

Area and Integration

Professor Richard Blecksmith

richard@math.niu.edu

Dept. of Mathematical Sciences

Northern Illinois University

<http://math.niu.edu/~richard/Math229>

1. TWO FUNDAMENTAL PROBLEMS OF CALCULUS

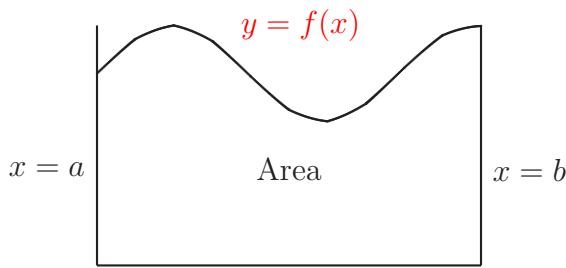
- First Problem: Study how one variable changes with respect to another
 - How does area change with respect to radius
 - How does temperature change with respect to altitude
 - How does y change with respect to x
- Second Problem: Find the area under a curve
- We use **derivatives** to solve Problem 1
- We use **integrals** to solve Problem 2

2. THE AREA PROBLEM

Given a curve $y = f(x)$

For now we will assume that $f(x)$ is ≥ 0

How do we find the area under the curve and above the x -axis, between two vertical sides $x = a$ and $x = b$?



