

DISC METHOD VERSUS SHELL METHOD

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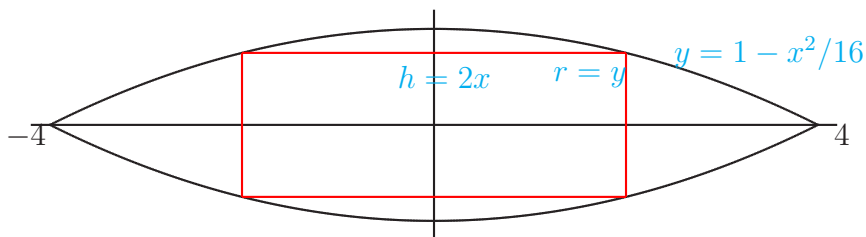
<http://math.niu.edu/~richard/Math230>

1. FOUR EXAMPLES

- First Example. Disc Method is Preferable
- Second Example. Shell Method is Preferable
- Third Example. Shell Method is necessary

2. DISC METHOD PREFERABLE

Rotate the region bounded by the parabola $y = 1 - \frac{x^2}{16}$, $-4 \leq x \leq 4$ about the x -axis.



In the shell Method,

$$r = y$$

$$h = 2x$$

$$V = \int_0^1 2\pi r h \, dy$$

$$= \int_0^1 2\pi(y) (2x) \, dy$$

3. SHELL METHOD CONTINUED

Change all variables to y .

$$y = 1 - x^2/16$$

$$\implies x^2 = 16(1 - y)$$

$$\implies x = 4\sqrt{1 - y}$$

$$\text{So } V = \int_0^1 2\pi(y) (2x) dy$$

$$= \int_0^1 2\pi(y) (8\sqrt{1 - y}) dy$$

$$= 16\pi \int_0^1 y \sqrt{1 - y} dy$$

Can we do this integral?

Yes, but it takes some work.

4. EVALUATE THE INDEFINITE INTEGRAL

Without the limits of integration, our integral is

$$I = \int y \sqrt{1 - y} dy$$

The trick is to make the following change of variable:

Substitute $u = 1 - y$

$$\frac{du}{dy} = -1$$

So $du = -dy$ or $dy = -du$

$$\text{So } I = \int y \sqrt{1 - y} dy = \int \textcolor{red}{y} \sqrt{u} (-1) du$$

What do we do about the y ?

From $u = 1 - y$ we have $y = 1 - u$.

$$\text{So } I = \int (\textcolor{red}{1} - \textcolor{red}{u}) \sqrt{u} (-1) du = \int (u - 1) \sqrt{u} du$$

5. FINISHING THE INTEGRATION

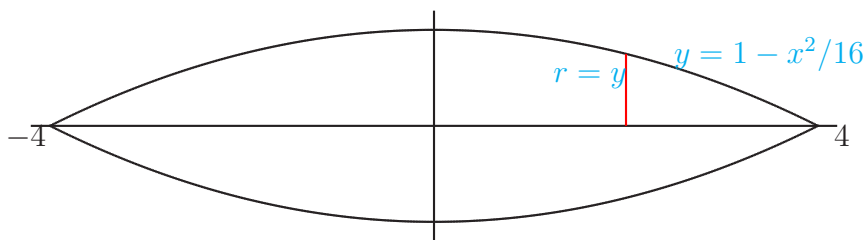
$$\begin{aligned} I &= \int (u - 1) \sqrt{u} du \\ &= \int u^{3/2} - u^{1/2} du \\ &= \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + C \\ &= \frac{2}{5} (1 - y)^{5/2} - \frac{2}{3} (1 - y)^{3/2} + C \end{aligned}$$

6. EVALUATING THE VOLUME

$$\begin{aligned} V &= 16\pi \int_0^1 y \sqrt{1 - y} dy \\ &= 16\pi \left[\frac{2}{5} (1 - y)^{5/2} - \frac{2}{3} (1 - y)^{3/2} \right]_0^1 \\ &= 16\pi \left[0 - \left(\frac{2}{5} - \frac{2}{3} \right) \right] \\ &= -16\pi \left(\frac{6}{15} - \frac{10}{15} \right) \\ &= -16\pi \frac{-4}{15} \\ &= \frac{64\pi}{15} \end{aligned}$$

