

MODULE 2

Logic

Logic is the beginning of wisdom, not the end. – Mr. Spock

Five habits of a mathematician are the ability to (1) develop intuition, (2) gain experience, (3) make connections, (4) think abstractly, and (5) look for structure. At the very foundation of all these characteristics of a mathematician is **logic**.

Elementary logic provides the basic rules for constructing sound, compelling arguments. The building blocks of logical arguments are logical statements. A **logical statement** is a declarative sentence which conveys factual information. If the information is correct then we say the statement is **true**; and if the information is incorrect, then we say the statement is **false**.

In this module we will study how logical statements are combined and used to make correct arguments.

1. Statements and negations

All logical statements are formed by combining simple statements.

A **simple statement** is a logical statement carrying one piece of information.

Example 2.1. Each of the following is a simple statement:

- (1) The United States Post Office delivers mail.
- (2) Paris is the capital of Tanzania.
- (3) The polynomial $x^2 - 1$ factors as $(x - 1)(x + 1)$.

Statements 1 and 3 are true. Statement 2 is false.

Not all short sentences are simple statements

Example 2.2.

- (1) *Close the window.* This is a command. It does not convey information and is not a logical statement.
- (2) *The United States Congress is doing an excellent job.* This is a statement of opinion, not a logical statement.

The **negation** of a logical statement is a new logical statement which says the opposite of the original statement. The statement

$$\text{not } p$$

is true exactly when the original statement p is false. To **negate** a logical statement means to find the statement's negation. Various ways to form the negation of a statement are discussed in the next example.

Example 2.3. We could negate the statement

the sky is blue

by forming the statement

it is not the case that the sky is blue.

We can negate any statement in this way, but such constructions are clumsy and sometimes unclear. The negation

the sky is *not* blue

is much clearer.

A statement containing a negative can often be negated by removing the negative.

Example 2.4.

- (1) The negation of

loitering is *not* permitted here

is

loitering is permitted here.
- (2) The negation of

e-mail is *unreliable*
 is
 e-mail is reliable.

When a logical statement is used to say something about a collection of objects, the statement must be **quantified**—that is, we must specify which objects in the collection the statement applies to. The basic distinctions between applications are reflected in the words **all** and **some**.

If a logical statement applies to *all* objects in a collection, then it is called a **universally quantified** statement. For example,

- (1) *Every* McDonald's serves french fries.
- (2) *All* math majors study calculus.

A logical statement which applies to *some* objects in a collection is called an **existentially quantified** statement. For example,

- (1) *Some* people attend college.
- (2) *There are* people who believe in UFO's.

As illustrated above, an existentially quantified statement asserts that an object of a particular nature exists.

We can contrast universally quantified statements and existentially quantified statements in the following way.

- **Universal quantification**—If every object in the collection fits the description, then the universally qualified statement is true.
- **Existential quantification**—If the existentially quantified statement accurately describes at least one object in the collection, then it is true.

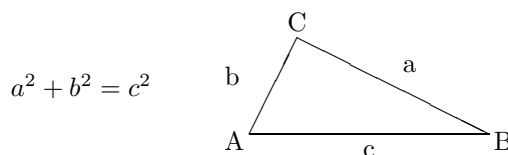
Example 2.5.

- (1) *Some* people are female. This existentially quantified statement applies to the collection of people, and states the existence of a person with the property of being female. It is clearly a true statement.

- (2) *All* people are female. This universally quantified statement applied to the collection of people. Since there are people who are not female, it is a false statement.
- (3) *Every* whole multiple of four is even. This universally quantified statement applies to the collection of all whole multiples of four. It is true since four is even, and every multiple of an even number is even.

Mathematical knowledge is often expressed with quantified statements:

- (1) For all real numbers x and y , $x + y = y + x$.
- (2) All pairs of non-parallel lines in the plane intersect.
- (3) Some numbers can be factored (for example, $6 = 2 \times 3$, $15 = 3 \times 5$)
- (4) All right triangles obey the Pythagorean Theorem



The negation of a *for all* statement is a *some* statement.

Example 2.6. The negation of

All birds can fly

is

Some birds *cannot* fly.

