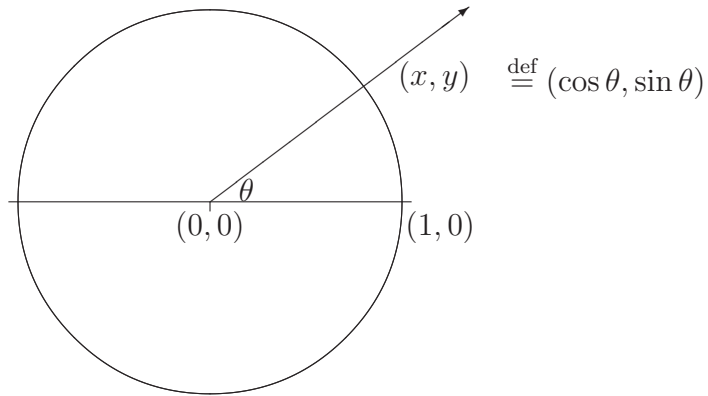


MATH 229 LECTURE NOTES: TRIGONOMETRY

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1. BASIC DEFINITION



Starting at the point $(1,0)$, as you travel counter-clockwise around the unit circle, $\cos(\theta)$ and $\sin(\theta)$ are defined to be the x and y coordinates of the point where the angle θ crosses the unit circle.

2. THE OTHER TRIG FUNCTION

Sine and cosine are used to define the four other trig functions:

$$\tan(\theta) \stackrel{\text{def}}{=} \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sec(\theta) \stackrel{\text{def}}{=} \frac{1}{\cos(\theta)}$$

$$\cot(\theta) \stackrel{\text{def}}{=} \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}$$

$$\csc(\theta) \stackrel{\text{def}}{=} \frac{1}{\sin(\theta)}$$

3. THE 3 BASIC TRIG FORMULAS

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Other identities may be derived from these:

$$\text{Start with } \sin^2(\theta) + \cos^2(\theta) = 1$$

and divide by $\cos^2(\theta)$:

$$\implies \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\implies \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\implies \tan^2(\theta) + 1 = \sec^2(\theta)$$

4. COMPLEMENTARY ANGLES

Start with

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

and let $A = \frac{\pi}{2}$ and $B = \theta$ to obtain

$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2}\right) \cos \theta - \cos\left(\frac{\pi}{2}\right) \sin \theta$$

$$= 1 \cdot \cos \theta - 0 \cdot \sin \theta$$

$$= \cos \theta$$

Similary,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

5. DOUBLE ANGLE FORMULAS

Start with

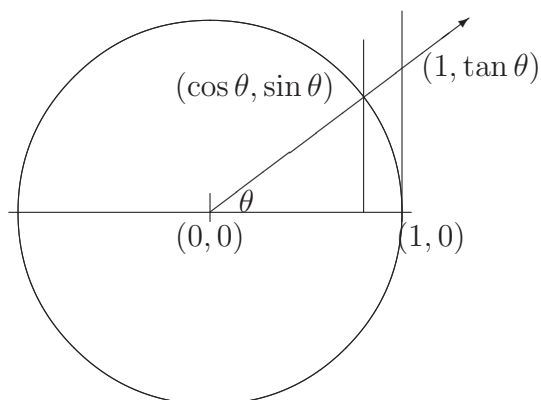
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

and let $A = \theta$ and $B = \theta$ to obtain

$$\sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta$$

$$\implies \sin(2\theta) = 2 \sin \theta \cos \theta$$

6. CIRCLES AND TRIANGLES PICTURE



$$\text{Area of little triangle} = \frac{1}{2} \cos \theta \cdot \sin \theta$$

$$\text{Area of big triangle} = \frac{1}{2} \cdot 1 \cdot \tan \theta$$

$$\text{Arclength of slice} = \theta \text{ (in radians)}$$

$$\text{Area of slice} = \frac{1}{2} \theta \text{ (in radians)}$$

7. A TRIG INEQUALITY

Little triangle $\subseteq$ Slice $\subseteq$ Big triangle

Area of little $\triangle <$ Area of slice $<$ Area of big $\triangle$

$$\frac{1}{2} \cos \theta \cdot \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \cdot \tan \theta$$

$$\cos \theta \cdot \sin \theta < \theta < \frac{\sin \theta}{\cos \theta}$$

$$\cos \theta \cdot \sin \theta < \theta \text{ implies } \frac{\sin \theta}{\theta} < \frac{1}{\cos \theta}$$

$$\theta < \frac{\sin \theta}{\cos \theta} \text{ implies } \cos \theta < \frac{\sin \theta}{\theta}$$

$$\text{Therefore, } \cos \theta < \frac{\sin \theta}{\theta} < \frac{1}{\cos \theta}$$

$$8. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$$

$$\cos \theta < \frac{\sin \theta}{\theta} < \frac{1}{\cos \theta}$$

What happens as $\theta \rightarrow 0$?