3. A CASE WE CAN DO

If the curve is linear, then we can use the formula for the area of a triangle

$$A = \frac{1}{2} \text{ base} \times \text{height}$$

Example. $y = f(x) = 2x$

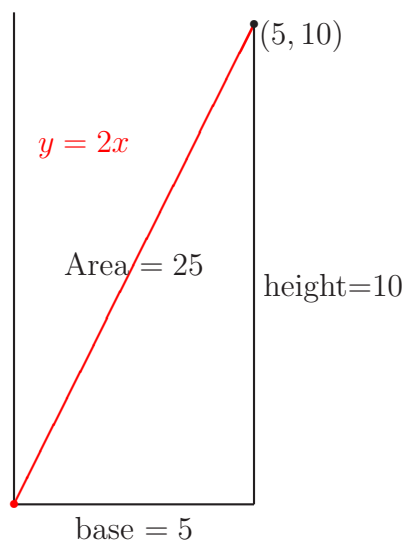
between $a = 0$ and $b = 5$

base = 5

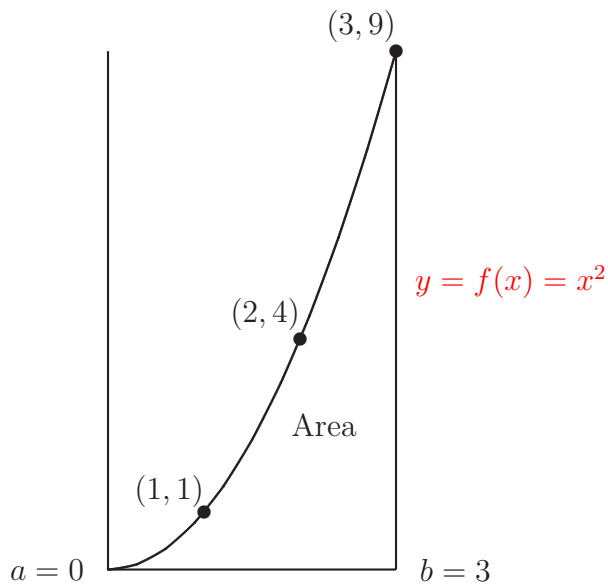
height = 10

$$A = \frac{1}{2} 5 \cdot 10 = 25$$

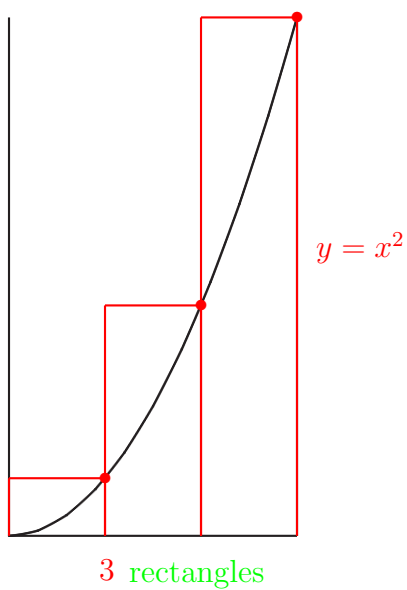
4. GRAPH OF $y = 2x$



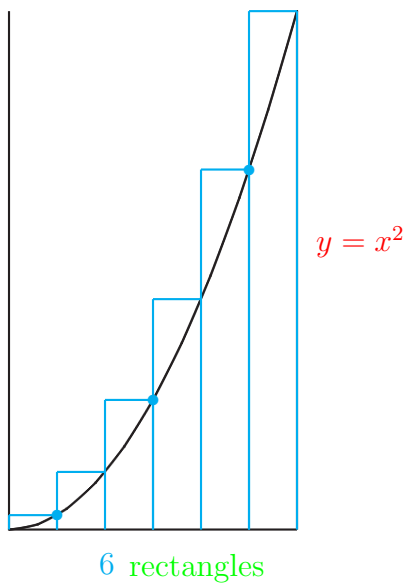
5. A CASE WE CAN'T DO



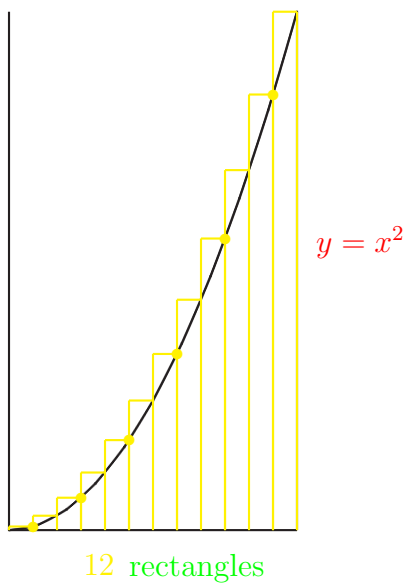
6. RECTANGULAR APPROXIMATION



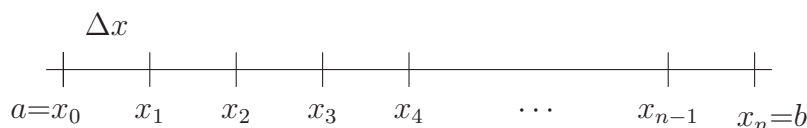
7. RECTANGULAR APPROXIMATION



8. RECTANGULAR APPROXIMATION

9. SUB-DIVIDING $[a, b]$

Set-up: divide the interval $[a, b]$ into n equally spaced segments, whose division points are $x_0, x_1, \dots, x_n$.



The width of each segment is

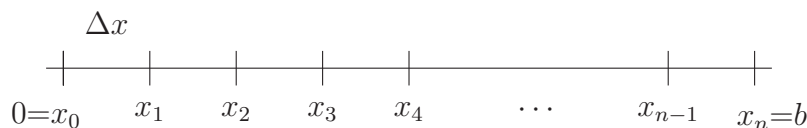
$$\Delta x = \frac{b - a}{n}$$

The division points x_k are $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $x_3 = a + 3\Delta x$ and, in general,

$$x_k = a + k\Delta x$$

10. SPECIAL CASE: $a = 0$

When $a = 0$,



these formulas become

$$\Delta x = \frac{b}{n} \quad \text{and} \quad x_k = k\Delta x$$

Note that $x_n = n\Delta x = n\frac{b}{n} = b$, as it should.

11. RIEMANN METHOD

In the Riemann method, we use n rectangles to approximate the area under the curve $y = f(x)$

Each rectangle has as its base the subinterval $[x_{k-1}, x_k]$ of width Δx

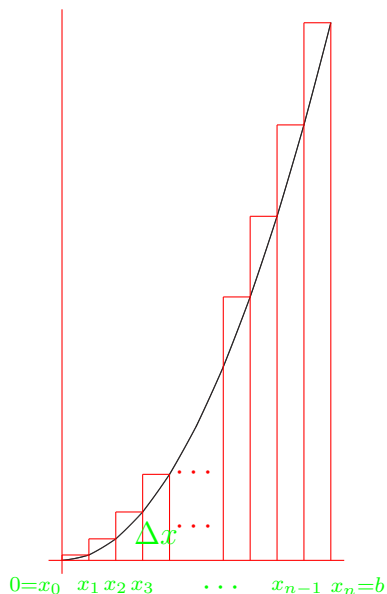
The height of each rectangle is $f(x_k)$

Since the area of a rectangle is base $\times$ height, the k -th rectangle has area

$$\Delta x \cdot f(x_k)$$

Our approximation, called a **Riemann Sum** is the sum of the areas of these n rectangles

12. RIEMANN METHOD PICTURE



13. USING 3 SUBDIVISIONS

For the function $f(x) = x^2$, and interval $[a, b] = [0, 3]$ suppose we use $n = 3$ rectangles