7. THE DISC METHOD

Recall the picture:



In the disc Method,

$$r = y$$

Because we double the volume of the half “football,”

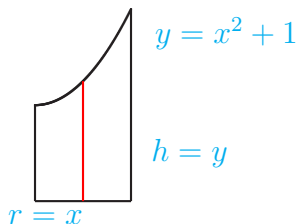
$$\begin{aligned} V &= 2 \int_0^4 \pi r^2 dx = \int_0^4 2\pi \left(1 - \frac{x^2}{16}\right)^2 dx \\ &= \int_0^4 2\pi \left(1 - 2\frac{x^2}{16} + \frac{x^4}{256}\right) dx \end{aligned}$$

8. THE DISC METHOD CONTINUED

$$\begin{aligned} V &= 2\pi \int_0^4 \left(1 - 2\frac{x^2}{16} + \frac{x^4}{256}\right) dx \\ &= 2\pi \int_0^4 \left(1 - \frac{x^2}{8} + \frac{x^4}{256}\right) dx \\ &= 2\pi \left[x - \frac{x^3}{3 \cdot 8} + \frac{x^5}{5 \cdot 256} \right]_0^4 \\ &= 2\pi \left[4 - \frac{4^3}{3 \cdot 8} + \frac{2^{10}}{5 \cdot 2^8} \right] \\ &= 2\pi \left[4 - \frac{8}{3} + \frac{4}{5} \right]_0^4 \\ &= 2\pi \left[\frac{60}{15} - \frac{40}{15} + \frac{12}{15} \right] \\ &= 2\pi \frac{32}{15} = \frac{64\pi}{15} \end{aligned}$$

9. SHELL METHOD PREFERABLE

Rotate the region bounded by the parabola $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y -axis.



In the shell Method

$r = x$ and $h = y$.

So $V = \int_0^1 2\pi r h \, dx$

$$= 2\pi \int_0^1 x(x^2 + 1) \, dx$$

It is easy to evaluate this integral.

10. INTEGRATION

$$V = 2\pi \int_0^1 x^3 + x \, dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^1$$

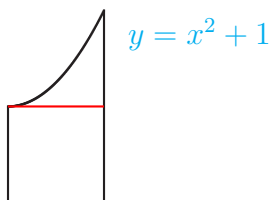
$$= 2\pi \left[\frac{1}{4} + \frac{1}{2} \right]$$

$$= 2\pi \frac{3}{4}$$

$$= \frac{3\pi}{2}$$

11. SHELL METHOD PREFERABLE

Rotate the region bounded by the parabola $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y -axis.



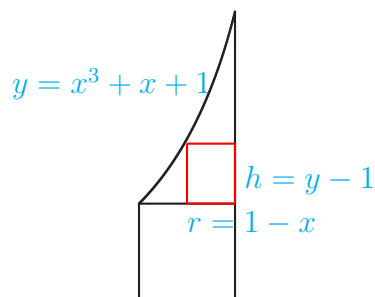
Note that the disc method requires two distinct integrals:

- (1) from $y = 0$ to $y = 1$ and
- (2) from $y = 1$ to $y = 2$.

12. SHELL METHOD NECESSARY

Rotate the region bounded by the curve $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 1$.

Here the problem with the disc method is solving the equation $y = x^3 + x + 1$ for x .



Here the problem with the disc method is solving the equation $y = x^3 + x + 1$ for x .

The shell method is straight-forward:

$$r = x \text{ and } h = y.$$

13. SHELL METHOD INTEGRAL

$$V = 2\pi \int_0^1 r h dx$$

$$= 2\pi \int_0^1 (1-x)(y-1) dx$$

$$= 2\pi \int_0^1 (1-x)(x^3+x) dx$$

Exercise: Work out his integral.