

Mathematical Heresy

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Three Deadly Sins

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1. Adding the numerators and denominators of two fractions.
2. Dividing by zero.

In this section we will commit a third deadly sin:

3. Taking the square root of a negative number.

The Mediant

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has many interesting uses, even though it **looks like** we are adding the fractions incorrectly.

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Thou shall not divide by 0.

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Answer 1: It's the law!

Answer 2: It has a proof.

Answer 3: Simple illustration of good and evil.

The proof

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So $(-x)y + (-x)(-y) = xy + (-x)y$
since both sides are 0.

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That was pretty boring, but it did get the job done.

Good and Evil

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Similarly When bad people move into your neighborhood, the result is negative.

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Similarly When bad people move into your neighborhood, the result is negative.

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Negative (bad people) times Negative (move out) is Positive.

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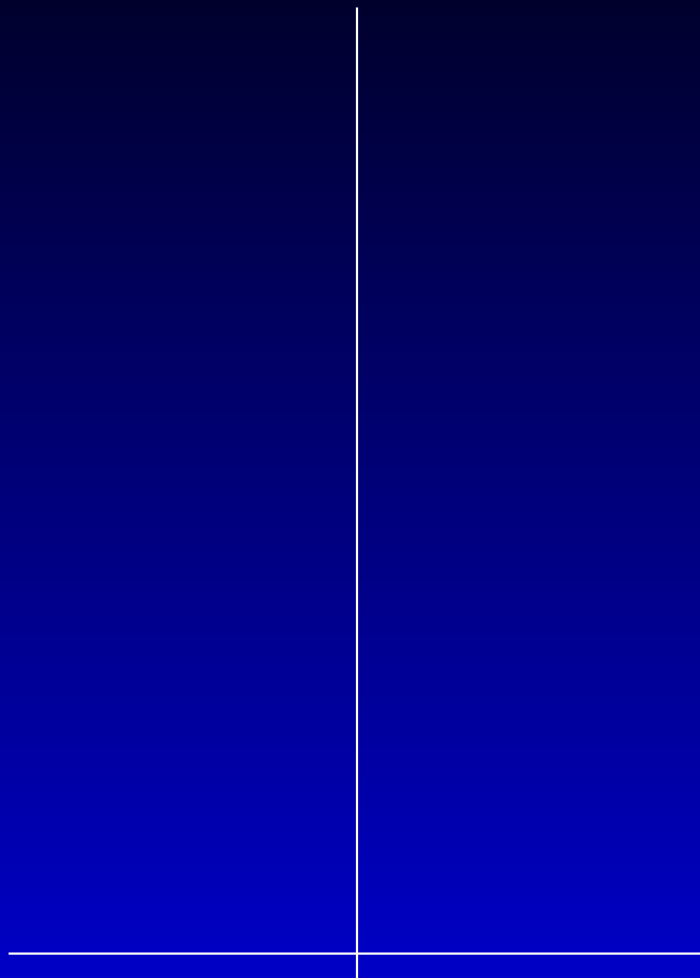
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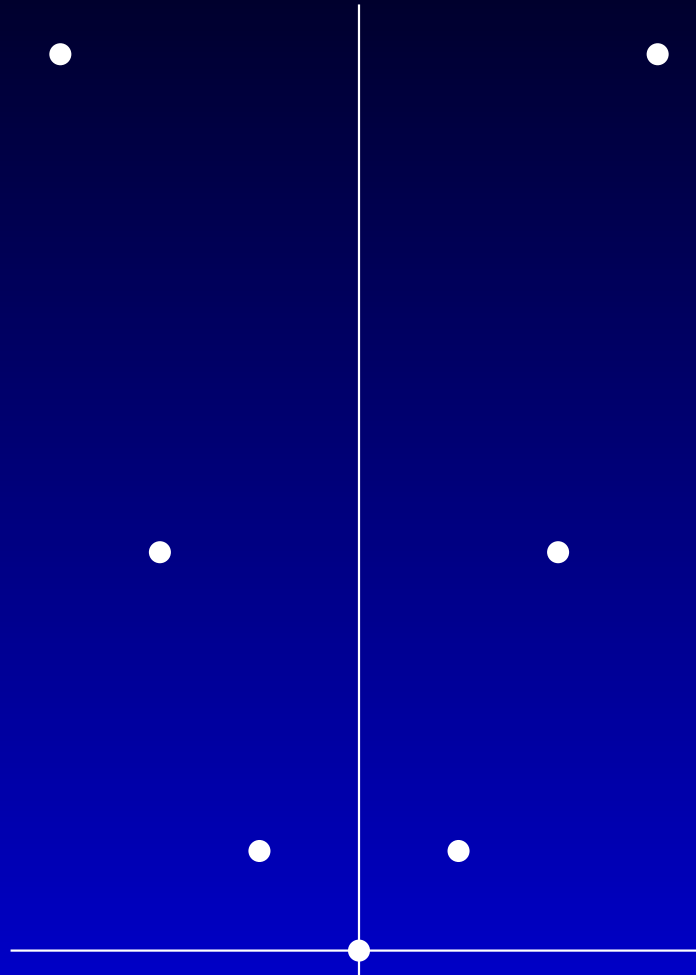
But if x is negative then x^2 is positive, since the product of two negatives is positive.

