

1. Comparing the size of two sets

Given two sets A and B , how can we tell whether they have the same number of elements? One answer, of course, is to just “count the number of elements” of both sets and see if the counts are the same.

3. Advantages to the Comparison Method

There are two great things about the “pairing” definition. The first is that it doesn’t require us to count the number of elements in either set. But the second, more important advantage is that our definition applies to infinite sets.

5. Example 1.

The set of odd integers is countable. The pairing is obvious:

$$\begin{array}{l} 1 \leftrightarrow 1 \\ 2 \leftrightarrow 3 \\ 3 \leftrightarrow 5 \\ \vdots \quad \vdots \\ n \leftrightarrow 2n - 1 \end{array}$$

No surprise here.

7. Unsettling thoughts

There is something a little unsettling about this example. It certainly seems like there are a lot more integers than perfect squares. After all, loads of numbers are not perfect squares: 2, 3, 5, 6, 7, etc. Up to one million we have a million numbers and only one thousand squares. Our discomfort stems from the fact that both sets are infinite. Obviously, strange things happen when we deal with infinity.

2. The “pairing” definition

Do you have to count the number of elements?

No! To tell whether the number of fingers on your left hand is the same as the number of fingers on your right hand, touch the fingers of your two hands together. If no finger is left out, the numbers are the same. (Of course, it is not too difficult to count to 5.)

If at a dance, everyone is dancing (with a partner of the opposite sex), then there are as many girls as boys at the dance.

4. Countable sets

The elements of an infinite set S are **countable** if every element of S can be paired with a unique positive integer.

6. Example 2.

The set of squares is countable:

$$\begin{array}{l} 1 \leftrightarrow 1 \\ 2 \leftrightarrow 4 \\ 3 \leftrightarrow 9 \\ \vdots \quad \vdots \\ n \leftrightarrow n^2 \end{array}$$

8. The set of prime numbers is countable

:

$$\begin{array}{l} 1 \leftrightarrow 2 \\ 2 \leftrightarrow 3 \\ 3 \leftrightarrow 5 \\ 4 \leftrightarrow 7 \\ 5 \leftrightarrow 11 \\ \vdots \quad \vdots \\ n \leftrightarrow p_n \end{array}$$

No one knows an explicit formula for the n th prime p_n .

9. Counting fractions

We can place every rational number, written as the ordered pair (i, j) on an infinite grid of rows. The row number is the numerator i and the column number is the denominator j . How do we count these ordered pairs?

Like going into the DMV to take a driving test. You pick a number, sit in a chair, and wait until your number is called.

Our goal is to find a way to assign to each fraction (i, j) , written in ordered pair notation, a unique positive integer **Fraction** (i, j) .

We cannot count along, say only the first row, since then we will completely miss rows 2, 3, etc. We need to include all rows in the counting process.

11. Wedding Buffets

Like being served at a wedding buffet, you need to wait until your particular table is called to the service line. For our problem, each fraction must wait until the number of their particular diagonal is called. Once your diagonal is being processed, the wait is almost over.

This method works because the wait is always finite. Only a finite number of other fractions lie ahead of any particular fraction.

13. A problem

There is a problem with this method. Do you see what it is?

Fraction $(1, 2)$ and Fraction $(2, 4)$ correspond to the same fraction, one half, since $\frac{2}{4}$ simplifies to $\frac{1}{2}$. (Many people say $\frac{2}{4}$ reduces to $\frac{1}{2}$, implying some kind of weight loss.) Indeed, there is an infinite chain of pairs equivalent to $\frac{1}{2}$:

$$(1, 2) = (2, 4) = (3, 6) = (4, 8) = \dots$$

Each of these fractions are assigned different numbers, yet, as fractions, they are all the same number, namely $\frac{1}{2}$. How do you suggest we fix this problem?

15. Exercise 2

What is the number assigned to Fraction $(7, 4)$?

Fraction $(30, 20)$?

[Do not worry about equivalent fractions.]

Hint: determine the diagonal that $(30, 20)$ lies on. The formula in Exercise 1 comes in handy.

10. Use diagonals

Clearly Fraction $(1, 1)$, should be assigned the number 1. Who's next? The trick is to go down the diagonals, starting at the top. Numbers 2 and 3 are assigned to the fractions $(1, 2)$ and $(2, 1)$, in that order. The third diagonal consists of the fractions $(1, 3)$, $(2, 2)$, and $(3, 1)$, which are assigned the numbers 4, 5, and 6. To decide which lucky passenger gets Room number 7, we go to the next diagonal at the top—the winner is Fraction $(1, 4)$, to be followed by $(2, 3)$, $(3, 2)$, and $(4, 1)$. Proceeding in this way, one diagonal at a time, every fraction will eventually be given a room.

12. Counting along Diagonals

# 1 (1,1)	# 2 (1,2)	# 4 (1,3)	# 7 (1,4)	# 11 (1,5)
# 3 (2,1)	# 5 (2,2)	# 8 (2,3)	# 12 (2,4)	(2,5)
# 6 (3,1)	# 9 (3,2)	# 13 (3,3)	(3,4)	(3,5)
# 10 (4,1)	# 14 (4,2)	(4,3)	(4,4)	(4,5)
# 15 (5,1)	(5,2)	(5,3)	(5,4)	(5,5)

14. Exercise 1. Useful formula

Find a formula for the sum

$$S = 1 + 2 + 3 + 4 + \dots + (n - 1) + n$$

by experimenting with small values of n .

16. Exercise 3

What fraction is assigned the number 100?