

Math 450/550 Introduction to Topology
Summer 2011

Richard Blecksmith

Contents

Chapter 1. Introduction	1
1. Goals of the Course	1
2. Rules of the Course	2
Chapter 2. Distance	3
1. Intuitive Notions	3
2. Distance in the plane	3
3. General Definition of Distance	4
Chapter 3. Metric Space Topology	7
1. Set Theory	7
2. Open and Closed Sets	8
3. Connectedness	11
4. Topology of the real numbers	12
5. Compactness	13
6. Functions	13
7. Continuity	14
8. Arcwise connected	14
9. Topological Equivalence	15
Chapter 4. General Topology	17
1. Introduction	17
2. Definitions and Examples	17
3. Base for a topology	18
4. Cartesian products and product spaces	19
5. Subspaces	19
6. Neighborhood, Interior, Exterior, and Boundary	20
7. Accumulation points	21
8. Separation axioms and Hausdorff spaces	21
9. Sequences and Convergence	22
10. Continuity and Homeomorphism	22
11. Compactness	23

12. Connectedness	24
13. Continuum	26
14. Number Theory and Topology	27

CHAPTER 1

Introduction

1. Goals of the Course

Since mathematics is a useful tool in everyday life and in many academic fields, it is taught in elementary courses in terms of its applications. Indeed most people think that mathematics is synonymous with applications of mathematics, such as balancing a check book, finding the area of a field, computing the stress on a beam, or determining the orbit of a satellite. These are examples of using mathematics to solve problems in other disciplines. They are not problems in mathematics itself. The view that applied mathematics is all of mathematics is no more a true picture of mathematics than it would be to think that commercial art is all of art. What, then, is mathematics? One of the goals of this course is to provide you with a clearer answer to this question.

A real mathematician is not just a user of mathematics, but is also an *inventor of mathematics*. This course will give you a glimpse of the work of a mathematician. You will invent your own proofs to the many theorems stated in the notes. You will discover your own counter-examples and solutions to the various questions and problems. This course requires you to *actively participate* at a level beyond that assumed in an ordinary course. In a standard course, the homework done outside of class reinforces the material presented in lectures or in the text. In this class, however, student generated proofs form the foundation of the course. Just as a music student could not possibly learn how to play an instrument by listening to someone else play, you cannot become a mathematician merely by seeing another person's proofs. To draw another analogy, this time from sports, the only way to learn to catch a baseball is to put on a glove and play catch. In this course we play ball in a game called topology, using the rules of logic.

2. Rules of the Course

There are five golden rules:

1. *Members of the class will present their own proofs at the board.* It is the job of the rest of the class to spot any errors in the proof being presented. As your instructor—and therefore a member of the class—I too can give my own proofs. In general, however, I hope to keep the time I spend in front of the class to a minimum. The more you do on your own, the more you will master this subject.

2. *Other textbooks are forbidden.* Proofs to most of the theorems in these notes can be found in standard textbooks on topology. Please do not give into the temptation to look at them. It will spoil the problem for you and other students in the class.

3. *No outside help.* Do not seek answers from faculty members, advanced graduate students, or other math majors not enrolled in the class. You may talk to me outside of class. Show me what you've done and I might be able to throw you a lifeline.

4. *Minimum collaboration.* Discussions during class about how to prove the theorems may naturally continue when class is over. I do not mind *limited* discussions outside of class, but I do want you to tackle these problems individually, not as teams.

5. *You can leave during a presentation.* Suppose you have been working for several days on a proof, when another student is ready to give a solution in front of the class. If you wish to keep working on the problem, you may elect to temporarily leave the classroom while the other proof is presented.

CHAPTER 2

Distance

1. Intuitive Notions

The concept with which we start the course is *distance*. Everyone knows what distance is, however, most people would be hard-pressed to give an adequate definition. Consider the following examples. The distance from the earth to the moon is 239,000 miles. The distance from San Francisco to New York is 3025 miles. The distance from Forty-second Street and Broadway to Fifty-seventh Street and Seventh Avenue in New York City is 11 blocks. We know from geometry class that the shortest distance between two points is measured along a straight line. Suppose we define distance to be the shortest distance and consider the previous three examples. In the statement about the earth and the moon, we certainly mean straight line distance. In the statement about San Francisco and New York, however, we don't mean straight line distance, since the straight line would pass through the earth, making travel along this line unfeasible, to say the least. Distance between two cities is measured along the earth's surface. Yet in the statement that it is 11 blocks from Forty-second and Broadway to Fifty-first and Seventh Avenue, we again don't mean straight line distance, nor do we mean the shortest distance on the earth's surface. We mean instead the shortest distance if we restrict ourselves to traveling along the network of streets. Thus, in everyday usage, the meaning of the word distance is not precise, but must be judged from context. Still, the various meanings of the word distance have certain things in common. The distance from A to B , for example, is certainly the same as the distance from B to A . The method of the mathematician is to abstract from the various meanings the essential properties they have in common and then to take these as the defining properties of distance.