The only case we are omitting is $x = 0$ and we know $0^2 = 0$.

Graph of $y = x^2$

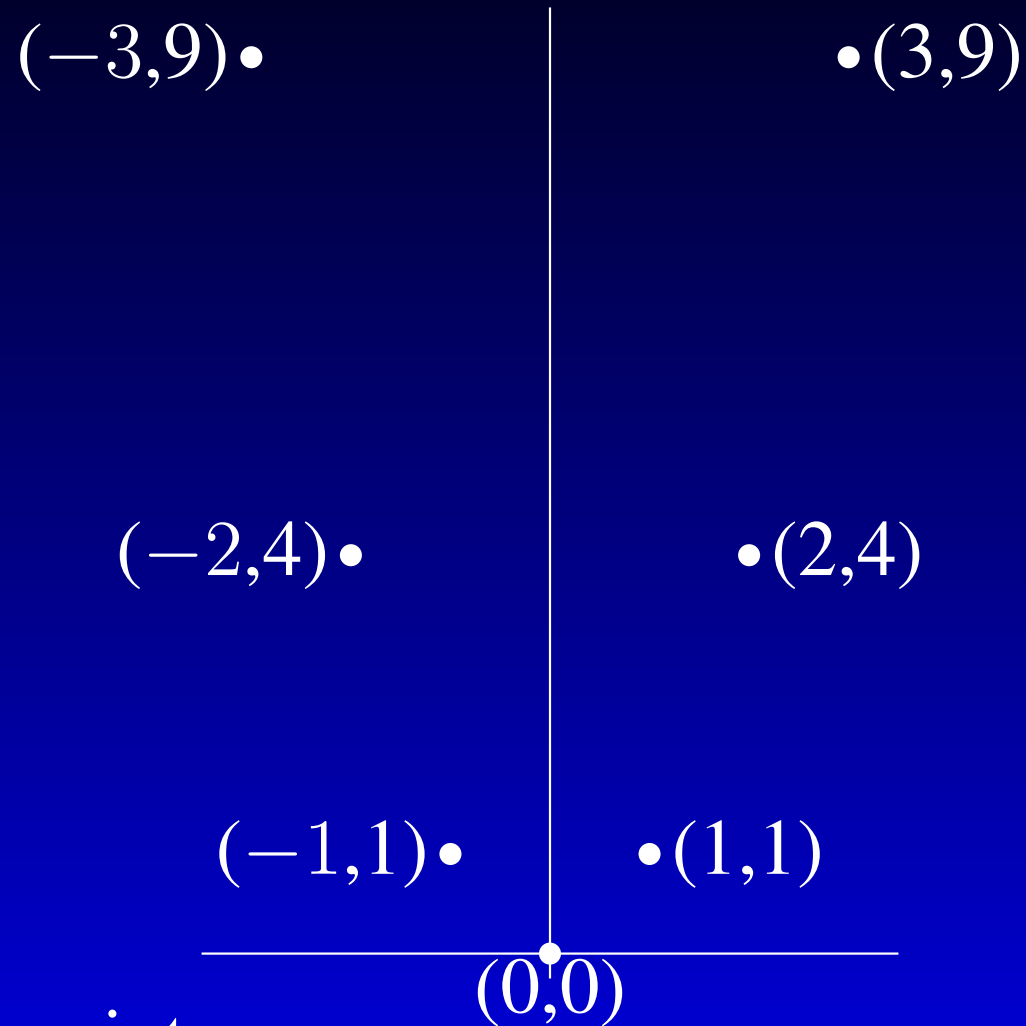


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