

MATH 229 LECTURE NOTES: CHAPTER 3
CONCAVITY AND SKETCHING GRAPHS

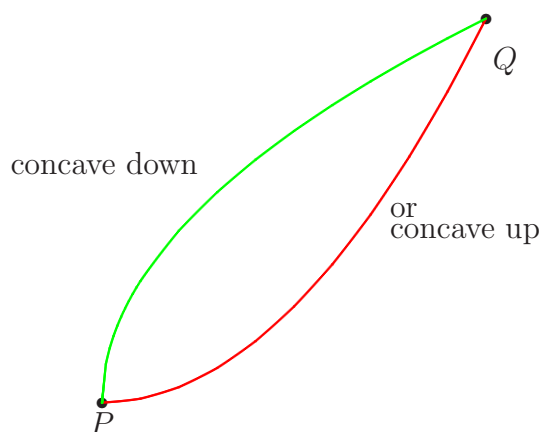
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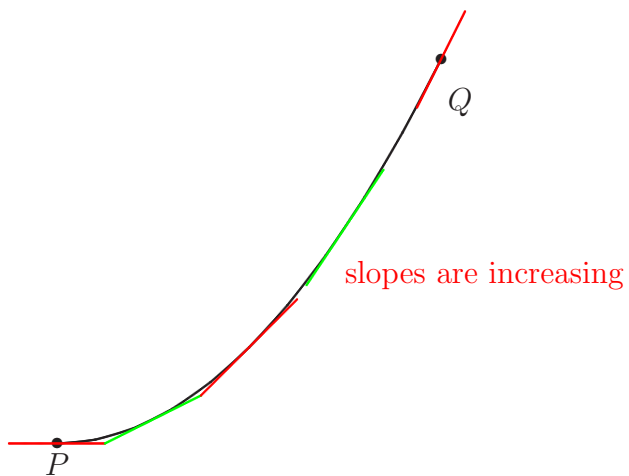
1. SHAPE OF A CURVE

There are two ways that a curve can increase from point P to point Q

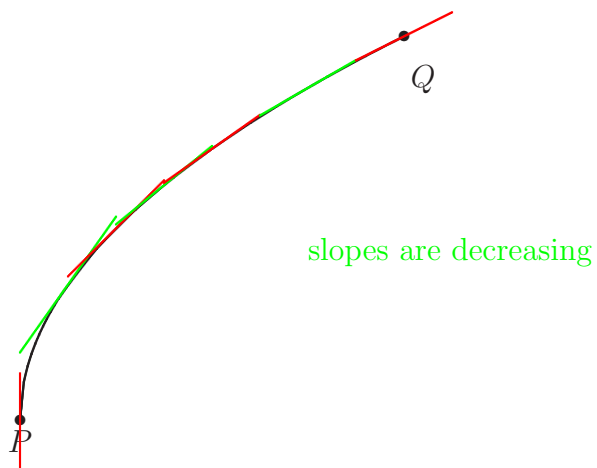


2. CONCAVE UP GRAPHS

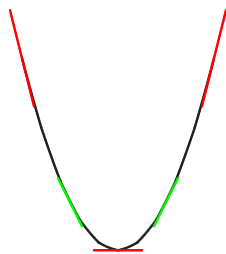
When a graph is concave up, the slopes get progressively larger.



3. CONCAVE DOWN CURVE



4. CONCAVE UP CURVE



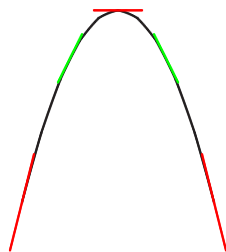
y' is increasing

$$\iff (y')' > 0$$

$$\Longleftrightarrow y'' > 0$$

Thus the graph is concave up if and only if $y'' > 0$

5. CONCAVE DOWN CURVE



y' is decreasing

$$\Longleftrightarrow (y')' < 0$$

$$\Longleftrightarrow y'' < 0$$

Thus the graph is concave down if and only if $y'' < 0$

A point at which concavity changes is called an **inflection point**

6. EXAMPLE - REVISITED

$$y = f(x) = x^3 - 6x^2 + 9x + 5$$

$$y' = f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3)$$

Based on this factorization we saw earlier that

Interval	Behavior of graph
$(-\infty, 1)$	increasing
$(1, 3)$	decreasing
$(3, \infty)$	increasing

7. DETERMINING CONCAVITY

$$y = f(x) = x^3 - 6x^2 + 9x + 5$$

$$y' = f'(x) = 3x^2 - 12x + 9$$

$$y'' = f''(x) = 6x - 12 = 6(x - 2)$$

To determine the sign of $f''(x)$ use the following chart

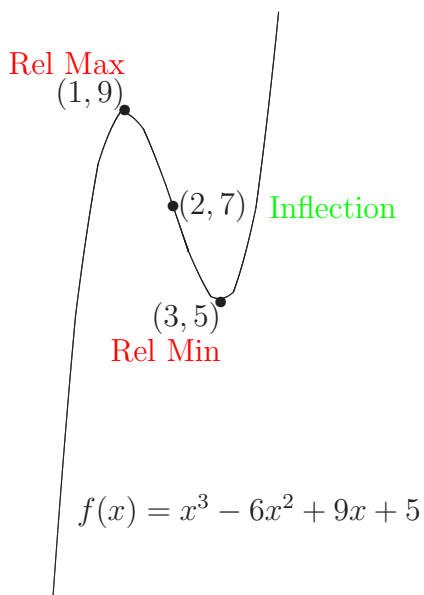
Interval	$(-\infty, 2)$	$(2, \infty)$
$x - 2$	neg	pos
Sign of y''	$y'' < 0$	$y'' > 0$
Interpretation	concave down	concave up

So the graph is concave down to the left of 2 and concave up to the right to 2.

