

CONVERGENCE TESTS

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These notes are taken from Calculus Vol I, by Tom M. Apostol, 1966.

1. NECESSARY VERSUS SUFFICIENT

Type I. Necessary Condition.

If $\sum a_n$ converges, then condition C is satisfied.

Type II. Sufficient Condition.

If condition C is satisfied, then $\sum a_n$ converges.

Type III. Necessary and Sufficient Conditions

$\sum a_n$ converges if and only if condition C is satisfied.

2. EASY EXAMPLES

Suppose you need to save \$100 for a winter coat.

Saving \$200 is a **sufficient** condition to purchase the coat.

Is it necessary?

Suppose you want to make eggs and ham.

Having eggs in the refrigerator is a **necessary** condition to cook eggs and ham.

Is it sufficient?

Having eggs, ham, and cheese is **sufficient**, but not necessary.

3. 2016 ELECTION

In the 2016 presidential election, Wisconsin has 10 electoral votes, Michigan has 16, and Pennsylvania has 20. Donald Trump won all three states.

Was winning these three states a necessary condition for his election?

Without these states, Clinton had 232 (pledged) electoral votes and Trump had 260.

Since 270 electoral votes (out of 538) are needed to win the presidential election,

a necessary condition for his victory was a win in *any one* of these three states.

Winning all three was a **sufficient, but not necessary condition** for a Trump victory.

Winning all three was a **necessary condition** for a Clinton win.

4. RESTAURANTS

A restaurant has a sign: No shoes, no shirt, no service.

What does this mean?

Is wearing a shirt and a pair of shoes a necessary condition for begin served in the restaurant?

Is it a sufficient condition?

5. MATH EXAMPLE

Let n be a positive integer.

Let A be the condition that the decimal representation of n ends in 5.

Let B be the condition that 5 divides n .

Let C be the condition that n is odd.

A is a (necessary/sufficient) condition for B.

A is a (necessary/sufficient) condition for C.

B is a (necessary/sufficient) condition for C.

What is a necessary and sufficient condition for B?

6. THE ZERO TEST

Suppose the series $\sum_{k=1}^{\infty} a_k$ converges to S .

Let $S_n = \sum_{k=1}^n a_k$ be the n -th partial sum.

Note that $a_n = S_n - S_{n-1}$.

Since the series converges,

$$\lim_{n \rightarrow \infty} S_n = S \text{ and } \lim_{n \rightarrow \infty} S_{n-1} = S$$

Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} S_n - S_{n-1} \\ &= \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = S - S = 0 \end{aligned}$$

This proves that the limit $\lim_{n \rightarrow \infty} a_n = 0$

7. NECESSARY OR SUFFICIENT

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$

Let A be the condition: $\sum_{k=1}^{\infty} a_k$ converges.

Let B be the condition: $\lim_{n \rightarrow \infty} a_n = 0$.

Is A a (necessary/sufficient) condition for B?

It is certainly necessary.

Is it sufficient?

Hold this thought.

8. EXAMPLE 3

Test for convergence:

$$\sum_1^{\infty} \frac{n}{2n+1}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2},$$

the series diverges.

9. DANGER DANGER

The zero limit test is a test for divergence.

It is **WRONG** in general to conclude that the series

$$\sum_1^{\infty} a_n \text{ converges just because } \lim_{n \rightarrow \infty} a_n = 0.$$

For a counter-example, consider the harmonic series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}.$$

What is $\lim_{n \rightarrow \infty} a_n$?

Does the harmonic series converge?

10. COMPARISON TEST

Assume $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq 1$.

If there exists a positive number c such that

$a_n \leq c b_n$ for all n , then

$\sum_1^{\infty} b_n$ converges implies $\sum_1^{\infty} a_n$ converges.

What is logically equivalent to this statement is

$\sum_1^{\infty} a_n$ diverges implies $\sum_1^{\infty} b_n$ diverges.

If you weigh less than a skinny person, then you are skinny.

If you weigh more than a fat person, then you are fat.

11. LIMIT COMPARISON TEST

Assume $a_n > 0$ and $b_n > 0$ for all $n \geq 1$.

Suppose $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$

where c is a positive constant (usually $C = 1$).

Then both series $\sum_1^{\infty} a_n$ and $\sum_1^{\infty} b_n$ converge or they both diverge.

12. EXAMPLE 4

Test for convergence:

$$\sum_1^{\infty} \frac{1}{2 + 3^n}.$$

13. EXAMPLE 5

Test for convergence:

$$\sum_1^{\infty} \frac{1}{\sqrt{n(n+10)}}.$$

14. EXAMPLE 6

Test for convergence:

$$\sum_1^{\infty} \sin \frac{1}{n}.$$

15. THE INTEGRAL TEST

Let $f(x)$ be a positive decreasing function, defined for all $x \geq 1$.

Then the series $\sum_1^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{\infty} f(x) dx$ converges.

16. p -SERIES

If $p > 0$, then the series $\sum_1^{\infty} \frac{1}{n^p}$ is called a p -series.

We know that the improper integral $\int_1^{\infty} \frac{1}{x^p} dx$ converges if and only if $p > 1$.

It follows from the integral test that

the p -series $\sum_1^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

17. RULE OF THUMB

The Limit Comparison Test works well for comparing “messy” algebraic series to a p -series.

In choosing the appropriate value of p , the rule of thumb is disregard all but the highest powers of n in the numerator and denominator.

18. 4 EXAMPLES

- (i) Given $\sum \frac{1}{3n^2 - 4n + 5}$, compare with $\sum \frac{1}{n^2}$
- (ii) Given $\sum \frac{1}{\sqrt{3n - 2}}$, compare with $\sum \frac{1}{\sqrt{n}}$
- (iii) Given $\sum \frac{n^2 - 10}{4n^5 + 7n^3}$, compare with $\sum \frac{1}{n^3}$
- (iv) Given $\sum \frac{\sqrt{n}}{\sqrt{n^3 + 1}}$, compare with $\sum \frac{1}{n}$

19. THE ROOT TEST

Given a series $\sum a_n$ of non-negative terms such that

$$\lim_{n \rightarrow \infty} a_n^{1/n} = L. \text{ Then}$$

- (i) if $L < 1$, then the series converges
- (ii) if $L > 1$, then the series diverges
- (iii) if $L = 1$, then the test is inconclusive.

20. PROOF OF STATEMENT (I)

$$\lim_{n \rightarrow \infty} a_n^{1/n} = L.$$

Assume $L < 1$.

Then there is a number r between L and 1, that is, $L < r < 1$.

There is a number N so large that for all $n \geq N$ we have

$$a_n^{1/n} < r.$$

Hence $a_n < r^n$ for all $n \geq N$.

The geometric series $\sum_{n=N}^{\infty} r^n$ converges. Why?

So the series $\sum_{n=N}^{\infty} a_n$ converges by the comparison test.

21. EXAMPLE 7

Test for convergence:

$$\sum_3^{\infty} \frac{1}{(\ln n)^n}$$

22. EXAMPLE 8

Test for convergence:

$$\sum_3^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

23. THE RATIO TEST

Given a series $\sum a_n$ of positive terms such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L. \text{ Then}$$

- (i) if $L < 1$, then the series converges
- (ii) if $L > 1$, then the series diverges
- (iii) if $L = 1$, then the test is inconclusive.

24. EXAMPLE 9

Test for convergence:

$$\sum_3^{\infty} \frac{n!}{n^n}$$