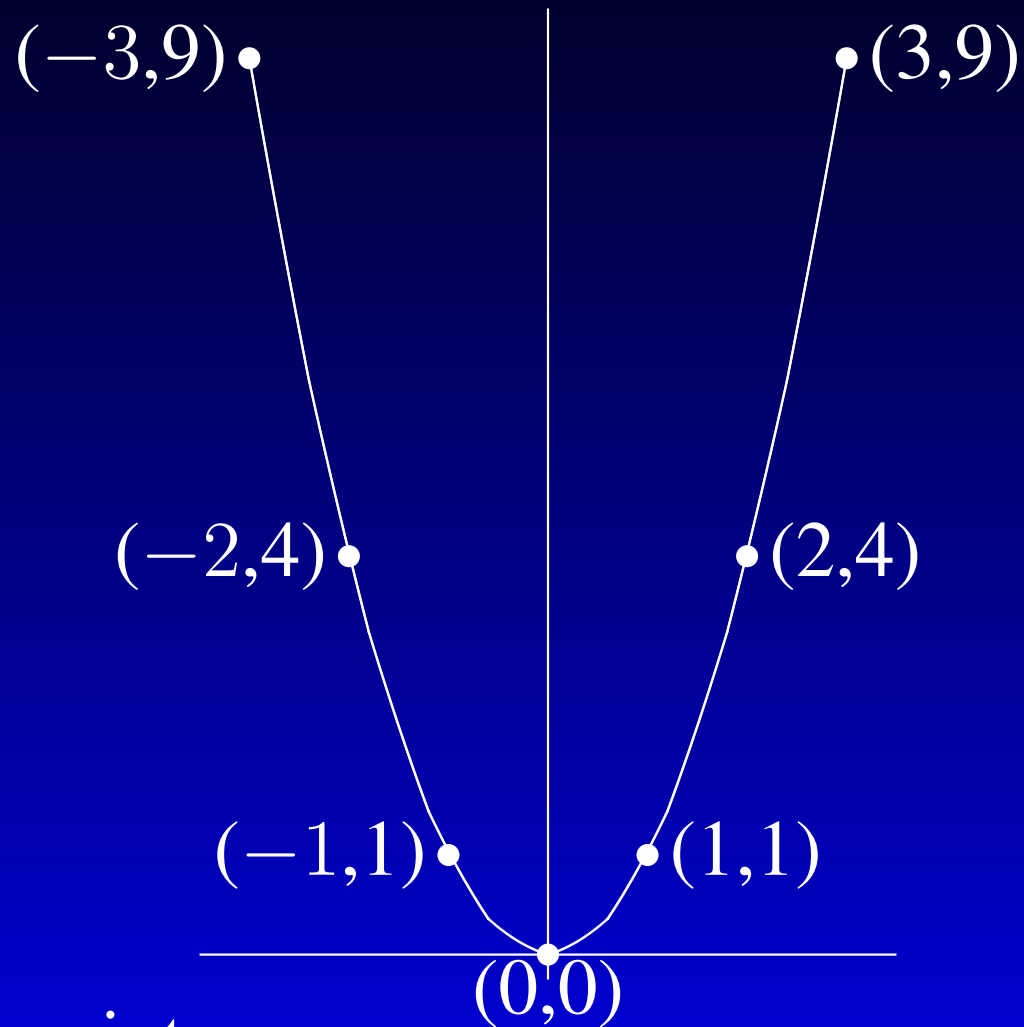
Plot some points.

Graph of $y = x^2$



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Graph of $y = x^2$



Plot some points.
Connect the dots.