The negation of a *some* statement is a *for all* statement.

Example 2.7. The negation of

There exists an honest man

is

All men are *dishonest*.

Example 2.8. Consider the statement

All numbers can be factored.

A universally quantified statement is false if it does not apply to at least one object in the collection. So the statement is false if some number cannot be factored. We know that

7 cannot be factored (with factors > 1), so the statement is false. The negation of this universally quantified statement is

Some numbers *cannot* be factored.

This existentially quantified statement is true.

2. Combining Logical Statements

We combine logical statements to form new logical statements by joining them with an *and* or an *or*.

and

The statement p *and* q is true exactly when both p and q are true.

It is convenient to display this information in the following chart:

The **and** Chart:

p	q	p <i>and</i> q
T	T	T
T	F	F
F	T	F
F	F	F

The chart lists the four possible truth values of the two sentences p and q :

- (1) both are true;
- (2) p is true, but q is false
- (3) p is false, but q is true
- (4) both are false.

In just the first case is the statement p *and* q true.

or

“Or” is used in two different ways in English:

- *exclusive use*: Tea *or* Coffee?
- *inclusive use*: Cream *or* Sugar?

When you are offered “tea or coffee,” it is assumed that you may choose one of the two, *but not both*. On the other hand, when offered “cream or sugar,” no eyebrows will be raised if you take them both.

Example 2.9. Determine whether the following uses of *or* are inclusive or exclusive.

- (1) You can kill a vampire by exposing him to sunlight *or* driving a wooden stake through his heart. Suppose you drive a stake through the heart of a vampire at exactly sunrise? Is he still dead?
- (2) You buy a car with a 3 year *or* 36,000 mile warranty. If more than 3 years pass *and* you drive the car more than 36,000 miles, is the warranty still expired?
- (3) A restaurant offers “soup or salad” with their dinner special. Can you order both?

The first two are examples of the inclusive use of *or*. The third is an example of the exclusive use. You won’t get both the soup and the salad without paying an extra charge.

In logic we always use “or” in the inclusive sense. In other words, for us

or means *and/or*.

The **or** Chart:

p	q	$p \text{ or } q$
T	T	T
T	F	T
F	T	T
F	F	F

The negation of an *and* statement is an *or* statement.

Example 2.10. The negation of

Sam likes green eggs *and* Sam likes ham

is

Sam does *not* like green eggs *or* Sam does *not* like ham.

The negation of an *or* statement is an *and* statement.

Example 2.11. The negation of

I am going to the movies on Saturday *or* Sunday

is

I am *not* going on Saturday *and* I am *not* going on Sunday.

More complicated logical statements may be formed by further combining statements already joined by *and* or *or*.

Example 2.12. The following nonpunctuated statement was found on a menu.

All dinners include vegetable *and* soup *or* salad.

Such a statement can be interpreted in two different ways:

- (1) You get the vegetable and you must decide between the soup or the salad.
- (2) You can must decide between the vegetable and soup combination or the salad.

Question: Which meaning do you think the menu intends?

3. Implication

The statement “*p implies q*” means that if *p* is true, then *q* must also be true. The statement “*p implies q*” is also written “if *p* then *q*” or sometimes “*q* if *p*.” Statement *p* is called the **premise** of the implication and *q* is called the **conclusion**.

Example 2.13. Each of the following statements is an implication:

- (1) *If* you score 85% or above in this class, *then* you will get an A.
- (2) *If* the U.S. discovers that the Taliban Government is involved in the terrorist attack, *then* it will retaliate against Afghanistan.
- (3) My thumb will hurt *if* I hit it with a hammer,
- (4) $x = 2$ *implies* $x + 1 = 3$.

You can view Statement 1 above as a promise. It says

You are *guaranteed* an A *provided* you score 85% or above.

Suppose you score a 90% in the class. If your final grade is an A, then the promise was kept and Statement 1 is true. If your grade is not an A, then the promise was broken and Statement 1 is false.

But what if you do not score 85% or above? Is Statement 1 true or false in this case? Statement 1 does not say what grade you will receive if you score less than 85%. If you score 75% in the class and receive a B, you cannot complain that the promise was broken. If you score 84% and end up with an A, you still cannot say that the promise was broken.

