

L'HÔPITAL'S RULE

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1. MOTIVATION

Suppose $f(x)$ and $g(x)$ are two functions with

$$f(a) = g(a) = 0.$$

Then for $x \neq a$

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

If the derivatives $f'(a)$ and $g'(a)$ exist,

and if $g'(a) \neq 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

2. EXAMPLE 1

Compute

$$\lim_{x \rightarrow 0} \frac{1 - e^{2x}}{x}$$

3. L'HÔPITAL'S RULE (0/0)

Assume $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

4. EXAMPLE 2

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{1} \\ &= \frac{1}{1} = 1 \end{aligned}$$

Why didn't we use this method in Calc I?

5. EXAMPLE 3

Compute $\lim_{x \rightarrow 0} \frac{x - \tan x}{x - \sin x}$

The ratio of the derivatives is

$$\frac{f'(x)}{g'(x)} = \frac{1 - \sec^2 x}{1 - \cos x}$$

By L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{x - \tan x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{1 - \sec^2 x}{1 - \cos x}$$

This second limit also assumes the form (0/0).

There are two solutions:

1. Use L'Hôpital's Rule a second time.
2. Remove the indeterminacy by writing

$$1 - \sec^2 x = 1 - \frac{1}{\cos^2 x} = -\frac{1 - \cos^2 x}{\cos^2 x}$$

Now factor $1 - \cos^2 x$

6. EXAMPLE 4

Compute

$$\lim_{x \rightarrow 1} \frac{x^c - cx + c - 1}{(x - 1)^2}$$

7. DANGER – DANGER

When using L'Hôpital's Rule, some care is needed to make certain the quotients under consideration are of the form $0/0$.

Example 5. Compute $\lim_{x \rightarrow 1} \frac{3x^2 - 2x - 1}{x^2 - x}$

By L'Hôpital's Rule (twice)

$$\lim_{x \rightarrow 1} \frac{3x^2 - 2x + 1}{x^2 - x} = \lim_{x \rightarrow 1} \frac{6x - 2}{2x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{6}{2} = 3$$

What is wrong here?

8. EXAMPLE 6

Compute

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x}$$

By L'Hôpital's Rule

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x} = \lim_{x \rightarrow 0^+} \frac{e^{-1/x} \cdot (x^{-2})}{1}$$

$$= \lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x^2}$$

Yikes. It got worse.

We will return to this problem later.

9. EXAMPLE 7

Sometimes a change of variable shortens the work.

Compute

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}}$$

One could apply L'Hôpital's Rule directly.

But it saves time to change variables to $t = \sqrt{x}$:

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}} &= \lim_{t \rightarrow 0^+} \frac{t}{1 - e^{2t}} \\ &= \lim_{t \rightarrow 0^+} \frac{1}{-2e^{2t}} \\ &= -\frac{1}{2} \end{aligned}$$

10. EXTENSIONS OF L'HÔPITAL'S RULE

We can allow a to be ∞ or $-\infty$.

Assume $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 0$. Then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

There is an analogous result as we consider limits as $x \rightarrow -\infty$.

11. A SECOND EXTENSION

Suppose $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$.

Then $\lim_{x \rightarrow a} \frac{1}{f(x)} = 0$ and $\lim_{x \rightarrow a} \frac{1}{g(x)} = 0$.

Note $\frac{d}{dx} \frac{1}{f(x)} = \frac{d}{dx} (f(x))^{-1} = (-1)(f(x))^{-2} \cdot f'(x)$

By L'Hôpital's Rule,

$$\lim_{x \rightarrow a} \frac{g(x)}{f(x)} = \lim_{x \rightarrow a} \frac{1}{f(x)} / \frac{1}{g(x)} =$$

$$\lim_{x \rightarrow a} \frac{-(f(x))^{-2} f'(x)}{-(g(x))^{-2} g'(x)} = \lim_{x \rightarrow a} \frac{(g(x))^2}{(f(x))^2} \cdot \frac{f'(x)}{g'(x)}$$

$$\text{So } \lim_{x \rightarrow a} \frac{g(x)}{f(x)} \cdot \frac{(f(x))^2}{(g(x))^2} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

$$\text{or } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Thus L'Hôpital's Rule hold when both limits are $\pm\infty$

12. EXAMPLE 8

Compute

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x}$$

By L'Hôpital's Rule ($\frac{\infty}{\infty}$ case)

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}e^x}$$

$$= 0$$

13. EXAMPLE 9

We return to Example 6: $\lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x}$

A direct application on L'Hôpital's Rule fails to determine the limit.

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x} = \lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x^2}$$

The trick is to rewrite the fraction as

$$\frac{1/x}{e^{1/x}} = \frac{t}{e^t} \quad \text{where } t = \frac{1}{x}.$$

As x tends to 0^+ , $t = \frac{1}{x}$ tends to ∞

$$\text{So } \lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x} = \lim_{t \rightarrow \infty} \frac{t}{e^t} = \lim_{t \rightarrow \infty} \frac{1}{e^t} = 0$$

14. EXAMPLE 10

Compute $\lim_{x \rightarrow \infty} x^{1/x}$

We don't see a quotient, so what should we do?

If $y = x^{1/x}$, the trick is to take the natural log of y :

$$\ln y = \ln(x^{1/x}) = \frac{1}{x} \cdot \ln x = \frac{\ln x}{x}$$

Now use L'Hôpital's Rule on this quotient:

$$\text{So } \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$$

Since $\ln y = \ln(x^{1/x})$, $x^{1/x} = e^y$,

If $\ln y \rightarrow 0$, then $x^{1/x} = e^y \rightarrow e^0 = 1$.