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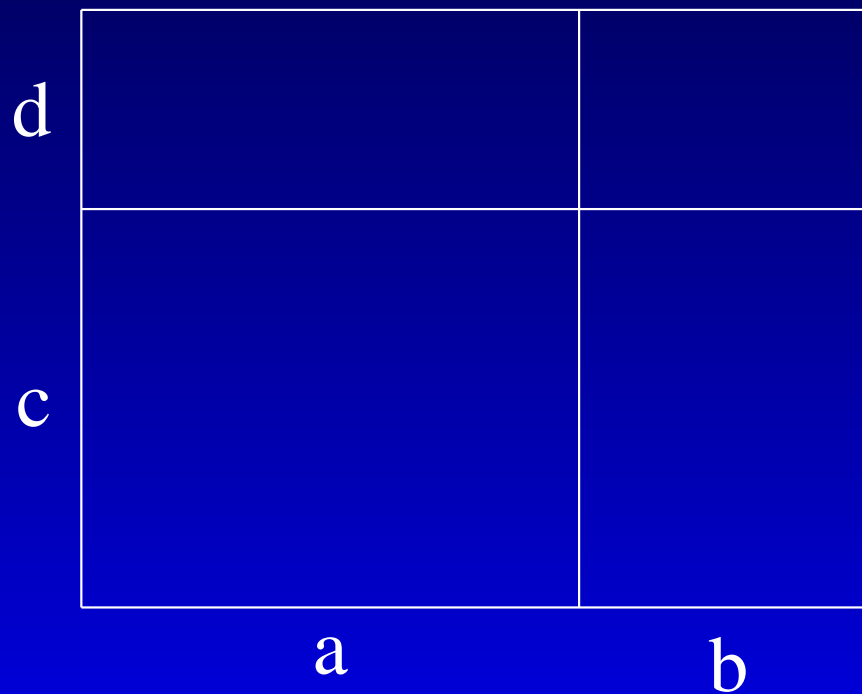
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Hence the letter “i” for “imaginary.”

Aren't all numbers imaginary in that they live in our minds?