When the premise p of the implication “ p *implies* q ” is false, we are forced into a corner. We cannot say that the implication is false, yet we have no evidence that it is true—because p didn’t happen. This situation is reminiscent of the following dialogue from the Three Stooges:

Larry: I couldn’t say ‘yes’ and I couldn’t say ‘no’.

Curly: Could you say ‘maybe’?

Larry: I might.

Moe: [Hits them both on the head.]

Logicians have decided to take an “innocent until proven guilty” stance on this issue. An *if—then* statement is considered true until proven false. Since we cannot call the statement p *implies* q false when p is false, our only alternative is to call it true.

So the chart for *implies* is:

The **if—then** Chart:

p	q	p <i>implies</i> q
T	T	T
T	F	F
F	T	T
F	F	T

We emphasize again the surprising fact that

a false statement *implies* anything.

Example 2.14.

- (1) Does $2 = 3$ *imply* $2 + 1 = 3 + 1$?

Yes, it’s an example of the rule $x = y$ *implies* $x + 1 = y + 1$.

- (2) Does $2 = 3$ *imply* $2 \cdot 0 = 3 \cdot 0$?

Yes, it’s an example of the rule $x = y$ *implies* $xz = yz$.

Example 2.15. Consider the following implications

- (1) If elephants can fly, then the Cubs will win the World Series this year.
- (2) If elephants can fly, then the Cubs will lose the World Series this year.

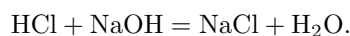
Both statements are true—assuming, of course, that elephants can't fly.

Warning to die hard Cubs fans: please do not take Statement 1 too seriously and push elephants out of trees, hoping that one might fly.

Example 2.16. Consider the following implications

- (1) If hydrochloric acid (HCl) and sodium hydroxide (NaOH) are combined, then table salt (NaCl) will be produced.
- (2) If March has 31 days, then dogs are mammals.

Both statements are true. The first statement is an example of cause and effect and reflects the chemical equation



In the second statement, there is clearly no causal relation between days of the month and dogs being mammals. Both the premise and the conclusion of Statement 2 are true, so according to the chart, the implication is true. The point here is that the use of *implies* in logic is very different from its use in everyday language to reflect causality.

From the chart we see that the implication *if p then q* is false when it happens that *p* is true, but *q* is false. This leads us to the surprising conclusion that

the negation of an implication is an *and* statement.

The negation of the statement

p implies q

is the statement

p and not q.

Example 2.17.

- (1) The negation of

if I hit my thumb with a hammer, then my thumb will hurt

is

I hit my thumb with a hammer *and* my thumb does *not* hurt.
- (2) The negation of

if Sosa is traded, *then* Cubs attendance will drop
is

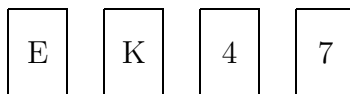
Sosa is traded *and* the Cubs attendance does *not* drop.

Example 2.18. The Four Card Problem

You are shown one side of four cards. You are told that each card has a number on one side and a letter on the other side. You are to test the rule:

if a card has a vowel on one side, then it has an even number on the other side.

You want to find out whether the rule is true or false. What are the only cards you need to turn over to test the rule (and why) if the cards pictured in front of you are:



[The answer is given at the end of this section.]

The *converse* of the statement

$p \text{ implies } q$

is the statement

$q \text{ implies } p.$

Example 2.19. The converse of

if an elephant walks in Mom's garden, *then* her tomato plants will be ruined
is

if Mom's tomato plants are ruined, *then* an elephant was walking in her garden.

It is clear from this example that a statement is **not** logically equivalent to its converse. There could be other causes for ruined tomato plants besides promenading pachyderms.

The **inverse** of the statement

$p \text{ implies } q$

is the statement

$\text{not } p \text{ implies not } q.$

The **contrapositive** of the statement

$p \text{ implies } q$

is the statement

$\text{not } q \text{ implies not } p.$

Example 2.20.

