theorem 4.6. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$

We would like to have an axiom which says that our rays “look like” rays on the real numbers line. This is accomplished by the following axiom.

The Ruler Axiom. Given a ray $\overrightarrow{AB}$ and a real number $t \geq 0$. Then there is a point X on $\overrightarrow{AB}$ such that

$$\boxed{AX = t.}$$

The Ruler Axiom can be used to prove the following theorem about midpoints:

Theorem 4.7. [midpoint] Given segment $\overline{AB}$ there is a unique point M on $\overline{AB}$, called the **midpoint** of $\overline{AB}$, such that

$$AM = MB.$$

4.4 Exercises

1. Write an equation of the form $y = mx + b$ to describe the segment $\overline{AB}$ in $\mathbb{R}^2$ with $A = (0, 0)$ and $B = (1, 2)$.
2. Let ℓ be the line $y = 2x + 1$ in the x - y plane. (a) Find two point on ℓ whose Euclidean distance from the point $(0, 1)$ is precisely 5. (b) Find two point on ℓ whose taxicab distance from the point $(0, 1)$ is precisely 5.
3. Is it possible to find two points X_1 and X_2 on $\overleftrightarrow{AB}$ whose distance from A is t ?
4. Is it possible to find two points X_1 and X_2 on $\overrightarrow{AB}$ whose distance from A is t ?
5. Prove Theorem 4.4. Hint: Use Theorem 4.2.
6. Prove Theorem 4.5. Hint: Use Theorem 4.3.
7. Prove Theorem 4.6. Hint: Use the Betweenness Axiom.

8. Prove Theorem 4.7.

5. Dimension

If we have a betweenness relation $B - A - C$ then the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called **opposite** rays. Notice that the ray $\overrightarrow{AB}$ together with the opposite ray $\overrightarrow{AC}$ comprise the entire line $\overleftrightarrow{AB}$. Since the two rays meet only at the endpoint A , we see that a point A on a line ℓ separates the line into three disjoint pieces:

1. the point A itself
2. the interior of the ray $\overrightarrow{AB}$ lying on ℓ
3. the interior of the opposite ray $\overrightarrow{AC}$ lying on ℓ .

In a similar way a line ℓ separates the plane into three disjoint pieces:

1. the line ℓ itself
2. a halfplane on one side of ℓ
3. the opposite halfplane on the other side of ℓ .

This separation fact is one of the basic axioms used in studying the foundations of geometry. Rather than try to state a precise definition of the separation axiom, we will just contend ourselves with this informal description.

It is separation by lines that makes our geometry two dimensional. Consider the three dimensional world we live in. To locate a point in our world we must specify three values: width, depth, and height. All the other axioms of geometry hold in three dimensional space: two points determine a line; the Betweenness Axiom; the Ruler Axiom. Segments and rays can be defined in three space using the notion of betweenness. A line, however, does not separate three dimensional space into three disjoint pieces. Separation in three space requires a plane, which acts like a wall, dividing the space into the wall itself and the two regions on either side of the wall.

The idea of dimension works as follows. A point has dimension 0; it has no width, depth, or height. A line is one dimensional; it can be separated into three pieces by a 0 dimensional point. A plane has two dimensions; it can be separated by a one dimensional line. Three dimensional space is separated by a two dimensional plane.

4.5 Exercises

1. Consider the $x - y$ plane and take ℓ to be the line $y = 2x + 1$. Describe the two halfplanes separated by ℓ .
2. Consider the sphere of the earth. What “halfplanes” are separated by the equator?
3. Given two random lines in three dimensional space, do you think it is very likely that they intersect?
4. Think of the fourth dimension as time. Explain how a three dimension subspace of space-time at time $t = 0$ separates space-time into three distinct parts.