

MATH 229 LECTURE NOTES: THE CHAIN RULE

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1. RELATIVE WEIGHTS

In terms of pounds,

animal	average weight
elephant	9,000 pounds
bear	600 pounds
human	180 pounds

Thus an elephant weighs 15 times more than a bear

a bear weighs $3\frac{1}{3}$ times more than a human

Question: How much more does an elephant weigh than a human?

Answer: Multiply: An elephant weighs

$$15 \times \frac{10}{3} = 50$$

times as much as a human

2. RELATIVE SPEEDS

In terms of mph (miles per hour),

vehicle	average speed
airplane	600 mph
car	60 mph
bicycle	15 mph
walker	3 mph

Thus a plane travels 10 times faster than a car

a car travels 4 times faster than a bike

a bike travels 5 times faster than a person walking

Question: How much faster is traveling in an airplane compared to walking?

Answer: Multiply: An airplane travels

$$10 \times 4 \times 5 = 200$$

times as fast as a person on foot.

3. THE MORAL

The moral to these two examples is that

Rates Multiply!

This is a famous rule of calculus, called the chain rule which says

If we have three variable x , y , and z ,

If z is changing m times faster than y

and y is changing n times faster than x ,

then z is changing $m \cdot n$ times faster than x .

4. THE CHAIN RULE

Since the derivate tells us the rate of change, the fact that rates multiply can be written succintly as

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

This formula is called the **Chain Rule**

When measuring weights, our first example becomes

$$\frac{d \text{ elephant}}{d \text{ human}} = \frac{d \text{ elephant}}{d \text{ bear}} \cdot \frac{d \text{ bear}}{d \text{ human}}$$

5. THE CHAIN RULE CONTINUED

When measuring speed, the second example shows that you can string several rates together:

$$\frac{d \text{ plane}}{d \text{ pedestrian}} = \frac{d \text{ plane}}{d \text{ car}} \cdot \frac{d \text{ car}}{d \text{ bike}} \cdot \frac{d \text{ bike}}{d \text{ pedestrian}}$$

which illustrates why the rule is called the chain rule.

6. USING THE CHAIN RULE

Suppose

$$z = y^2 \text{ and } y = x^3 + 11$$

Use the chain rule to find $\frac{dz}{dx}$

$$\begin{aligned} \frac{dz}{dx} &= \frac{dz}{dy} \cdot \frac{dy}{dx} \\ &= 2y \cdot 3x^2 = 2(x^3 + 11) \cdot 3x^2 \end{aligned}$$

7. DIRECT CALCULATION

$$z = y^2 \text{ and } y = x^3 + 11$$

$$\text{So } z = (x^3 + 11)^2 = x^6 + 22x^3 + 121$$

$$\text{So } \frac{dz}{dx} = 6x^5 + 66x^2$$

This is equivalent to our previous solution:

$$\begin{aligned} \frac{dz}{dx} &= 2(x^3 + 11) \cdot 3x^2 = 6x^2(x^3 + 11) \\ &= 6x^5 + 66x^2 \end{aligned}$$

8. PRIME NOTATION VERSION

Suppose we combine two functions f and g to get

$$f \circ g(x) = f(g(x))$$

Then the Chain Rules becomes

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

In words this says: the derivative of a composition is the derivative of the outer function evaluated at the inner function **times** the derivative of the inside part.

9. EXAMPLES

Suppose $f(x) = x^2$ and $g(x) = x^3 + 11$

Then $f'(x) = 2x$ and $g'(x) = 3x^2$

$$\begin{aligned} \text{So } \frac{d}{dx}f(g(x)) &= f'(g(x)) \cdot g'(x) \\ &= 2g(x) \cdot 3x^2 \\ &= 2(x^3 + 11) \cdot 3x^2 \end{aligned}$$

We have seen this example before.

10. SECOND EXAMPLE

Differentiate $y = \sqrt{7x^3 + x^2 + 1}$ We are using two functions:
 $f(x) = \sqrt{x}$ and $g(x) = 7x^3 + x^2 + 1$.

Since $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = 21x^2 + 2x$

$$\begin{aligned} \text{So } \frac{d}{dx}f(g(x)) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2}(7x^3 + x^2 + 1)^{-1/2} \cdot (21x^2 + 2x) \end{aligned}$$

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11. CHAINING THE CHAIN RULE

Differentiate $f(x) = \sqrt{x + \sqrt{x + 1}}$

Here we must use the chain rule twice.

The secret is to write the square roots as exponents:

Differentiate $f(x) = (x + (x + 1)^{\frac{1}{2}})^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(x + (x + 1)^{\frac{1}{2}})^{-\frac{1}{2}} \cdot \frac{d}{dx}(x + (x + 1)^{1/2})$$

$$f'(x) = \frac{1}{2}(x + (x + 1)^{\frac{1}{2}})^{-\frac{1}{2}} \cdot \left(1 + \frac{d}{dx}(x + 1)^{1/2}\right)$$

$$f'(x) = \frac{1}{2}(x + (x + 1)^{\frac{1}{2}})^{-\frac{1}{2}} \cdot \left(1 + \frac{1}{2}(x + 1)^{-1/2}\right)$$

12. THE RULES COMBINED

Differentiate $f(x) = x\sqrt{2x + 1}$

By the product rule , chain rule, and power rule,

$$\begin{aligned} f'(x) &= 1 \cdot \sqrt{2x + 1} + x \cdot \frac{d}{dx}(2x + 1)^{1/2} \\ &= \sqrt{2x + 1} + x \cdot \frac{1}{2}(2x + 1)^{-1/2} \cdot \frac{d}{dx}(2x + 1) \\ &= \sqrt{2x + 1} + \frac{1}{2}x(2x + 1)^{-\frac{1}{2}} \cdot 2 \\ &= \sqrt{2x + 1} + x(2x + 1)^{-\frac{1}{2}} \end{aligned}$$

13. THE RULES COMBINED II

Differentiate $f(x) = \sqrt{\frac{x}{x + 1}}$

By the chain rule , power rule, and quotient rule,

$$f'(x) = \frac{1}{2} \left(\frac{x}{x + 1} \right)^{-1/2} \cdot \frac{d}{dx} \left(\frac{x}{x + 1} \right)$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{x}{x+1} \right)^{-\frac{1}{2}} \frac{\frac{d}{dx}(x) \cdot (x+1) - x \cdot \frac{d}{dx}(x+1)}{(x+1)^2} \\
&= \frac{1}{2} \left(\frac{x}{x+1} \right)^{-\frac{1}{2}} \frac{1(x+1) - x(1)}{(x+1)^2} \\
&= \frac{1}{2} \left(\frac{x}{x+1} \right)^{-\frac{1}{2}} \frac{1}{(x+1)^2}
\end{aligned}$$