Since the concavity changes at $x = 2$, $(2, f(2))$ is an inflection point. Here,

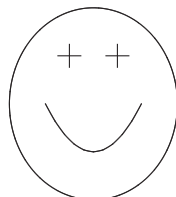
$$f(2) = 2^3 - 6 \cdot 2^2 + 9 \cdot 2 + 5 = 8 - 24 + 18 + 5 = 7$$

8. PICTURE OF GRAPH



9. TWO FACES

Happy Face



concave up mouth

++ eyes

pos second deriv

Sad Face



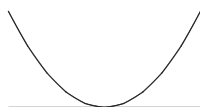
concave down mouth

-- eyes

neg second deriv

10. MAX VERSUS MIN

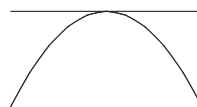
Concavity gives us a way to tell maximums from minimums when dealing with critical points



Local Min

concave up

$$y'' > 0$$



Relative Max

concave down

$$y'' < 0$$

11. SECOND DERIVATIVE TEST

Suppose $f(x)$ is continuous on a closed interval $[a, b]$ and c is a critical point of $f(x)$ in the open interval (a, b) for which $f'(c) = 0$

$f''(c)$	Conclusion
pos	Minimum
neg	Maximum
zero	Don't Know

12. TEST FOR $x^3 - 12x$

$$f(x) = x^3 - 12x$$

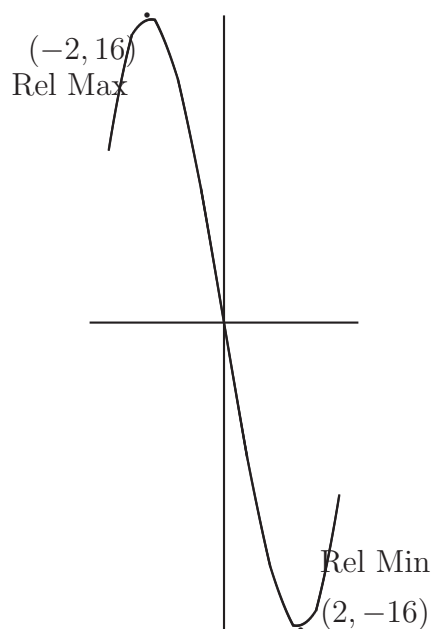
$$f'(x) = 3x^2 - 12$$

$$= 3(x - 2)(x + 2)$$

Critical points: 2 and -2

$$f''(x) = 6x$$

x	$f(x) = x^3 - 12x$	$f''(x) = 6x$	Sign	Concl
-2	$-8 + 24 = 16$	-12	neg	Max
2	$8 - 24 = -16$	12	pos	Min

13. GRAPH OF $x^3 - 12x$ 

14. WARNING

No conclusion can be made if $f''(c) = 0$ as the following three examples show.

For each of these three functions,

$$f'(0) = 0 \text{ and } f''(0) = 0$$

$f(x)$	$f'(x)$	$f''(x)$	Behavior
x^4	$4x^3$	$12x^2$	Min
$-x^4$	$-4x^3$	$-12x^2$	Max
x^3	$3x^2$	$6x$	Neither

For $f(x) = x^4$, $x = 0$ is a minimum

For $f(x) = -x^4$, $x = 0$ is a maximum

For $f(x) = x^3$, $x = 0$ is neither a max nor min

15. SECOND WARNING

The statement

inflection points occur when the second derivative is zero

is slightly misleading.

Although this statement is true in practice for the functions you are likely to see in this course.

Consider the function $f(x) = x^4$.

Here $f''(x) = 12x^2 = 0$ at $x = 0$

But $x = 0$ is not an inflection point, because

This function is always concave up.

It never changes concavity.

16. STRATEGY FOR SKETCHING GRAPHS

Step 1. [Derivatives] Find $f'(x)$ and $f''(x)$

Step 2. [Domain and symmetry] Note the domain of $f(x)$ and look for possible symmetries:

(a) $f(-x) = f(x)$ implies symmetry about y -axis

(b) $f(-x) = -f(x)$ implies symmetry about origin

Step 3. [Critical points] Determine critical points where the derivative is zero or undefined. These are candidates for max/min points.

Step 4. [Increasing/Decreasing] Determine the sign of the derivative in the intervals between successive critical points.

(a) $y' > 0$ means graph is increasing

(b) $y' < 0$ means graph is decreasing

(c) Identify the maximum versus the minimums using the first derivative test.

17. SKETCHING GRAPHS CONTINUED

Step 5. [Inflection points] Determine when $f''(x) = 0$. These are candidates for inflection points.

Step 6. [Concavity] Determine the sign of the second derivative in the intervals between successive points where $f''(x) = 0$.

(a) $y'' > 0$ means graph is concave up

(b) $y'' < 0$ means graph is concave down

(c) Inflection points occur when concavity changes

Step 7. [Table of Points] Make a table of the y values of the critical and inflection points.

Step 8. [Complete the sketch] Connect the points in your graph using the information about increasing/decreasing and concavity.