

The Fundamental Theorem of Calculus

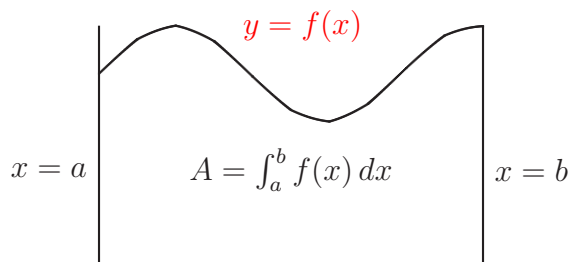
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1. THE DEFINITE INTEGRAL

We define the **definite integral** of a continuous function $f(x)$ from a to b , written

$$\int_a^b f(x) dx$$

to be the area under the curve $y = f(x)$ between two vertical sides $x = a$ and $x = b$.



2. THE FUNDAMENTAL THEOREM OF CALCULUS PART I

If $A(x)$ = the area under the curve $y = f(x)$ from a to x ,

$$A(x) = \int_a^x f(t) dt,$$

then

$$A'(x) = f(x).$$

So in order to solve the area problem, we must find a function whose derivative is $f(x)$.

Such functions are just **anti-derivatives**, which we have seen before.

3. CONNECTION TO ANTI-DERIVATIVES

Let $A(x) = \int_a^x f(t) dt$, the area under the curve $f(t)$ starting at a and ending at x .

Note that we have written the function as $f(t)$ rather than as $f(x)$, because we are using x as the stopping value.

Now let $F(x)$ be any anti-derivative of $f(x)$.

That is, $F(x)$ has the property

$$F'(x) = f(x)$$

We have seen that

$$A'(x) = f(x)$$

So $F(x)$ and $A(x)$ both have the same derivative $f(x)$.

But functions with the same derivative differ by a constant.

So $A(x) = F(x) + C$ for some constant C

4. CONNECTION TO ANTI-DERIVATIVES II

If $F(x)$ is an anti-derivative of $f(x)$, then

$$A(x) = \int_a^x f(t) dt = F(x) + C$$

To determine C , plug in $x = a$ into the equation

$$A(a) = \int_a^a f(t) dt = F(a) + C$$

Clearly $A(a) = \int_a^a f(t) dt = 0$ **Why?** because our interval $[a, a]$ is just the single point a .

$$\text{So } F(a) + C = 0$$

$$\text{or } C = -F(a)$$

$$\text{and hence } A(x) = \int_a^x f(t) dt = F(x) - F(a)$$

Switching letters b for x and x for t

$$\int_a^b f(x) dx = F(b) - F(a)$$

5. THE FUNDAMENTAL THEOREM OF CALCULUS

$$\boxed{\int_a^b f(x) dx = F(b) - F(a)}$$

where $F(x)$ is an anti-derivative of $f(x)$

Calculus was born when Sir Issac Newton **the fig guy?** and Gottfried Wilhelm Leibniz independently discovered this essential connection between integrals (areas) and derivatives (rates of change).

The Fundamental Theorem of Calculus says that in order to find the area A under the curve $y = f(x)$ from a to b ,

first find an anti-derivative of $f(x)$, that is, a function $F(x)$ whose derivative is $f(x)$

then the area is $A = F(b) - F(a)$

6. EASY EXAMPLE

Compute the definite integral $\int_1^3 2x dx$

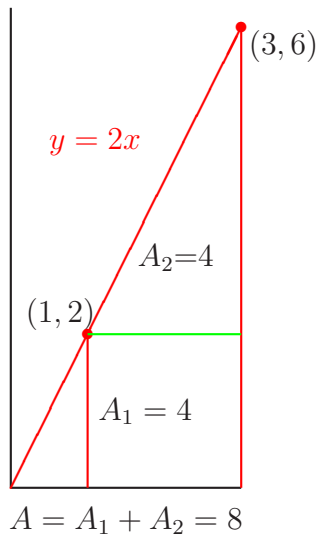
An anti-derivative of $f(x) = 2x$ is $F(x) = x^2$

So by the Fundamental Theorem of Calculus

$$\int_1^3 2x dx = F(3) - F(1) = 3^2 - 1^2 = 8$$

The area under the line $y = 2x$ for 1 to 3 is 8, as the next picture shows.