Geometric Proof of FOIL

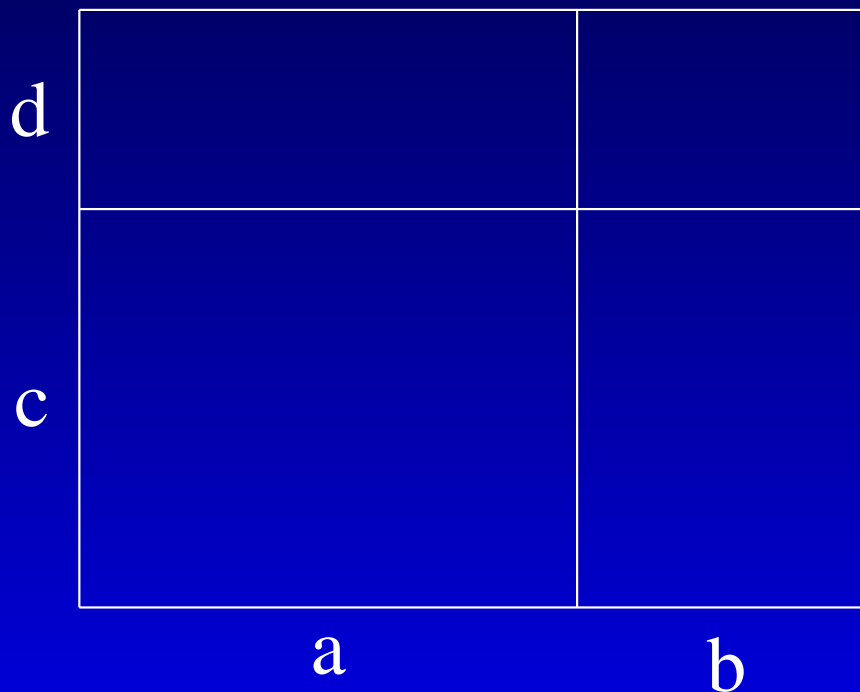
$$(a + b)(c + d) =$$



Geometric Proof of FOIL

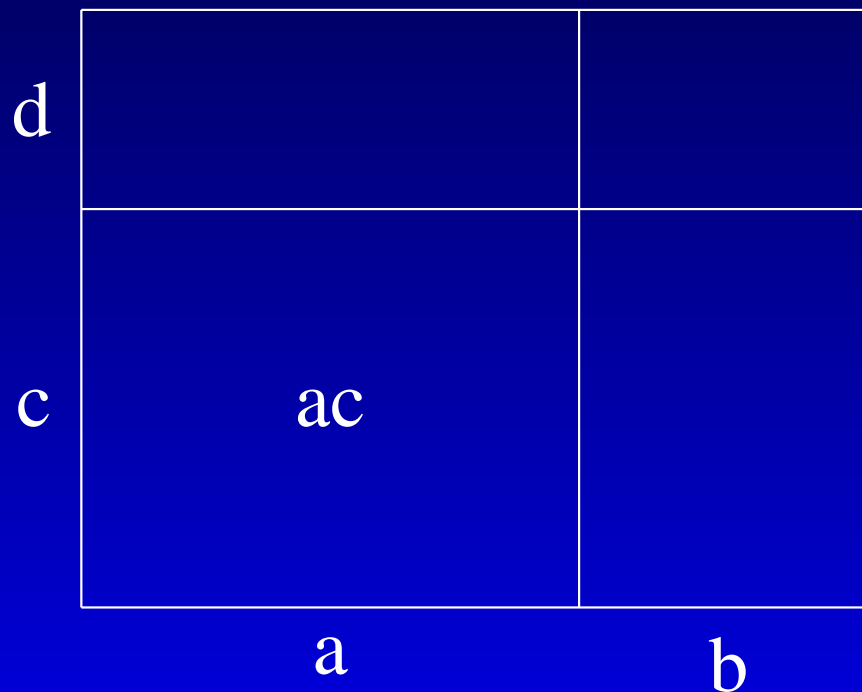
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FIRST



