

1. Simplest element

Simplest element

Let S be a set of reduced rational numbers with distinct denominators greater than zero.

The element of S with least denominator is said to be the **simplest** element of S .

What is the simplest element of

$$S = \left\{ \frac{109}{83}, \frac{2}{91}, \frac{108}{1237}, \frac{23107}{30} \right\}$$

Answer: $\frac{23107}{30}$

3. Summary

Given two fractions $\frac{a}{b} < \frac{c}{d}$, the closest distance that they can be from each other is $\frac{1}{bd}$.

This minimum distance occurs when

$$bc - ad = 1.$$

5. Tape Measure

How can you measure the fraction $\frac{1}{3}$ on a tape measure where the tick marks are in 16ths of an inch? That is, what is the closest fraction on a tape measure to $\frac{1}{3}$?

We know $\frac{1}{3} = 0.3333 \dots$

Multiplying by 16, $\frac{16}{3} = 5.3333 \dots$

So $\frac{1}{3} = \frac{5.3333 \dots}{16}$

It looks like the closest fraction to $\frac{1}{3}$ is $\frac{5}{16}$, a mark on a tape measure.

7. Contiguous Example

Two fractions $\frac{a}{b} < \frac{c}{d}$ which satisfy $ad + 1 = bc$ are called **contiguous**.

Show that the fractions $\frac{a}{b} = \frac{2}{5}$ and $\frac{c}{d} = \frac{3}{7}$ are contiguous.

Discussion Problem

How are the fractions $\frac{a}{b} = \frac{2}{5}$ and $\frac{c}{d} = \frac{3}{7}$

related to the Pails of Water Problem for 5 and 7?

2. Closest fractions

Given two fractions $\frac{a}{b} < \frac{c}{d} \in \mathbb{Q}$, what is the closest distance that can be from each other?

Using the common denominator bd , we have

$$\frac{a}{b} = \frac{ad}{bd} \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}.$$

The distance between these fractions is

$$\frac{c}{d} - \frac{a}{b} = \frac{bc}{bd} - \frac{ad}{bd} = \frac{bc - ad}{bd}.$$

The numerator $bc - ad$ is positive [Why?]

and is an integer. [Why?]

So the smallest value for the $bc - ad$ is 1. [Why?]

4. Contiguous fractions

Contiguous fractions

Two fractions $\frac{a}{b} < \frac{c}{d}$ which satisfy $ad + 1 = bc$ are called **contiguous**.

Note that $\frac{a}{b} + \frac{1}{bd} = \frac{c}{d}$

6. Check it out!

Look at the equation

$$\frac{5}{16} < \frac{1}{3} < \frac{3}{8}$$

Are these fractions contiguous?

How do you suppose a carpenter would mark off one third of an inch?

8. Exploration

If $\frac{a}{b} = \frac{2}{5}$ and $\frac{c}{d} = \frac{3}{7}$, show that the mediant $\frac{a}{b} \oplus \frac{c}{d}$ is contiguous to both $\frac{a}{b}$ and $\frac{c}{d}$.

Think of your own example of two contiguous fractions $\frac{a}{b}$ and $\frac{c}{d}$. Is it true that the mediant is always contiguous to the endpoints $\frac{a}{b}$ and $\frac{c}{d}$ for the example you thought up?

9. General case

Contiguous fractions Mediant is Contiguous Theorem
 It turns out that this behavior is not an accident.
 It is always the case that the mediant of two contiguous fractions is contiguous to both of them.

The “Mediant is Contiguous” Theorem.

If $\frac{a}{b} < \frac{c}{d}$ are contiguous, then the mediant $\frac{a+c}{b+d}$ is contiguous to both $\frac{a}{b}$ and $\frac{c}{d}$.

I challenge you to prove this theorem.

11. Example

Take the fractions $\frac{2}{5}$ and $\frac{3}{7}$.

They are contiguous and their mediant is $\frac{2+3}{5+7} = \frac{5}{12}$.
 So 12 is the smallest denominator possible for any fraction between $\frac{2}{5}$ and $\frac{3}{7}$.

13. More busy work

Check the denominators:

$d = 4$: Show that no fraction with denominator 4 lies between $\frac{2}{5}$ and $\frac{3}{7}$.

$d = 5$: Show that no fraction with denominator 5 lies between $\frac{2}{5}$ and $\frac{3}{7}$.

$d = 10$: Show that no fraction with denominator 10 lies between $\frac{2}{5}$ and $\frac{3}{7}$.

10. There's more

It turns out that if $\frac{a}{b} < \frac{c}{d}$ are contiguous, then the mediant $\frac{a+c}{b+d}$ is the **simplest** rational number between them, meaning it is the fraction with smallest denominator lying between $\frac{a}{b}$ and $\frac{c}{d}$.

12. A direct check

Let's check this last statement:

12 is the smallest denominator for any fraction between $\frac{2}{5}$ and $\frac{3}{7}$ by examining each choice for denominator d less than 12:

$d = 2$:

Both $\frac{2}{5}$ and $\frac{3}{7}$ are less than $\frac{1}{2}$.

So $\frac{1}{2}$ is not between them.

$d = 3$:

$\frac{1}{3} < \frac{2}{5}$ and $\frac{3}{7} < \frac{2}{3}$.

So neither $\frac{1}{3}$ nor $\frac{2}{3}$ is between them.

14. Is it true in general?

What if we drop the condition that $\frac{a}{b}$ and $\frac{c}{d}$ are contiguous?

Is it still true that the mediant is the simplest fraction between them?

You might try some examples, like $\frac{a}{b} = \frac{1}{3}$ and $\frac{c}{d} = \frac{4}{5}$.