2. Distance in the plane

The usual definition of distance between two points $p = (x_1, y_1)$ and $q = (x_2, y_2)$ in the plane is

$$d_1(p, q) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}.$$

Questions.

1. What kind of a mathematical entity is distance in the plane?
2. What are some of its important properties?
3. Can you think of some other way of defining distance between two points in the plane besides the above formula which would satisfy all of the properties you gave in your answer to Question 2?
4. What is the distance between two points p and q in three dimensional space? in n dimensional space?
5. State some undesirable properties of the following ways of defining “distance” from $p = (x_1, y_1)$ to $q = (x_2, y_2)$:
 - (a) $d(p, q) = (x_1, y_1)$ (ordered pair)
 - (b) $d(p, q) = (x_1 + y_1) - (x_2 + y_2)$
 - (c) $d(p, q) = (x_1 - x_2) + (y_1 - y_2)$
 - (d) $d(p, q) = [(x_1 - x_2) + (y_1 - y_2)]^2$
 - (e) $d(p, q)$ = the angle in radians between the vectors p and q .

3. General Definition of Distance

DEFINITION 1. Let X be a non-empty set and let $\mathbb{R}$ denote the field of real numbers. A function $d : X \times X \rightarrow \mathbb{R}$ is called a *distance function* or a *metric* for X if and only if for all a, b , and c in X , the following properties hold:

- (i) $d(a, b) \geq 0$
- (ii) $d(a, b) = 0$ if and only if $a = b$
- (iii) $d(a, b) = d(b, a)$ [symmetry]
- (iv) $d(a, c) \leq d(a, b) + d(b, c)$ [triangle inequality].

Notice how general this definition is. It could apply to any non-empty set X , not just to real numbers or points in the plane. Moreover, different functions d could be distance functions for the same set X . For example, instead of the usual definition of distance from $p = (x_1, y_1)$ to $q = (x_2, y_2)$ in the plane, we could take

$$d_1(p, q) = [2(x_1 - x_2)^2 + 3(y_1 - y_2)^2]^{1/2}.$$

Exercises.

1. Show that the function d_1 above is a distance formula.
2. If $X = \mathbb{R}$, the set of real numbers, determine whether or not each of the following functions is a distance function. Here a and b are real numbers.

$$(a) \quad d(a, b) = a^2 + b^2$$

$$(b) \quad d(a, b) = |a - b|$$

$$(c) \quad d(a, b) = \min(|b - a|, 1)$$

$$(d) \quad d(a, b) = |a^2 - b^2|$$

$$(e) \quad d(a, b) = 0 \quad \text{for all } a, b.$$

In each case, find $d(-1, 2)$ and determine the set $\{a : d(a, 0) = 1\}$.

3. If $X = \mathbb{R}^2$, the plane, determine whether or not each of the following functions is a distance function. Here $a = (x_1, y_1)$ and $b = (x_2, y_2)$.

$$(a) \quad d(a, b) = |x_1 - x_2| + |y_1 - y_2|$$

$$(b) \quad d(a, b) = |(x_1^2 + x_2^2) - (y_1^2 + y_2^2)|$$

$$(c) \quad d(a, b) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

$$(d) \quad d(a, b) = \min(|x_1 - x_2|, |y_1 - y_2|).$$

In each case, find $d((-1, 3), (2, 1))$ and determine the set $\{a : d(a, (0, 0)) = 1\}$.

4. Determine other metrics for the set of points in the plane.
5. If X is *any* set, show that the following function is a distance function:

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b. \end{cases}$$

This function is called the *discrete* metric on X .

6. Let X be the set of continuous functions from the interval $[0, 1]$ to the set of real numbers. A natural way to define the distance $d(f, g)$ between two functions f and g in X is to use the area between their graphs. Express this area as an integral and show that it satisfies the four properties of a distance function.