7. PICTURE $\int_1^3 2x = 8$



8. SOME NOTATION

Instead of writing $F(b) - F(a)$, we use notation

$$F(x) \Big|_a^b = F(b) - F(a)$$

Evaluate the definite integral $\int_2^5 3x^2 dx$

An anti-derivative of $f(x) = 3x^2$ is $F(x) = x^3$

So by the Fundamental Theorem of Calculus

$$\int_2^5 3x^2 dx = x^3 \Big|_2^5 = 5^3 - 2^3 = 125 - 8 = 117$$

9. DEFINITE VS INDEFINITE INTEGRAL

The integral sign $\int$ has two meanings:

(1) without limits a and b , $\int f(x) dx$ denotes the antiderivative of $f(x)$.

(2) with limits a and b , $\int_a^b f(x) dx$ denotes the area under the curve $y = f(x)$ from a to b .

Type (1) integrals are called **indefinite** integrals.

Type (2) integrals are called **definite** integrals.

10. THE $+C$

When employing the Fundamental Theorem of Calculus

$$\int_a^b f(x) dx = F(b) - F(a)$$

the anti-derivative $F(x)$ is just the indefinite integral $F(x) = \int f(x) dx$.

In this case, we do not need to write the $+C$ at the end of the integral because it will end up being cancelled.

For example,

$$\int_1^3 2x dx = (x^2 + C) \Big|_1^3 = (3^2 + C) - (1^2 + C) = (9 + C) - (1 + C) = 9 - 1 + C - C = 8$$

11. PC (PLUS “C”) CORRECTNESS

It is **VERY WRONG!!!** to write

$$\int_1^3 2x dx = 8 + C$$

After all, the area is the specific number 8. You cannot add any constant you like to it.

On the other hand, it is also wrong to forget the $+C$ when writing indefinite integrals,

$$\int 2x dx = x^2 + C$$

12. A MATH JOKE

Two male calculus students are sitting in the Junction Restaurant in DeKalb.

“Math 229 sure is hard,” says one of them.

“Oh, I don’t know,” says the other. “I bet anybody could learn calculus.”

“Really? What about our waitress?”

“I bet I could teach her some calculus in a couple of minutes.”

“Okay, it’s a bet. The loser buys lunch.”

So the second fellow wanders off to where the waitress (who happens to be a blonde) is filling some drinks at her station and says to her:

13. END OF THE JOKE

When you bring us our check, I’ll ask you a question. I would like you to answer: one third x cubed. Can you do that? It’s worth an extra tip.

“Sure.” The girl giggles and repeats several times: “one third x cubed, one third x cubed, one third x cubed, . . .”

When the waitress brings the guys their bill, one of them says, “I have a little bet with my friend here. Can you tell us what the integral of x squared is?”

She replies: “one third x cubed.”

Then, as she leaves the table, the waitress says over her shoulder,

”plus, of course, an undertermined constant.”

14. AREA PROBLEM 1

Find the area under the curve $y = \sqrt{x}$ from 1 to 9

Solution: By the Fundamental Theorem of Calculus, this area is

$$\int_1^9 x^{1/2} dx = \frac{2}{3} x^{3/2} \Big|_1^9 = \frac{2}{3} [\sqrt{9^3} - \sqrt{1^3}] = \frac{2}{3} (27 - 1) = \frac{52}{3}$$

15. AREA PROBLEM 2

Find the area under the curve $y = \sin(x)$ from 0 to π

Solution: In order to use the Fundamental Theorem of Calculus, we need an anti-derivative of $\sin x$.

Happily we know a function whose derivative is $\sin x$,

namely the negative of the cosine $-\cos x$.