On the left, $\cos \theta$ tends to $\cos(0) = 1$.

On the right, $\frac{1}{\cos \theta}$ tends to $\frac{1}{\cos(0)} = 1$.

The function in the middle, $\frac{\sin \theta}{\theta}$, is **squeezed** between two functions ($\cos(\theta)$ and $1/\cos(\theta)$) which both approach 1 as θ goes to 0.

$$\text{Conclusion: } \boxed{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1}$$

Note: we often use the letter h or sometimes x instead of θ .

$$9. \lim_{h \rightarrow 0} 1 - \frac{\cos h}{h}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} &= \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \frac{1 + \cos h}{1 + \cos h} \\ &= \lim_{h \rightarrow 0} \frac{1 - \cos^2 h}{h(1 + \cos h)} \\ &= \lim_{h \rightarrow 0} \frac{\sin^2 h}{h(1 + \cos h)} \\ &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \frac{\sin h}{1 + \cos h} \\ &= 1 \cdot \frac{0}{1 + 1} \\ &= 0 \end{aligned}$$

10. THE TWO BASIC TRIG LIMITS

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

$$\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0$$

11. DERIVATIVE OF $f(x) = \sin(x)$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ [\sin(A+B) &= \sin A \cos B + \cos A \sin B] \quad = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 = \cos x \end{aligned}$$

12. Group Problem

Part I. Use the limit formulas

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0$$

and the sum formula

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

to find the derivative of $f(x) = \cos x$ using the limit definition.

Part II. Use the quotient rule (and the derivatives of $\sin$ and $\cos$) to find the derivative of

$$y = \tan x = \frac{\sin x}{\cos x}.$$

13. ANOTHER WAY

Recall that $\sin x = \cos(\frac{\pi}{2} - x)$ and $\cos x = \sin(\frac{\pi}{2} - x)$.

Thus,

$$\begin{aligned} \frac{d}{dx} \cos x &= \frac{d}{dx} \sin\left(\frac{\pi}{2} - x\right) \\ &= \cos\left(\frac{\pi}{2} - x\right) \cdot \frac{d}{dx}\left(\frac{\pi}{2} - x\right) \quad [\text{by the chain rule}] \\ &= \sin(x) \cdot (-1) \\ &= -\sin x \end{aligned}$$

Sneaky, huh?

14. The Tangent

To find the derivative of the tangent function, first write

$$y = \tan x = \frac{\sin x}{\cos x}$$

then use the quotient rule:

$$\begin{aligned}
\frac{d}{dx} \tan x &= \frac{\left(\frac{d}{dx} \sin x\right) \cdot \cos x - \sin x \cdot \left(\frac{d}{dx} \cos x\right)}{(\cos x)^2} \\
&= \frac{(\cos x) \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

15. COOK BOOK CALCULUS

Addition Rule: $(f + g)' = f' + g'$

Product Rule: $(f \cdot g)' = f'g + fg'$

Constant Rule: $(cf)' = cf'$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Extended Product Rule: $(fgh)' = f'gh + fg'h + fgh'$

16. TRIG FORMULAS

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$

Co-function Rule: You may replace every trig function in a derivative formula with its co-function, provided you multiply by -1 on the right.

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$17. \ y = \sqrt{\sin x}$$

Find $\frac{dy}{dx}$.

Rewrite $y = (\sin x)^{1/2}$

We are using two functions:

$f(x) = x^{1/2}$ and $g(x) = \sin x$.

Since $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = \cos x$

So $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$

$$= \frac{1}{2} (\sin x)^{-1/2} \cdot \cos x$$

$$18. \ y = \frac{\cos x + 1}{x}$$

Find $\frac{dy}{dx}$.

By the quotient rule $\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x + 1) \cdot x - (\cos x + 1) \cdot \frac{d}{dx}(x)}{x^2}$

$$= \frac{(-\sin x) \cdot x - (\cos x + 1) \cdot 1}{x^2}$$

$$= \frac{-x \sin x - \cos x - 1}{x^2}$$