We conclude this chapter with our first

THEOREM 1. For every a, b, c in X ,

$$|d(a, b) - d(b, c)| \leq d(a, c).$$

CHAPTER 3

Metric Space Topology

If X is any non-empty set and d is a distance function on X , then (X, d) , that is, X together with the function d , is called a *metric space*. We shall study some of the properties of a metric space (X, d) .

1. Set Theory

The terms “set,” “collection,” and “family” are synonymous. If A is a set, then “ $x \in A$ ” means that x is an *element* (or *member* of A , or that x belongs to A). The notation $x \notin A$ indicates x is *not* an element of A . Sets A and B are *equal*, $A = B$, if and only if they have the same elements.

Curly braces are used for set description. Sets may be specified by listing, for example $\{1, 2, 3\}$, $\{1, 2, 3, \dots\}$. If $P(x)$ is a proposition about x , $\{x : P(x)\}$ is the set of exactly those x for which $P(x)$ is true.

The *empty set* is denoted by $\emptyset$. Generally, we use “set” to mean “nonempty set.”

A set A is a *subset* of a set B if and only if each element of A is also an element of B . $A \subseteq B$ means that A is a subset of B .

Unions of sets are indicated by “ $\cup$.” Thus $A \cup B$, $A \cup B \cup C$, $A_1 \cup A_2 \cup \dots \cup A_n$, and $\bigcup_{i=1}^n A_i$ denote unions. If $\mathcal{A}$ is a family of sets, $\bigcup\{A : A \in \mathcal{A}\}$ is the union of the sets of $\mathcal{A}$:

$$\bigcup\{A : A \in \mathcal{A}\} \stackrel{\text{def}}{=} \{x : x \in A \text{ for some } A \in \mathcal{A}\}.$$

The union of a family of sets $\mathcal{A}$ is denoted more concisely as $\bigcup \mathcal{A}$.

Intersections of sets are indicated by “ $\cap$,” with usage analogous to those for “ $\cup$.” If $\mathcal{A}$ is a family of sets

$$\bigcap \mathcal{A} = \bigcap\{A : A \in \mathcal{A}\} \stackrel{\text{def}}{=} \{x : x \in A \text{ for every } A \in \mathcal{A}\}.$$

If A and B are sets, $A - B = \{x : x \in A \text{ and } x \notin B\}$. Thus $A - B$ is a *relative difference*.

Frequently one speaks of *complements* of sets. In our discussion, the set X is considered as the universal set. If $A \subseteq X$, the complement of A with respect to X is $X - A$, denoted more simply as A^c .

THEOREM 2 (DeMorgan's Laws). For any nonempty collection of sets $\mathcal{A}$

(a) $(\cap \mathcal{A})^c = \cup \{A^c : A \in \mathcal{A}\}.$

(b) $(\cup \mathcal{A})^c = \cap \{A^c : A \in \mathcal{A}\}.$

Interpret these statements when the collection $\mathcal{A}$ consists of two sets A and B .

2. Open and Closed Sets

2.1. ϵ -balls.

DEFINITION 2. Let ϵ be a positive real number and let $a \in X$. Then the set

$$B_\epsilon(a) \stackrel{\text{def}}{=} \{x \in X : d(a, x) < \epsilon\}$$

is called an open ϵ -ball or just *ball* about a .

2.2. Interior, Exterior, and Boundary.

DEFINITION 3. Let A be a subset of X . A point $a \in X$ is an *interior point* of A if and only if there exists an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq A$. The set of all interior points of A is called the *interior* of A and is denoted by $\text{Int } A$.

DEFINITION 4. If $A \subseteq X$, a point $b \in X$ is a *boundary point* of A if and only if for all $\epsilon > 0$, $B_\epsilon(b) \cap A \neq \emptyset$ and $B_\epsilon(b) \cap A^c \neq \emptyset$. The set of all boundary points of A is called the *boundary* of A and is denoted by $\text{Bd } A$.

DEFINITION 5. If $A \subseteq X$, a point $c \in X$ is an *exterior point* of A if and only if there exists an $\epsilon > 0$ such that $B_\epsilon(c) \subseteq A^c$. The set of all exterior points of A is called the *exterior* of A and is denoted by $\text{Ext } A$.

THEOREM 3.

(a) $\text{Bd } A = \text{Bd } A^c$

(b) $\text{Int } A = \text{Ext } A^c.$

2.3. Definition of open and closed sets.

DEFINITION 6. A set $A \subseteq X$ is an *open set* if and only if $A \subseteq \text{Int } A$, that is, every point in A is an interior point of A .