We divide $[0, 3]$ into 3 equally spaced subintervals, each of width $\Delta x = 3/3 = 1$

with division points $x_k = k \Delta x = k$

$$0=x_0 \quad | \quad x_1=1 \quad | \quad x_2=2 \quad | \quad x_3=3$$

Notice that the height of the k th rectangle with base $[x_{k-1}, x_k]$ is $f(x_k) = x_k^2$

k	1	2	3
x_k	1	2	3
$f(x_k)$	1	4	9

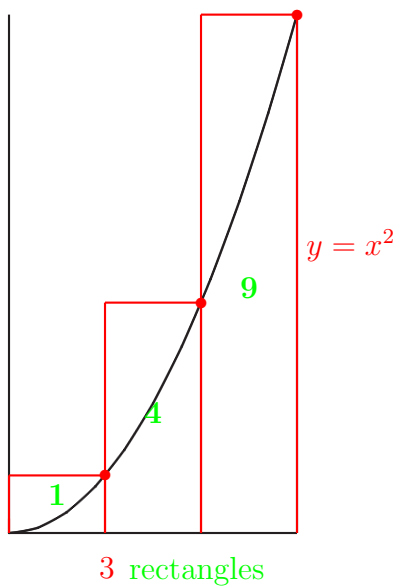
14. 3 RECTANGLES

Each rectangle has width $\Delta x = 1$ and height $f(x_k) = x_k^2$

The area of the 3 red rectangles is

$$\begin{aligned}
 A(3) &= \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) \\
 &= \Delta x \cdot x_1^2 + \Delta x \cdot x_2^2 + \Delta x \cdot x_3^2 \\
 &= 1 \cdot 1^2 + 1 \cdot 2^2 + 1 \cdot 3^2 \\
 &= 1 + 4 + 9 \\
 &= 14
 \end{aligned}$$

15. RECTANGULAR APPROXIMATION



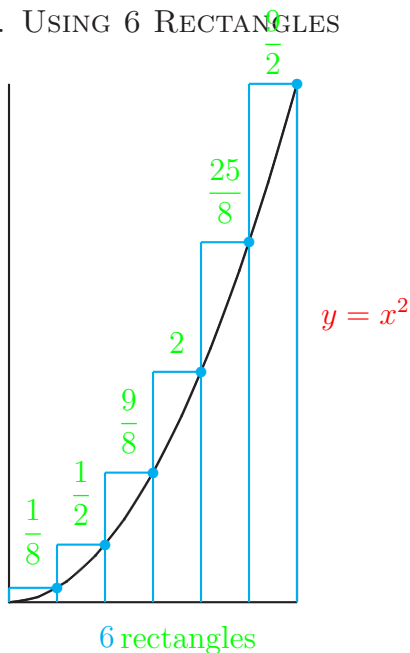
16. 6 RECTANGLES

Each rectangle has width $\Delta x = \frac{1}{2}$, subdivision point $x_k = k \Delta x = \frac{1}{2}k$, and height $f(x_k) = x_k^2$

The area of the 6 bluish green rectangles is

$$\begin{aligned}
 A(6) &= \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \Delta x \cdot f(x_4) + \Delta x \cdot f(x_5) + \Delta x \cdot f(x_6) \\
 &= \Delta x \cdot x_1^2 + \Delta x \cdot x_2^2 + \Delta x \cdot x_3^2 + \Delta x \cdot x_4^2 + \Delta x \cdot x_5^2 + \Delta x \cdot x_6^2 \\
 &= \Delta x \cdot \left(\frac{1}{2}\right)^2 + \Delta x \cdot (1)^2 + \Delta x \cdot \left(\frac{3}{2}\right)^2 + \Delta x \cdot (2)^2 + \Delta x \cdot \left(\frac{5}{2}\right)^2 + \Delta x \cdot (3)^2 \\
 &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{9}{4} + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot \frac{25}{4} + \frac{1}{2} \cdot 9 \\
 &= \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 + \frac{25}{8} + \frac{9}{2} = \frac{35}{8} + 7 = 11\frac{3}{8}
 \end{aligned}$$

17. USING 6 RECTANGLES



18. NOTATION TIME OUT

It is getting very tedious writing these sums

The sum with 12 rectangles will be even worse!

To ease the writing, mathematicians invented the summation symbol $\sum$

It works like this:

- underneath $\sum$ we write the starting index
- above $\sum$ we write the ending index
- to the right of $\sum$ we write the form of the things we are adding

19. SO WHAT?

$\sum_{k=1}^3 a_k$ is the sum $a_1 + a_2 + a_3$

It doesn't seem so bad to write these out!

