

1. The Farey Sequence

How do you go about finding contiguous numbers, say in the interval $[0, 1]$?

Constructive Method: Filter by the size of the denominators.

Our goal is to construct a sequence of contiguous fractions $\mathcal{F}_n$, called the **Farey Sequence**, whose denominator is $\leq n$.

3. Find $\mathcal{F}_2$

Starting with $\mathcal{F}_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$, what is the simplest fraction between the endpoints $\frac{0}{1}$ and $\frac{1}{1}$?

Answer: By the previous Theorem, it is $\frac{0+1}{1+1} = \frac{1}{2}$.

Put $\mathcal{F}_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$.

5. Find $\mathcal{F}_4$

Compute the median of each pair of successive fractions in $\mathcal{F}_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$.

If the denominator of the median is ≤ 4 , then include it in $\mathcal{F}_4$. Otherwise, discard it.

The median of $\frac{0}{1}$ and $\frac{1}{3}$ is $\frac{0+1}{1+3} = \frac{1}{4}$. Include it.

The median of $\frac{1}{3}$ and $\frac{1}{2}$ is $\frac{1+1}{3+2} = \frac{2}{5}$. Since $5 > 4$, do not include it.

The median of $\frac{1}{2}$ and $\frac{2}{3}$ is $\frac{3}{5}$. Do not include it.

The median of $\frac{2}{3}$ and $\frac{1}{1}$ is $\frac{3}{4}$. Include it.

So $\mathcal{F}_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$.

7. Busy Work

Find

$\mathcal{F}_5$

$\mathcal{F}_6$

$\mathcal{F}_7$

$\mathcal{F}_8$

Take a break, then find

$\mathcal{F}_9$

$\mathcal{F}_{10}$.

2. Starting out

The only two fractions with denominator 1 between 0 and 1 are the endpoints 0 and 1 themselves.

Note that $\frac{0}{1}$ and $\frac{1}{1}$ are contiguous: $\frac{0}{1} + \frac{1}{1 \cdot 1} = \frac{1}{1}$

So we start with $\mathcal{F}_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$.

4. Find $\mathcal{F}_3$

Start with $\mathcal{F}_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$.

The median of $\frac{0}{1}$ and $\frac{1}{2}$ is $\frac{0+1}{1+2} = \frac{1}{3}$.

The median of $\frac{1}{2}$ and $\frac{1}{1}$ is $\frac{1+1}{2+1} = \frac{2}{3}$.

So $\mathcal{F}_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$.

6. The Farey Sequence $\mathcal{F}_n$

Theorem. $\mathcal{F}_n$ consists of all reduced fractions $\frac{a}{b}$ in the interval $[0, 1]$, whose denominator b is less than or equal to n , arranged in increasing order.

$\mathcal{F}_n$ is called the **Farey sequence** of order n .

It is formed by taking all medians between successive elements of $\mathcal{F}_{n-1}$ so long as the resulting denominator is $\leq n$.

8. Challenge Question

Find the fraction $\frac{a}{b}$ immediately before and the fraction $\frac{c}{d}$ immediately after $\frac{9}{31}$ in the Farey sequence $\mathcal{F}_{31}$.

Hint 1: Pails of Water will get one solution.

Hint 2: Use the median equation to get the other solution.