DEFINITION 7. A set $A \subseteq X$ is a *closed set* if and only if $\text{Bd } A \subseteq A$, that is, A contains all of its boundary points.

PROBLEM 4. Give an example of a set A in a metric space X which is neither open nor closed.

THEOREM 5. For every set A ,

- (a) $\text{Int } A \subseteq A \subseteq A \cup \text{Bd } A$.
- (b) A is open if and only if $A = \text{Int } A$.
- (c) A is closed if and only if $A = A \cup \text{Bd } A$.

THEOREM 6.

- (a) $\emptyset$ is both open and closed.
- (b) X is both open and closed.

THEOREM 7. Any ϵ -ball $B_\epsilon(a)$ is open.

QUESTION 8.

- (a) Must every singleton $\{a\}$ be a closed set?
- (b) Can a singleton $\{a\}$ ever be an open set?

The collection of all open sets in a space is called the *topology* for the space.

2.4. Properties of Int, Ext, and Bd.

THEOREM 9. Let A be a given set in a metric space X . Then

- (a) $\text{Int } A$ is an open set.
- (b) $\text{Ext } A$ is an open set.
- (c) $\text{Bd } A$ is a closed set.

THEOREM 10. Let A be a given set and let $b \in X$.

- (a) If $b \in A$, then b is not an exterior point of A .
- (b) If $b \notin A$, then b is not an interior point of A .
- (c) $\text{Bd } A \cap \text{Int } A = \emptyset$ and $\text{Bd } A \cap \text{Ext } A = \emptyset$.
- (d) The point b is either an interior point of A , a boundary point of A , or an exterior point of A .

THEOREM 11. X is the disjoint union of $\text{Int } A \cup \text{Bd } A \cup \text{Ext } A$.

2.5. Set Theoretic Properties.

THEOREM 12. Let A and B both be open sets. Then $A \cap B$ is open.

QUESTION 13. How can we use the previous theorem to show that the intersection of finitely many open sets is open?

THEOREM 14. Let $\mathcal{A}$ be a collection of open sets in X . Then $\cup \mathcal{A} = \cup \{A : A \in \mathcal{A}\}$ is open.

PROBLEM 15. If $\mathcal{A}$ is a collection of open sets, show that $\cap \mathcal{A} = \cap \{A : A \in \mathcal{A}\}$ is not necessarily an open set.

THEOREM 16. If A is an open set, then there exists a collection $\mathcal{A}$ of ϵ -balls in A such that $A = \cup \mathcal{A}$.

THEOREM 17. For any set A , A is open if and only if A^c is closed.

THEOREM 18. If $\mathcal{A}$ is a collection of closed sets, then $\cap \mathcal{A} = \cap \{A : A \in \mathcal{A}\}$ is closed.

THEOREM 19. If A and B are both closed, then $A \cup B$ is closed.

PROBLEM 20. If $\mathcal{A}$ is a collection of closed sets, show that $\cup \mathcal{A}$ is not necessarily closed.

THEOREM 21. Let $A \subseteq X$ be a finite set, i.e., A has only finitely many elements. Then A^c is an open set.

2.6. Closure.

DEFINITION 8. The *closure* of a set A is $A \cup \text{Bd} A$. It is denoted by $\overline{A}$.

THEOREM 22.

$$\overline{A} = \text{Int } A \cup \text{Bd } A = (\text{Ext } A)^c.$$

THEOREM 23. Let A be a given set.

- (a) $\overline{A}$ is closed.
- (b) $\overline{\overline{A}} = \overline{A}$, i.e., the closure of the closure of A equals the closure of A .
- (c) If A is closed and $B \subseteq A$, then $\overline{B} \subseteq A$.
- (d) If $\epsilon > 0$, then for any $x \in X$, the set $\{y : d(x, y) \leq \epsilon\}$ is closed.

2.7. Distance Between Sets.

DEFINITION 9. If A and B are non-empty sets, then the distance between A and B is defined to be

$$d(A, B) = \text{glb}\{d(a, b) : a \in A \text{ and } b \in B\}.$$

Here glb means the greatest lower bound.

DEFINITION 10. A set A is *bounded* if and only if there exists a real number M such that for all $x, y \in A$, $d(x, y) \leq M$. If A is bounded, then the *diameter* of A is the glb of the set of numbers M such that $d(x, y) \leq M$ for all $x, y \in A$.

