

THE NATURAL LOGARITHM

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1. THE FUNDAMENTAL THEOREM OF CALCULUS

If $f(t)$ is continuous, a is a constant, and

$$A(x) = \int_a^x f(t) dt,$$

then $A'(x) = f(x)$.

For example, if $f(t) = t^2$, and $a = 0$, then

$$A(x) = \int_0^x t^2 dt = x^3/3,$$

$$\text{Then } A'(x) = \frac{d}{dx} x^3/3 = x^2.$$

2. APPLICATION OF THE FUND THM

Find a function whose derivative is $1/x$.

Define

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

For now, think of “ln” as standing for the “logical necessity” function.

By the Fundamental Theorem of Calculus,

$$\text{Then } \ln'(x) = \frac{1}{x}.$$

3. SOLVING PROBLEMS BY NAMING THINGS

So we have solved the problem of finding a function whose derivative is $1/x$.
or have we?

Another way to think about it is, we wanted a function whose derivative is $1/x$, so we just gave it a name—“ln”—and acted like we settled the question.

As an analogy, suppose we wish to find a number x whose square is 7.

Then we use the square root symbol $x = \sqrt{7}$.

We have just given a name to the thing we wanted to find.

4. STUFF WE WANT TO KNOW ABOUT SQUARE ROOTS

How do we know square roots exist? We could write $\sqrt{-1}$, but no number when squared will equal -1 .

How do we compute square roots? Some are easy, like $\sqrt{4}$ or $\sqrt{25}$. Others are not so easy.

What is the decimal value of $\sqrt{7}$?

What are the properties of the square root function?

For example,

$$\sqrt{1} = 1$$

$$\sqrt{xy} = \sqrt{x} \sqrt{y} \quad x, y \geq 0$$

$$(\sqrt{x})^2 = x$$

What's the derivative of $f(x) = \sqrt{x}$?

5. STUFF WE WANT TO KNOW ABOUT ln

How do we know the ln function exists?

How do we compute $\ln(x)$? One value of x in particular is easy:

$$\ln(1) = \int_1^1 \frac{1}{t} dt = 0$$

What is the decimal value of $\ln 7$

All we know is that $\ln(7)$ is the area under the reciprocal curve from $a = 1$ to $b = 7$. How can we find this area?

What are the properties of the $\ln$ function?

What's the derivative of $f(x) = \ln x$?

We know from the Fundamental Theorem that $\frac{d}{dx} \ln(x) = \frac{1}{x}$

6. A ROSE BY ANY OTHER NAME

Actually l.n. does not stand for “logical necessity”

nor “lost nation”

The function $\ln$ is called

the natural logarithm

Reminiscent of French: Logarithme naturel.

Why is it called a logarithm?

Weren't logarithms a topic from high school algebra?

The answer requires some study into the property of this mysterious function.

7. FUNDAMENTAL PROPERTY OF $\ln$

$$\ln(ab) = \ln a + \ln b$$

Why?

Split the interval $[1, ab]$ into two intervals $[1, a]$ and $[a, ab]$. (We are assuming both a and b are > 1 .)

$$\begin{aligned} \ln(ab) &= \int_1^{ab} \frac{1}{t} dt \\ &= \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt \end{aligned}$$

$$= \ln(a) + \int_a^{ab} \frac{1}{t} dt$$

8. EVALUATE THE SECOND INTEGRAL

To evaluate this second integral, $\int_a^{ab} \frac{1}{t} dt$,

substitute $u = \frac{1}{a}t$. Then

$$du = \frac{1}{a} dt, \text{ so } dt = a du.$$

Also, when $t = a$, $u = ?$ $a/a = 1$.

When $t = ab$, $u = ?$ $ab/a = b$.

$$\text{So } \int_a^{ab} \frac{1}{t} dt = \int_1^b \frac{1}{au} a du$$

$$= \int_1^b \frac{1}{u} du = \ln(b)$$

9. LN VERSUS LOG BASE 10

It follows that

$$\ln(ab) = \int_1^{ab} \frac{1}{t} dt$$

$$= \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt$$

$$= \ln(a) + \int_a^{ab} \frac{1}{t} dt$$

$$= \ln(a) + \ln(b)$$

Recall that the $\log_{10}$ function has this exact same property:

$$\log_{10}(ab) = \log_{10}(a) + \log_{10}(b).$$

The function $\ln$ is called a “logarithm” because it acts like one.

10. PLAYING MATHEMATICIAN

Suppose we have a function L that has the property

$$L(ab) = L(a) + L(b)$$

Prove the following:

1. $L(1) = 0$
2. $L(1/b) = -L(b)$
3. $L(a/b) = L(a) - L(b)$

11. MORE PLAY

For L with the property $L(ab) = L(a) + L(b)$

Find $L'(x)$

Write

$$(1) \quad L(ax) = L(a) + L(x)$$

where a is a constant.

By chain rule,

$$\frac{d}{dx}L(ax) = L'(ax) \cdot a$$

By equation (1),

$$\frac{d}{dx}L(ax) = L'(x)$$

since $L(a)$ is a constant.

$$\text{So } L'(x) = a L'(ax)$$

Set $x = 1$

$$\text{Then } a L'(a) = L'(1)$$

12. GRAND FINALE

$$L'(a) = \frac{L'(1)}{a}$$

Replacing a by x ,

$$L'(x) = \frac{L'(1)}{x}$$

So the derivative is a constant divided by x

For the $\ln$ function, can we show geometrically that $\ln'(1) = 1$?

Hint: draw a picture and use the Mean Value Theorem for Integrals.