

1. MATH 353

Betweenness of Points Axiom and Rule of Insertion

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2. FOUR WAY BETWEENNESS

If A , B , C , and D are four distinct **collinear** points, then $A - B - C - D$ means all four betweenness relations, where one of the four points is omitted:

(i) $A - B - C$

(ii) $A - B - D$

(iii) $A - C - D$ and

(iv) $B - C - D$

are true.

3. PROPOSITION

Hypothesis:

$A - B - C - D$

Conclusion: $AB + BC + CD = AD$.

Proof: You do it.

4. QUESTION

Is the converse of this Proposition true?

That is, if $AB + BC + CD = AD$, then is it true that $A - B - C - D$?

The answer is “Yes” but we need two tools before we can prove it:

- (1) the Triangle Inequality for lines and
- (2) the Rule of Insertion.

The proof of the triangle inequality for lines requires a new axiom, called the Betweenness of Points Axiom.

5. BETWEENNESS OF POINTS AXIOM [BP]

Given three distinct, collinear points A , B , and C .

If $AB + BC \leq \omega$, then there exists a betweenness relation among A , B , and C .

What if $AC + CB \leq \omega$?

or if $BA + AC \leq \omega$?

The associativity rule for addition states that we can reparenthesize:

$$(x + y) + z = x + (y + z)$$

We do not need a second rule stating

$$(x + z) + y = x + (z + y)$$

In other words, if a betweenness relation **could** happen, then a betweenness relation **does** happen.

6. BP AXIOM WHEN $\omega = \infty$

Given three distinct, collinear points A , B , and C .

When ω is finite, there may not be a betweenness relation among A , B , and C .

Just take A , B , and C to be equally spaced point on the equator in $\mathbb{S}$.

Note that when $\omega = \infty$, the condition $AB + BC < \omega$ is always satisfied.

If $\omega = \infty$, there exists a betweenness relation among A , B , and C .

Question: Does the Inside-out model satisfy BP?

7. TRIANGLE INEQUALITY

If A , B , and C are three distinct **collinear** points, then

$$AB + BC \geq AC$$

Proof: Exercise.

Note that the triangle inequality requires that the points are collinear.

8. RULE OF INSERTION

This is called Theorem 7.4 in the text.

Hypothesis:

$$A - B - C$$

Part(i): If $A - X - B$ then $A - X - B - C$

Part(ii): If $B - X - C$ then $A - B - X - C$

We call this theorem the **Insertion Theorem**

9. DANGER – DANGER

Warning: There are **false** versions of this theorem which you **cannot** use!

There are mushrooms which look like choice edible mushrooms, but will kill you. The Insertion Theorem does **not** say:

If $A - B - C$ and $B - C - X$ then $A - B - C - X$

which is **FALSE**.

You can construct a counter-example on the sphere.

10. PROOF OF PART (I)

Hypothesis: $A - B - C$ and $A - X - B$

Conclusion: $A - X - B - C$

To prove $A - X - B - C$ we must show that each of the three-way betweenness relations hold:

$$(1) A - X - B$$

$$(2) A - X - C$$

$$(3) A - B - C$$

$$(4) X - B - C$$

The hypothesis gives us (1) and (3).

So it remains to prove (2) and (4).

11. PROOF OF (2)

We want to prove

$$(2) A - X - C$$

From the hypothesis we know

$$A - X - B \implies AX + XB = AB$$

$$\text{and } A - B - C \implies AB + BC = AC$$

Hence

$$AX + XB + BC = AC$$

12. PROOF OF (2) CONTINUED

$$AX + XB + BC = AC$$

$XC \leq XB + BC$ by the Triangle Inequality for Lines

So $AX + XC \leq AX + XB + BC = AC$

Thus $AX + XC \leq AC$

Using the triangle inequality again,

$AX + XC \geq AC$

Thus,

$AX + XC = AC$

This proves $A - X - C$.

13. PROOF OF (4)

(4) $X - B - C$

Recall $AX + XB + BC = AC$

Since $A - X - C$, which we just proved we have

$AX + XC = AC$

So $AX + XB + BC = AX + XC$

Cancel AX to get

$XB + BC = XC$

This proves $X - B - C$

14. PROOF OF PART (II)

Part (ii) of Theorem 7.4:

Hypothesis: $A - B - C$ and $B - X - C$

Conclusion: $A - B - X - C$

Proof:

$A - B - C$ and $B - X - C$

$$\implies C - B - A \text{ and } C - X - B$$

$$\implies C - X - B - A \text{ by Part (i)}$$

$$\implies A - B - X - C$$

This completes the proof of Thm 7.4.

15. CHALLENGE

Use the Triangle Inequality and Rule of Insertion to prove:

if $AB + BC + CD = AD$, then $A - B - C - D$?