$\sum_{k=1}^{50} a_k$ is the sum

$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} +$
 $a_{16} + a_{17} + a_{18} + a_{19} + a_{20} + a_{21} + a_{22} + a_{23} + a_{24} + a_{25} + a_{26} + a_{27} + a_{28} + a_{29} + a_{30} +$
 $a_{31} + a_{32} + a_{33} + a_{34} + a_{35} + a_{36} + a_{37} + a_{38} + a_{39} + a_{40} + a_{41} + a_{42} + a_{43} +$
 $a_{44} + a_{45} + a_{46} + a_{47} + a_{48} + a_{49} + a_{50}$

Nobody wants to write all 50 terms.

Though, in fairness we could have written this last sum as $a_1 + a_2 + a_3 + \dots + a_{49} + a_{50}$

20. SUMMATION EXAMPLES

The sum of the first 10 integers can be written

$$S = \sum_{k=1}^{10} k$$

Evaluate this sum

$$S = \sum_{k=1}^{10} k = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

21. SUMMATION EXAMPLES

What is $S = \sum_{k=1}^{10} 2k$

This is the sum of the first 10 even integers:

$$S = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 = 110$$

Can you think of an easier way, using our previous calculation $S = \sum_{k=1}^{10} k = 55$?

Factor out the 2 from each term

$$S = \sum_{k=1}^{10} 2k = 2 \sum_{k=1}^{10} k = 2 \cdot 55 = 110$$

22. RECTANGULAR APPROX OF AREA

Using summation notation, we can write the sum of the areas of the 6 rectangles used to approximate the area under the curve

$$A(6) = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \Delta x \cdot f(x_4) + \Delta x \cdot f(x_5) + \Delta x \cdot f(x_6)$$

$$\text{as } A(6) = \sum_{k=1}^6 \Delta x f(x_k)$$

To go from 6 to 12 rectangles is easy:

$$A(12) = \sum_{k=1}^{12} \Delta x f(x_k) = \Delta x \sum_{k=1}^{12} f(x_k)$$

by factoring out Δx .

23. EXAMPLE $y = x^2$ REVISITED

$$y = f(x) = x^2$$

n = number of rectangles

$$[a, b] = [0, 3]$$

$$\Delta x = \frac{3}{n}$$

$$x_k = \Delta x \cdot k$$

$$A(n) = \Delta x \sum_{k=1}^n f(x_k)$$

24. COMPUTING THE EXACT AREA

The following table gives the values of $A(n)$ for various values of n :

n	$A(n)$
3	14
6	11.375
12	10.15625
24	9.5703125
100	9.13545
1000	9.0135045
10000	9.001350045
1000000	9.0000135

It looks like the exact area under the curve $y = x^2$ between 0 and 3 is 9

25. CALCULATING AREAS

The exact area A under the curve $y = f(x)$ from a to b is defined to be the limit of the Riemann Sums $A(n)$ as the number of rectangles n goes to infinity.

That is,

$$A = \lim_{n \rightarrow \infty} A(n) = \lim_{n \rightarrow \infty} \Delta x \sum_{k=1}^n f(x_k)$$

For many simple functions, such as $f(x) = x^n$, the limit that gives the area under the curve has been worked out.

26. EXAMPLE $y = x^2$

For example, suppose $f(x) = x^2$ and we wish to compute the area from $a = 0$ to b .

We begin by computing the area under the n rectangles, using right end points.

Here

$$\Delta x = \frac{b}{n} \text{ and } x_k = k \Delta x$$

$$\begin{aligned} A(n) &= \sum_{k=1}^n \Delta x \cdot f(x_k) \\ &= \Delta x \sum_{k=1}^n f(x_k) \quad [\text{factor out } \Delta x] \end{aligned}$$

27. EXAMPLE PART 2

$$\begin{aligned} A(n) &= \Delta x \sum_{k=1}^n f(x_k) = \Delta x \sum_{k=1}^n x_k^2 \quad [f(x) = x^2] \\ &= \Delta x \sum_{k=1}^n (k \Delta x)^2 \quad [\text{because } x_k = k \Delta x] \\ &= \Delta x \sum_{k=1}^n k^2 \Delta x^2 \quad [\text{algebra}] \\ &= (\Delta x)^3 \sum_{k=1}^n k^2 \quad [\text{factor out } (\Delta x)^2] \\ &= \frac{b^3}{n^3} \sum_{k=1}^n k^2 \quad [\text{because } \Delta x = \frac{b}{n}] \end{aligned}$$

28. EXAMPLE PART 3

At this point we need a formula for $\sum_{k=1}^n k^2$.

