

An Elementary Approach to Primes

Professor Richard Blecksmith
Dept. of Mathematical Sciences
Northern Illinois University

<http://math.niu.edu/~richard/Math230>

Primes and Composites

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But not $4 = 2 \times 2$, $6 = 2 \times 3$, etc.

A positive integer greater than 1 which factors is called a **composite** number.

The number 1 is called a **unit**. It is neither prime nor composite.

Question

What are the primes good for?

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They are useful in many arithmetic problems.

Grade School Problem

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So we can use 18 as a common denominator:

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So we can use 18 as a common denominator:

$$\frac{1}{6} + \frac{1}{9} = \frac{3}{18} + \frac{2}{18} = \frac{5}{18}$$

Least Common Multiple

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2. $\text{LCM}[25, 35] = 175$

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1. $\text{LCM}[6, 18] = 18$
2. $\text{LCM}[25, 35] = 175$
3. $\text{LCM}[100, 210] = 2100$

Greatest Common Divisor

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Examples:

$$\text{LCM}[6, 18] = \frac{6 \cdot 18}{\text{GCD}(6, 18)} = \frac{6 \cdot 18}{6} = 18.$$

$$\text{LCM}[25, 35] = \frac{25 \cdot 35}{\text{GCD}(25, 35)} = \frac{25 \cdot 35}{5} = 175.$$

Finding GCD

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$$25 = 5 \times 5 \text{ and } 35 = 5 \times 7.$$

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$$\text{So GCD}(30, 18) = 2 \times 3$$

Find GCD [25, 35]

$$25 = 5 \times 5 \text{ and } 35 = 5 \times 7.$$

$$\text{So GCD}(25, 35) = 5$$

Katie's Problem

Find the GCD (99, 100).

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$$99 = 9 \times 11 = 3 \times 3 \times 11$$

$$100 = 10 \times 10 = 2 \times 2 \times 5 \times 5$$

Katie's Problem

Find the GCD (99, 100).

Fifth Grade Textbook Solution: Factor 99 and 100 into primes:

$$99 = 9 \times 11 = 3 \times 3 \times 11$$

$$100 = 10 \times 10 = 2 \times 2 \times 5 \times 5$$

Since none of the primes factors of 99 (3 and 11) overlap with the prime factors of 100 (2 and 5),

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$$100 = 10 \times 10 = 2 \times 2 \times 5 \times 5$$

Since none of the primes factors of 99 (3 and 11) overlap with the prime factors of 100 (2 and 5),
 $\text{GCD}(99, 100) = 1$

Katie's Dad's Solution:

Any number which divides both 99 and 100 must divide their difference

$$100 - 99 = 1.$$

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But the only number which divides 1 is 1 itself.
So

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Another example:

Find $\text{GCD}(17463287493, 17463287494) = 1$

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Numbers with no common factor (except 1) are called
relatively prime.

Are primes plentiful or scarce?

What do you think?

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We will consider both points of view:

Are primes plentiful or scarce?

What do you think?

We will consider both points of view:

1. The primes are scarce.

Are primes plentiful or scarce?

What do you think?

We will consider both points of view:

1. The primes are scarce.
2. The primes are plentiful.

Handling Complaints

Whenever a student complains:

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Whenever a student complains:

“What good is all this math we’re learning?”

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my standard response is:

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We are going to take a little sojourn into the world of word problems.

Solving Word Problems

- Create Variables

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- Create Variables
- Organize Data
- Draw a diagram
- Translate Words $\rightarrow$ Equation (English to Math)
- Solve the equation.

Building Bookcases

You wish to build a bookcase.

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You are using lumber which is $\frac{3}{4}$ inch thick.

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The dimensions are 36 inches wide, 48 inches high.

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You want to install 2 interior shelves so that the shelves are equally spaced.