Geometric Proof of FOIL

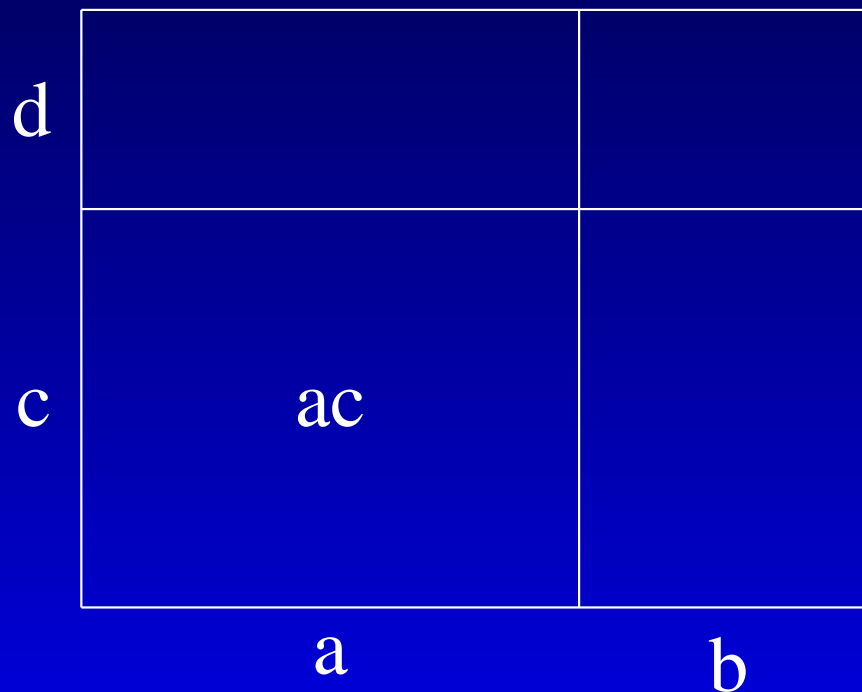
$$(a + b)(c + d) = ac +$$



Geometric Proof of FOIL

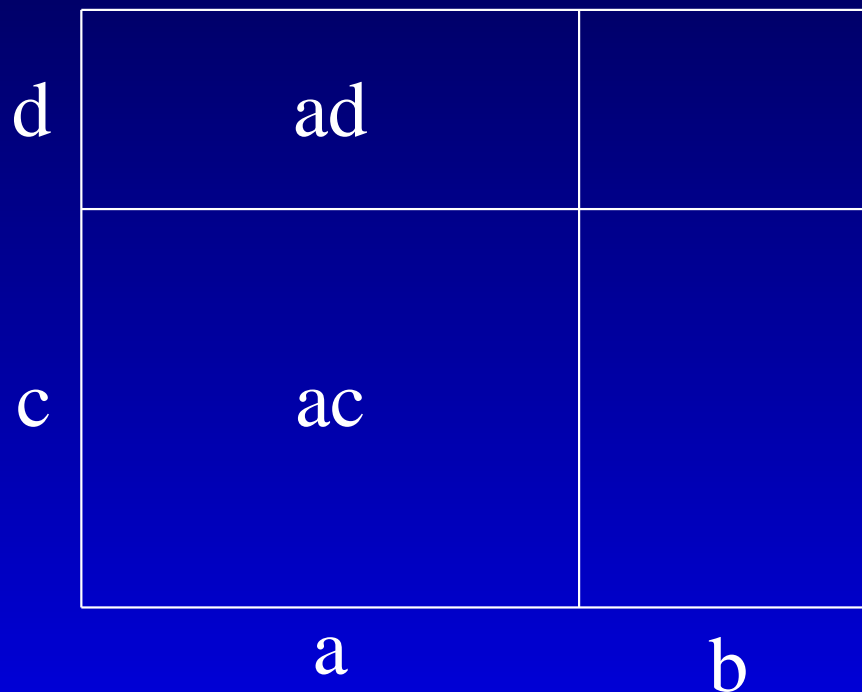
$$(\textcolor{red}{a} + b)(c + \textcolor{red}{d}) = ac +$$