Statement:	If the gloves fit, then the jury will acquit.
Converse:	If the jury acquits, then the gloves fit.
Inverse:	If the gloves don't fit, then the jury won't acquit.
Contrapositive:	If the jury doesn't acquit, then the gloves don't fit.

It can be shown by using truth charts that

a statement is logically equivalent to its contrapositive

and

the converse is logically equivalent to the inverse.

Example 2.21. The statement

if I hit my thumb with a hammer, *then* my thumb will hurt

is logically equivalent to

if my thumb doesn't hurt, *then* I didn't hit it with a hammer.

Example 2.22. Stated as an *if-then* sentence, the Golden Rule becomes

if you want other people to act in a certain way to you, then you should act that way towards them.

The inverse of the Golden Rule is

if you don't want other people to act in a certain way to you, then you should not act that way towards them.

Many people consider the inverse (or equivalently, the converse) of the Golden Rule to be a more reasonable moral law.

The **biconditional** statement

$$p \text{ if and only if } q$$

means that both p and q are true or else they are both false.

The **if and only if** Chart:

p	q	p if and only if q
T	T	T
T	F	F
F	T	F
F	F	T

The biconditional p if and only if q is logically equivalent to saying p implies q and q implies p .

Example 2.23. You are eligible to vote in a United States election if and only if you are a United States citizen, 18 years or older, and not a convicted felon.

The answer to the four-card problem is: you only need to turn over cards ‘E’ and ‘7’.

2 Exercises

1. Discuss the difference between the following two statements:

- (1) All geometry books are not alike.
- (2) Not all geometry books are alike.

2. Write an “or” statement which is the negation of the following

- (1) Daily brushing and regular checkups help prevent tooth decay.
- (2) You will go to the dance and have a good time.
- (3) The number n is prime and the number n even.
- (4) Sticks and stones will break my bones, but words will never hurt me.

3. Write an “and” statement which is the negation of the following

- (1) Your prize will be a color tv, a laptop computer, or a coffee mug.
- (2) The defendant is either not telling the truth or else he is not very smart.
- (3) I will study math on Saturday or Sunday.
- (4) If the two diagonals have the same length, then all four angles are 90° .
- (5) If a man answers, then the caller hangs up.

4. Find the negation of the following

- (1) All students study for exams.
- (2) If the sky is green, then a tornado has touched down.

(3) Some politicians are honest and sincere.

5. True or false? (Assume standard facts.)

- (1) George W. Bush is president or McDonald's is a restaurant.
- (2) If Al Gore is president then the Cubs won the 2003 World Series.
- (3) If Al Gore is president then the Cubs did not win the 2003 World Series.

6. Assume the truth values of the following statements:

“Bleebbs are quills” is true

“Blarbs are snarfs” is false

“Blairs are mares” is true.

Determine whether the following statements are true or false:

- (1) Bleebbs are quills or Blairs are mares.
- (2) If Blairs are mares then Blarbs are snarfs.
- (3) If Blarbs are snarfs then Blairs are mares.
- (4) If Bleebbs are not quills then both Blarbs are not snarfs and Blairs are not mares.

7. Find the converse, inverse, and contrapositive of the following sentence

- (1) If weapons of mass destruction are stored in Iraq, then the U.S. troops will find them.
- (2) If it's raining, then the streets are wet.
- (3) If blarbs are snarfs, then hameeds are steeds.

8. State as an “if ... then ...” sentence:

- (1) Where there's smoke, there's fire.
- (2) Beer sales end after the 7th inning.
- (3) People who go to college make more money than people who have not.
- (4) You may purchase beer provided you are 21 or older.
- (5) One gets wiser with age.
- (6) The Golden Rule
- (7) The converse of the Golden Rule
- (8) No shoes, no shirt, no service.

9. Determine whether the following pairs of statements are connected by an implication, biconditional, or neither:

- (1) two angles of a triangle are congruent
and
two sides of a triangle are congruent
- (2) $ABCD$ is a square
and
 $ABCD$ is a rectangle
- (3) x and y are even
and
 $x + y$ is even
- (4) x is a prime number
and
 x is an odd number
- (5) the real number x has a repeating decimal expansion
and
 x is a rational number