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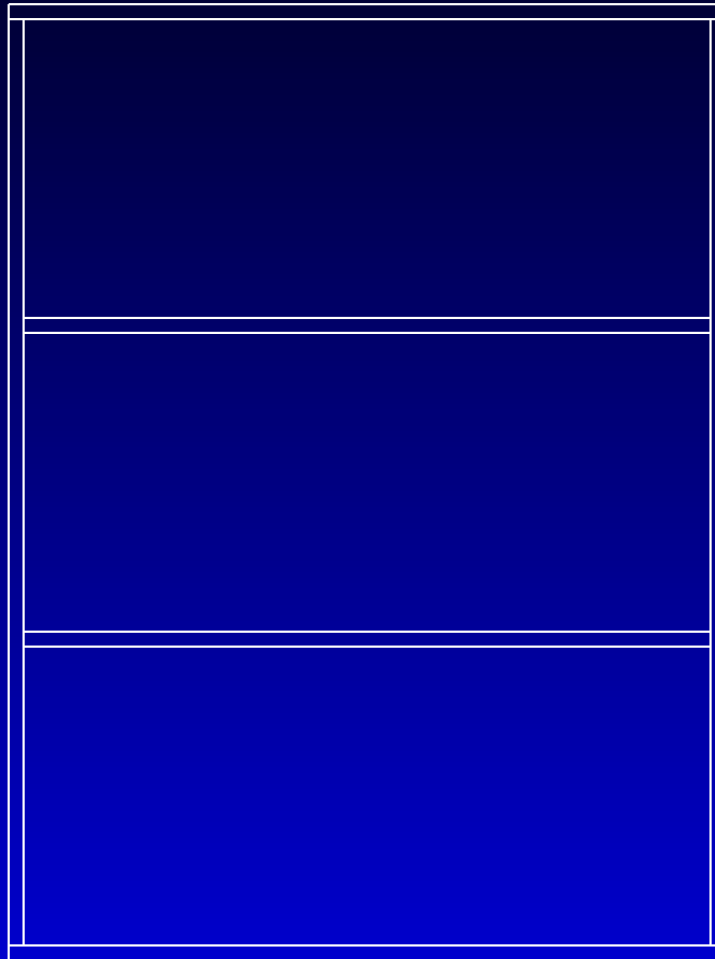
You are using lumber which is $\frac{3}{4}$ inch thick.

The dimensions are 36 inches wide, 48 inches high.

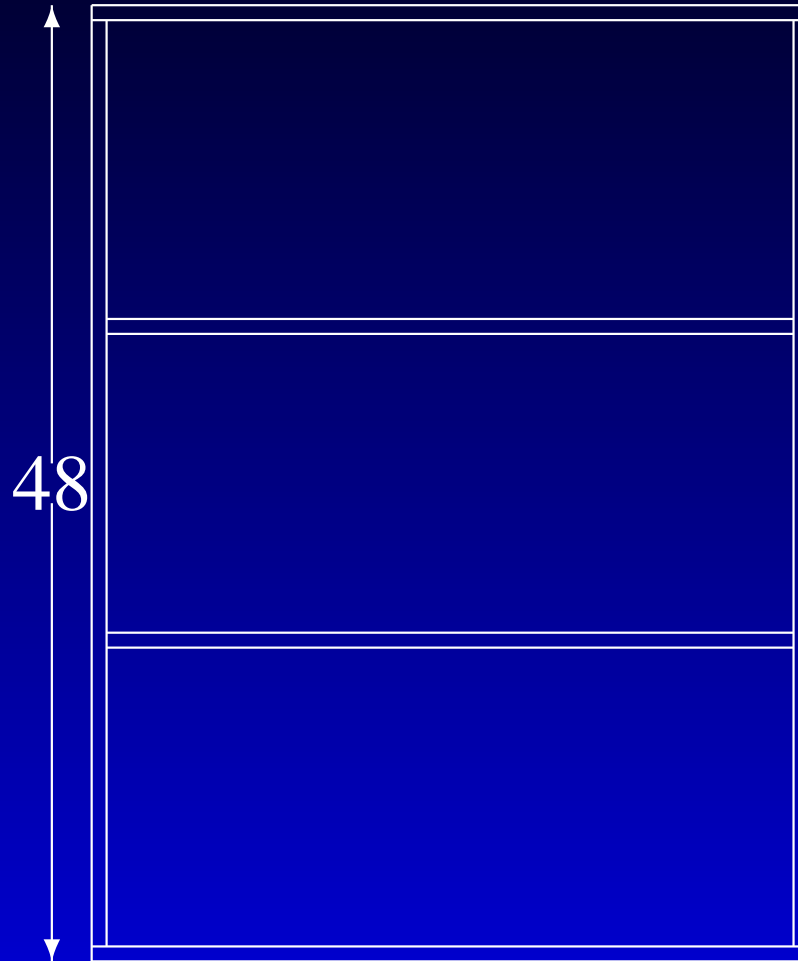
You want to install 2 interior shelves so that the shelves are equally spaced.

What is the spacing between the shelves?