OUTER



Geometric Proof of FOIL

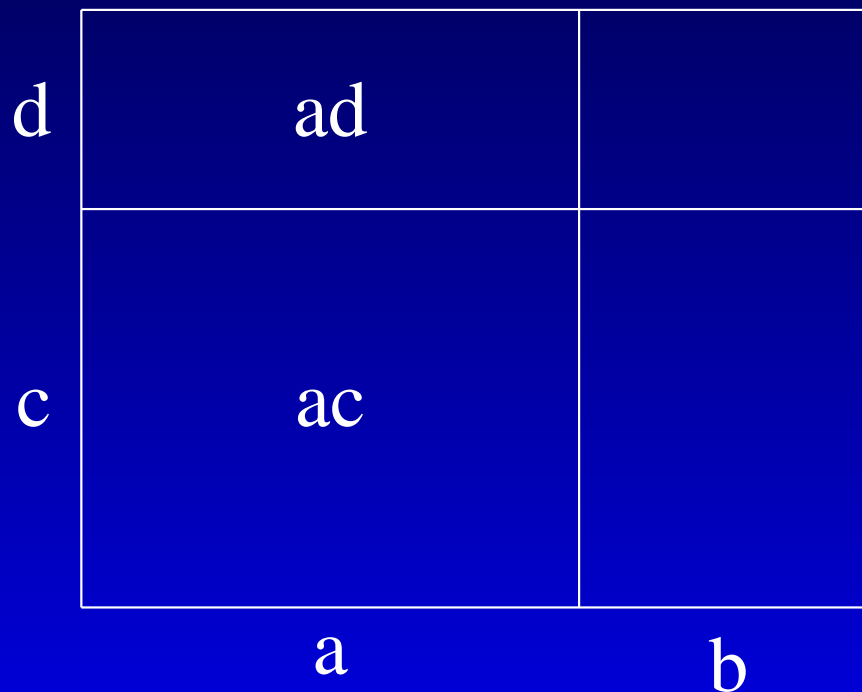
$$(a + b)(c + d) = ac + ad +$$



Geometric Proof of FOIL

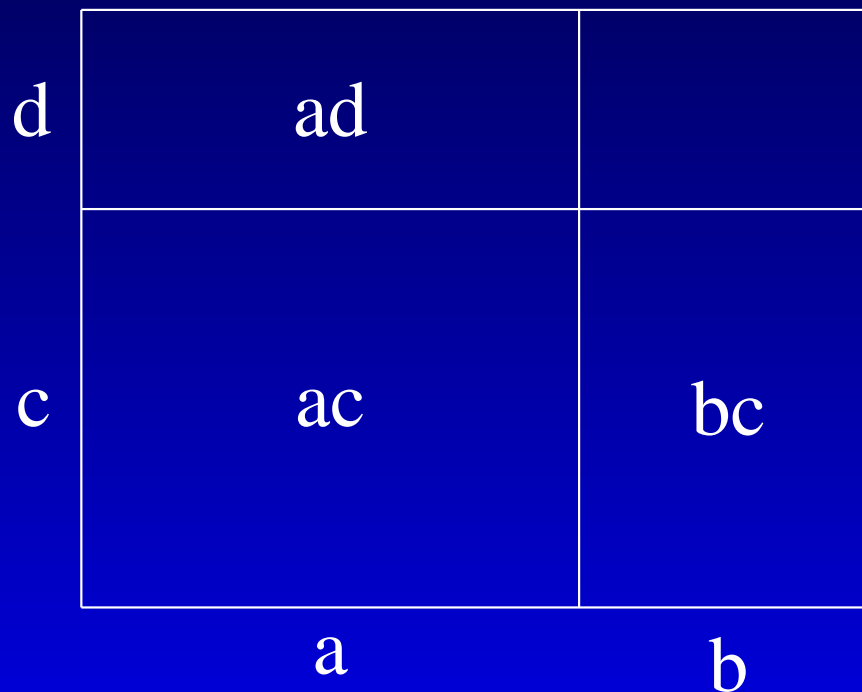
$$(a + b)(c + d) = ac + ad +$$

INNER



Geometric Proof of FOIL

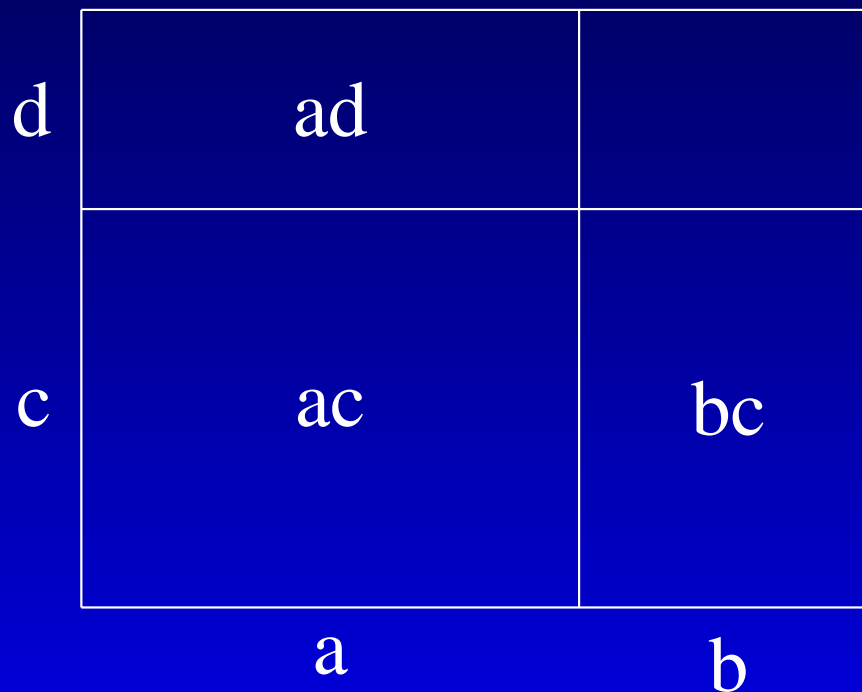
$$(a + b)(c + d) = ac + ad + bc +$$



Geometric Proof of FOIL

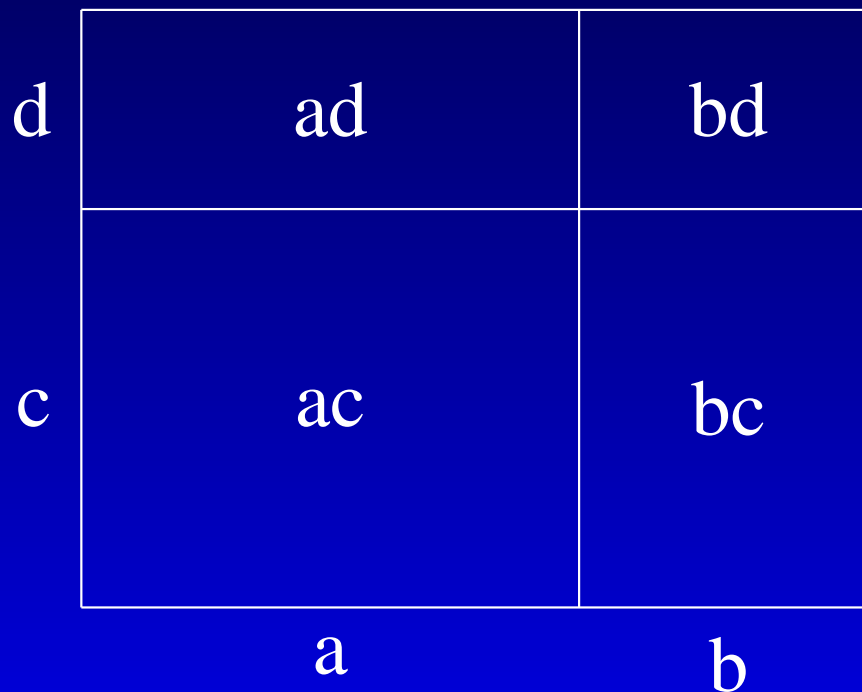
$$(a + b)(c + d) = ac + ad + bc +$$

LAST



Geometric Proof of FOIL

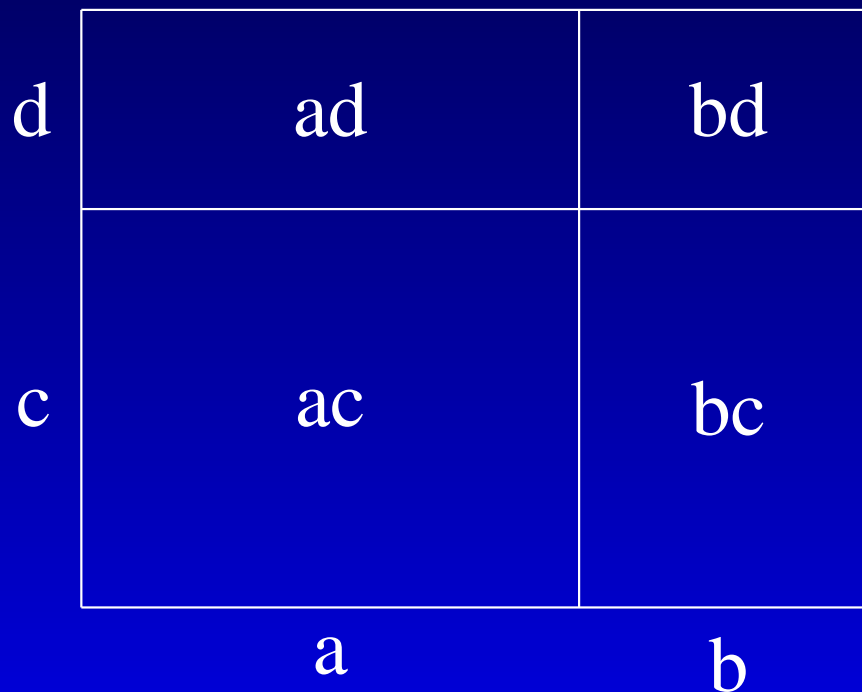
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Geometric Proof of FOIL

$$(a + b)(c + d) = ac + ad + bc + bd$$

F O I L



Some FOIL Magic

Using FOIL,

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$$(a + bi)(a - bi) = a^2 -$$

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$$= a^2 - b^2i^2$$

$$= a^2 - b^2(-1) \quad \text{since } i^2 = -1$$

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Does this method work for the prime 7?

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Which of these primes can be written as a sum of two squares and which cannot?

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Write your answer in two lists:

In the first list, write all primes (like 13) which can be written as a sum of two squares.

In the second list, write all primes (like 7) which cannot be written as a sum of two squares.

DISCOVERY: General Rule

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