That is,

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int_0^\pi \sin(x) dx = -\cos(x) \Big|_0^\pi = -\cos(\pi) - (-\cos(0)) = 1 + 1 = 2$$

16. AREA PROBLEM 3

Find the area under the curve $y = 2x - 10$ from 3 to 6

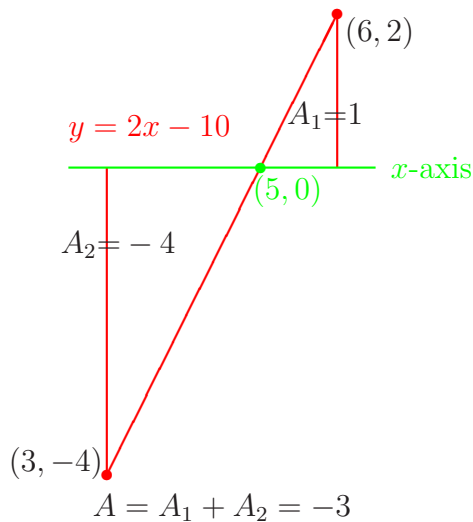
$$\begin{aligned} \int_3^6 2x - 10 dx &= x^2 - 10x \Big|_3^6 \\ &= 36 - 60 - (9 - 30) = -24 + 21 = -3 \end{aligned}$$

Wait a minute.

How can area be negative?

The answer is that integrals compute the areas of regions below the x -axis as negative.

17. PICTURE $\int_3^6 2x - 10 \, dx = -3$



18. AREA BETWEEN TWO CURVES

The area between two curves from a to b is given by the integral

$$A = \int_a^b f(x) - g(x) \, dx$$

provided $f(x) \geq g(x)$ on the interval $[a, b]$

That is, to find the area between two curves, we always subtract the curve on the bottom from the one on the top.

19. EXAMPLE

Find the area $y = f(x) = x^2 - 6x + 11$ and $y = g(x) = -x + 7$.

To find where these curve cross, we set $f(x) = g(x)$

$$x^2 - 6x + 11 = -x + 7$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$x = 1 \text{ or } x = 4$$

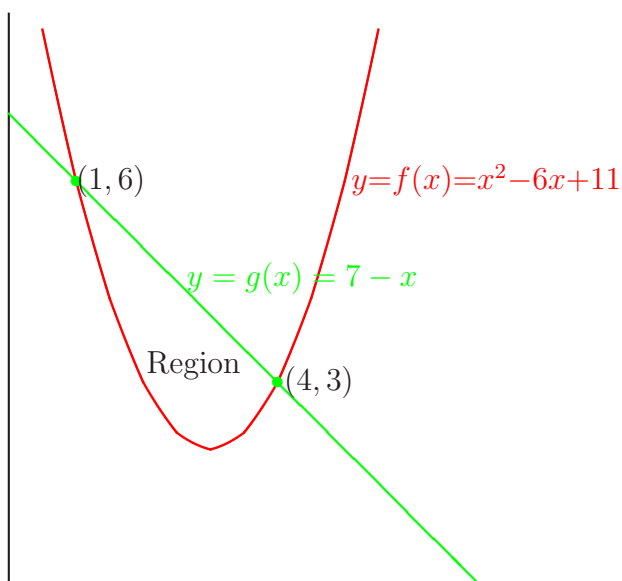
$$\text{When } x = 1, y = f(1) = g(1) = 6$$

$$\text{When } x = 4, y = f(4) = g(4) = 3$$

So the intersection points are $(1, 6)$ and $(4, 3)$

Now draw the graph to determine which curve is on top.

20. TALE OF TWO GRAPHS



21. AREA BETWEEN 2 CURVES

We see from the graph that $g(x) = 7 - x$ lies above $f(x) = x^2 - 6x + 11$

The area between these two curves is

$$\int_1^4 g(x) - f(x) \, dx = \int_1^4 (7 - x) - (x^2 - 6x + 11) \, dx$$

$$= \int_1^4 7 - x - x^2 + 6x - 11 \, dx$$

$$= \int_1^4 -4 + 5x - x^2 \, dx$$

$$\begin{aligned} &= \left(-4x + \frac{5}{2}x^2 - \frac{1}{3}x^3\right)\bigg|_1^4 \\ &= \left(-16 + \frac{5}{2}16 - \frac{1}{3}64\right) - \left(-4 + \frac{5}{2} - \frac{1}{3}\right) \\ &= -16 + 40 - \frac{64}{3} + 4 - \frac{5}{2} + \frac{1}{3} \\ &= 28 - \frac{63}{3} - \frac{5}{2} \\ &= 7 - \frac{5}{2} \\ &= \frac{9}{2} \end{aligned}$$