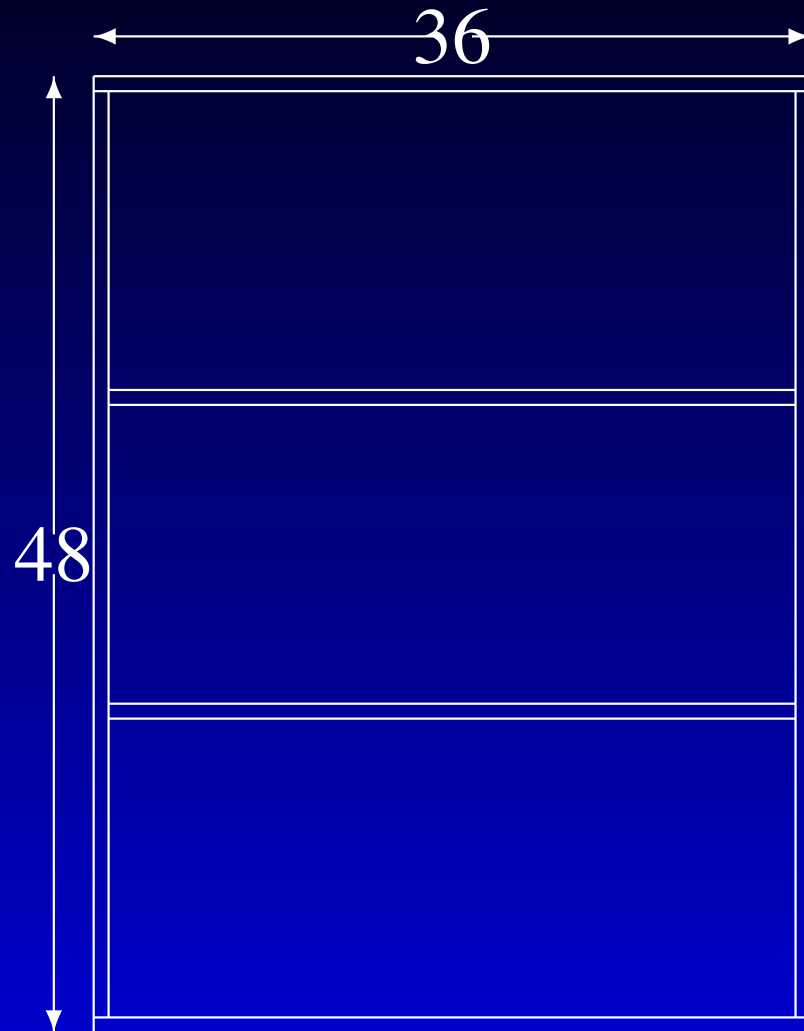
Bookcase Picture



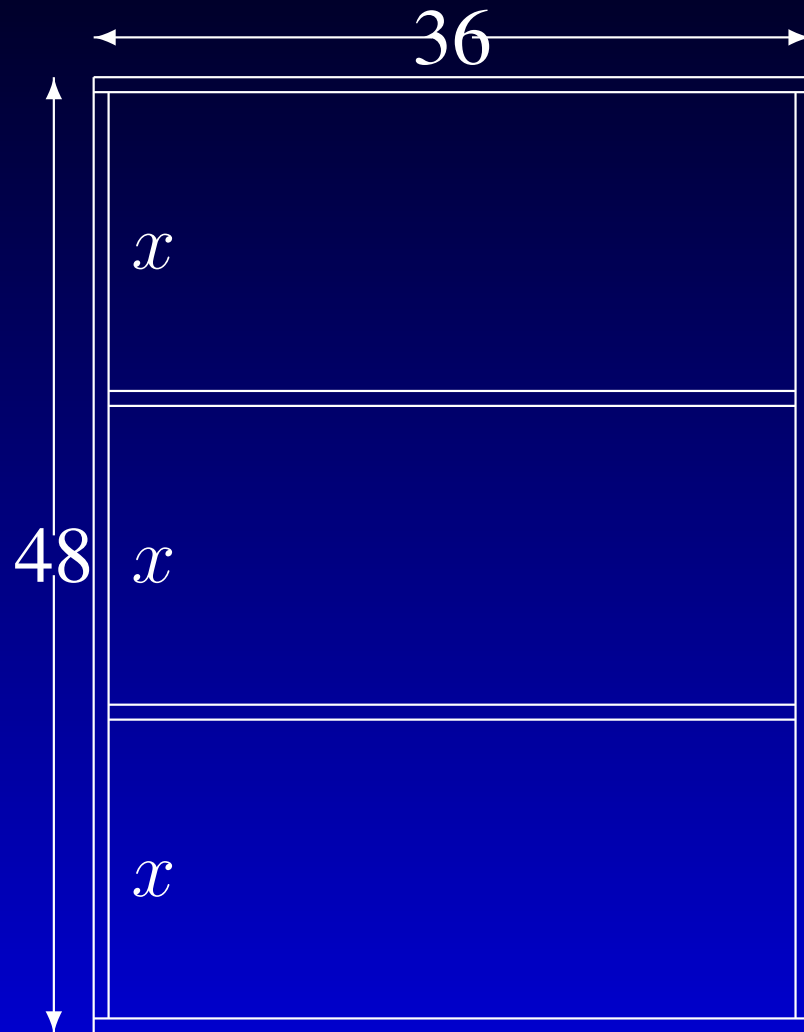
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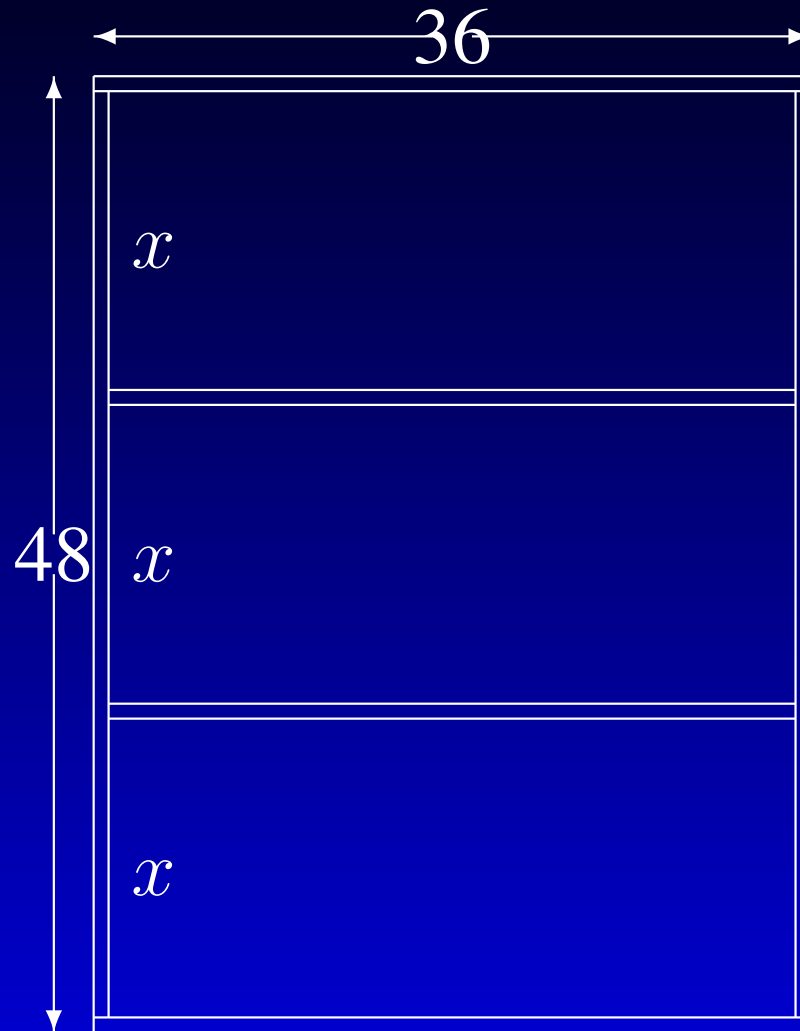
Bookcase Picture



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$$3x + 4(.75) = 48$$

Bookcase Solution

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$$x = 15$$

Bookcase Solution

$$3x + 4(.75) = 48$$

$$3x + 3 = 48$$

$$3x = 45$$

$$x = 15$$

Make the shelves 15 inches apart.

Professors and Students

Consider the sentence:

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For every professor at a small liberal arts college,
there are two students.

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If P = the number of professors and

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(a) $P = 2S$

(b) $S = 2P$

(c) Some other answer

Word Problems - Yuck

Are the following word problems solved in the same way:

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Painting Problem. Bob can paint a room in one hour.

His brother Bill can paint the room in 45 minutes.

How long will it take them, working together, to paint the room?

Word Problems - Yuck

Are the following word problems solved in the same way:

Painting Problem. Bob can paint a room in one hour. His brother Bill can paint the room in 45 minutes. How long will it take them, working together, to paint the room?

Laundry Problem. Joan can wash a load of laundry in one hour. Her mother Mary can do the laundry in 45 minutes. How long will it take them, working together, to do a load of laundry?