Page 309, formula [6]: $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$

Verify formula [6] for $n = 1, 2, 3, 4, 5$.

$$\begin{aligned}\text{Thus, } A(n) &= \frac{b^3}{n^3} \sum_{k=1}^n k^2 \\ &= \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= \frac{b^3}{6} \left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right).\end{aligned}$$

[note n cancels.]

29. EXAMPLE CONCLUSION

The exact area A under the curve is the limit of the rectangular approximations $A(n)$ as $n \rightarrow \infty$:

$$\begin{aligned}A &= \lim_{n \rightarrow \infty} A(n) \\ &= \lim_{n \rightarrow \infty} \frac{b^3}{6} \left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{b^3}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \\ &= \frac{b^3}{6} (1+0) (2+0) = \frac{b^3}{3}\end{aligned}$$

When $b = 3$, $A = \frac{3^3}{3} = 9$, a fact that we laboriously worked out earlier.

30. VARIABLES VERSUS CONSTANTS

In mathematics, variables represent quantities which are allowed to vary, while constants are quantities that never change.

We usually use letter at the beginning of the alphabet, such as “a”, “b”, “c” to represent constants, and variables close to “x” (the unknown) to represent variables.

In computing the area from a to b , if we want to allow the right endpoint b to vary, we should probably rename it to a variable letter, such as x .

This creates a problem, however, since we usually use the letter x when we write the function as $f(x)$.

To avoid confusion, we rewrite the function as $f(t)$.

That is, simultaneously, search and replace

$$b \rightarrow x \text{ and } x \rightarrow t.$$

31. CHANGING LETTERS

For example, suppose $f(t) = t^2$, the squaring function, and we wish to calculate the area from $a = 0$ to $b = x$.

Note that we are using the letter “**x**” for the right endpoint.

To avoid confusion, we are using the letter “**t**” to describe the function.

Using x instead of b for the right endpoint, we have

$$A = \lim_{n \rightarrow \infty} A(n) = \lim_{n \rightarrow \infty} \frac{x^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) = \frac{x^3}{6} (1 + 0) (2 + 0) = \frac{x^3}{3}$$

32. AREA UNDER $y = x^n$

Suppose $f(t) = t^n$ for any non-negative integer.

We want to find the area A under the curve from $a = 0$ to $b = x$.

Case: $n = 0$

Our function is just $f(t) = t^0 = 1$.

The region is one long rectangle, whose dimensions are x by 1.

The area of this rectangle is $A = x$.

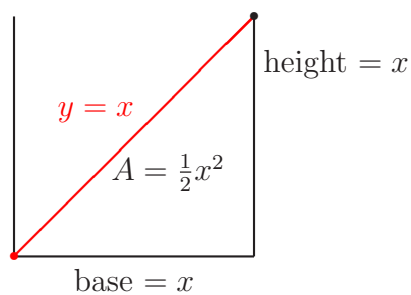
33. AREA UNDER $y = x^1$

Case: $n = 1$

Our function is just $f(t) = t^1 = t$. (5, 10)

The region is an isosceles triangle whose dimensions are x by x .

The area of this rectangle is $A = \frac{1}{2}x^2$.



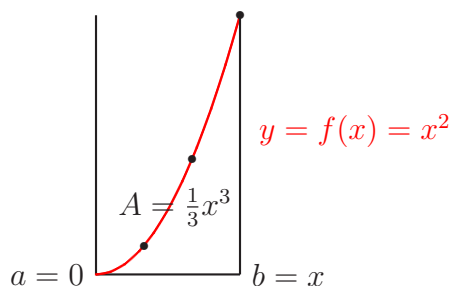
34. AREA UNDER $y = x^2$

Case: $n = 2$

Our function is just $f(t) = t^2$.

We have already worked this out.

The area of this region is $A = \frac{1}{3}x^3$.



35. AREA UNDER $y = x^n$, $n = 0, \dots, 6$

For every possible value of the n , the area under the curve $y = f(x) = x^n$ from 0 to x has been calculated.

These areas are given in the following chart:

$f(x)$	A
$x^0 = 1$	x
x^1	$\frac{1}{2}x^2$
x^2	$\frac{1}{3}x^3$
x^3	$\frac{1}{4}x^4$
x^4	$\frac{1}{5}x^5$
x^5	$\frac{1}{6}x^6$
x^6	$\frac{1}{7}x^7$

Differentiate the second column and compare with the first column.

36. BIG SURPRISE!

The second column is the anti-derivative of the first column.

Equivalently, the derivative of the area $A(x)$ is just the function $f(x)$.

This connection between area and derivatives is so, well fundamental, that it is given the following name:

The Fundamental Theorem of Calculus