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Are the following word problems solved in the same way:

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Laundry Problem. Joan can wash a load of laundry in one hour. Her mother Mary can do the laundry in 45 minutes. How long will it take them, working together, to do a load of laundry?
Explain your answer.

Mary Jane

Lucy and Lottie Hill are 90 years old.

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How old is Mary Jane?

This is why people hate math.

Mary Jane Cont'd

Let x = Mary Jane's age.

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How old was she on her last birthday?

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How old was she on her last birthday? x

Half as old as she was on her last birthday is $x/2$

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Lacking 5 years of being half as old as she was on her last birthday is

Mary Jane Cont'd

Let x = Mary Jane's age.

How old was she on her last birthday? x

Half as old as she was on her last birthday is $x/2$

Lacking 5 years of being half as old as she was on her last birthday is $x/2 - 5$

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How old was she on her last birthday? x

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Half again means 50% more,

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For example, if gas costs 2 dollars and next year it costs half again as much as this year, then next year a gallon of gas will cost

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For example, if gas costs 2 dollars and next year it costs half again as much as this year, then next year a gallon of gas will cost $\frac{3}{2} \times 2 = 3$ dollars

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Mary Jane Equation

Lucy and Lottie Hill are 90 years old. Mary Jane, on the other hand, is half again as old as $(\frac{3}{2} \text{ times})$ when she was half again as old as $(\text{another } \frac{3}{2} \text{ times})$ when she lacked 5 years of being half as old as was on her last birthday $(\frac{x}{2} - 5)$.

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Solve this equation for x .

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Equation:
$$x = \frac{3}{2} \frac{3}{2} \left(\frac{x}{2} - 5 \right)$$

Solve this equation for x .

Why does the problem begin with Lucy and Lottie Hill?

Charise's Problem:

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Can you solve this problem by coming up with an equation to solve for x ?

Answer for Charise

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Mary's age	Gramps' age
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Mary's age	Gramps' age
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1	
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Mary's age	Gramps' age
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Mary's age	Gramps' age
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---	----

3	
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Mary's age	Gramps' age
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4	
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Mary's age	Gramps' age
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2	
---	--

3	
---	--

4	
---	--

5	
---	--

Answer for Charise

How old was Mary's grandfather when she was born if for her first five consecutive birthdays, Mary's age exactly divided her grandfather's age?

Answer: Gramps was 60 when Mary was born.

Verification

Mary's age	Gramps' age
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1	61
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2	62
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Note that $60 = LCM[1, 3, 4, 5]$

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This would be a very old Grandpaw.

Prime Deserts

A group of n consecutive composite integers is called a **prime desert** of length n .

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How long can prime deserts be?

Shoot for 100

Problem Find 100 consecutive composite numbers.

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Answer: Let

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Answer: Let

$$x = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 100 \times 101.$$

Then the numbers

$$x + 2, x + 3, x + 4, \dots, x + 101$$

are all composite.

How do you know?

To see this, simply observe that

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What is the length of this list?

Arbitrarily Long Prime Deserts

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What does “arbitrarily long” mean?

Conclusion: Primes are scarce.

Euclid's Theorem

Over two thousand years ago Euclid (the geometry dude) showed that we never run out of primes.

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$$x = n! = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times n.$$

Explain how Katie's Theorem shows that none of the primes less than n divide $x + 1$.

An Infinity of Primes

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Theorem There are infinitely many primes.

Since there are **infinitely** many primes, they must be plentiful.

Second thought

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There are only 12 powers of 10 up to one trillion.

We want to know: Are big primes as rare as finding a parking space on campus?
or as winning the lottery?

Enter Probability

The question “Are the primes plentiful or scarce?”
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If I pick a large integer (at random) what is the probability it will be prime?

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Example: The probability a one hundred digit number chosen at random is prime is roughly

$$1/230$$

Application to Encryption

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If you avoid even numbers, the probability increases to $1/115$.

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Why?

Primes of form $4x + 1$

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x $4x + 1$ Prime or Composite?

Primes of form $4x + 1$

x	$4x + 1$	Prime or Composite?
-----	----------	---------------------

1		
---	--	--

Primes of form $4x + 1$

x	$4x + 1$	Prime or Composite?
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1	5	
---	---	--

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2		
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2	9	
---	---	--

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6	25	

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It has been proven that this list consists of infinitely many primes.

Twin Primes

Primes sometimes come in pairs, two apart:

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Such numbers are called **twin primes**.

It is unknown (that is, unproven) whether there are infinitely many twin primes.

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If you examine the table for $y = x^2 + 1$, you will find many primes.

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No one knows whether there are infinitely many primes in this list.