THEOREM 24. If $x \in X$ and $y \in X$, then

$$d(\{x\}, \{y\}) = d(x, y).$$

Because of Theorem 24, we can and will use the notation $d(x, B)$ for $d(\{x\}, B)$.

THEOREM 25. A set A is closed if and only if for every $x \in X$,

$$d(x, A) = 0 \text{ implies } x \in A.$$

3. Connectedness

DEFINITION 11. A set A is *disconnected* if and only if there exist open sets U and V with the properties:

- (i) $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$
- (ii) $A \subseteq U \cup V$
- (iii) $U \cap V = \emptyset$.

A set A is *connected* if and only if it is not disconnected.

THEOREM 26. If A is connected and B is a subset of $\text{Bd } A$, then $A \cup B$ is connected.

THEOREM 27. Let $\mathcal{A}$ be a collection of connected sets and let $x_0 \in X$. If for every $A \in \mathcal{A}$, $x_0 \in A$, then $\cup \mathcal{A}$ is connected.

THEOREM 28. The whole space X is disconnected if and only if there exists a proper subset $B \subseteq X$ (proper subset means $B \neq X$ and $B \neq \emptyset$) such that B is both open and closed.

DEFINITION 12. A *component* of a set A is a maximal connected set, i.e., a subset B such that B is connected, but if $B \subsetneq C \subseteq A$ and C is connected, then $B = C$.

THEOREM 29. A nonempty set A is connected if and only if A has exactly one component.

4. Topology of the real numbers

4.1. LUB Axiom. By the *real numbers* we mean the metric space $\mathbb{R}$ consisting of the set of real numbers together with the usual metric

$$d(a, b) = |a - b|.$$

We shall need the following axiom for the real numbers:

AXIOM . Every nonempty set $A \subseteq \mathbb{R}$ which has an upper bound has a least upper bound.

4.2. Connected subsets of $\mathbb{R}$.

THEOREM 30. If $\emptyset \neq A \subseteq \mathbb{R}$ has an upper bound and $x_o = \text{lub } A$, then $x_o \in \text{Bd } A$.

THEOREM 31. $\mathbb{R}$ is connected.

THEOREM 32. The space $\mathbb{Q}$ of rational numbers is disconnected.

THEOREM 33. If $b > a$, then the interval $[a, b]$ is connected.

DEFINITION 13. A nonempty set A is *totally disconnected* if and only if for each $x \in A$, $\{x\}$ is a component.

THEOREM 34. The rational numbers $\mathbb{Q}$ form a totally disconnected subset of $\mathbb{R}$.

4.3. Separability.

DEFINITION 14. A subset D is *dense* in X if and only if every non-empty open set U contains some element of D .

DEFINITION 15. A metric space X is *separable* if and only if X has a countable dense subset $D = \{a_1, a_2, a_3, \dots\}$.

PROBLEM 35. Find a countable dense subset for $\mathbb{R}$.

4.4. Nested Sets.

DEFINITION 16. By a *nested sequence* of sets $\{A_n\}$, we mean an infinite collection of sets $A_1, A_2, A_3, \dots$ such that for each positive integer n , $A_{n+1} \subseteq A_n$.

THEOREM 36. Let $\{A_n\}$ be a nested sequence of non-empty closed and bounded sets of real numbers. Then $\cap_{n=1}^{\infty} A_n$ is non-empty.

PROBLEM 37. Give an example of a nested sequence of nonempty open sets of real numbers whose intersection is empty.

PROBLEM 38. Give an example of a nested sequence of nonempty closed sets of real numbers whose intersection is empty.

5. Compactness

DEFINITION 17. A collection $\mathcal{A}$ of sets is called a *covering* of a set B if and only if $B \subseteq \cup \mathcal{A}$. If each $A \in \mathcal{A}$ is open, it is called an *open covering* of B . If $\mathcal{A}$ is a finite collection of sets, it is called a *finite covering* of B . If $\mathcal{A}_1 \subseteq \mathcal{A}$ is also a covering of B , it is called a *subcovering* of B .

DEFINITION 18. A set B is *compact* if and only if every open covering of B has a finite subcovering.

THEOREM 39. If A is compact, then A is bounded.

THEOREM 40. If A is compact, then A is closed.

THEOREM 41. Let A be a non-empty subset of the real numbers. Then A is compact if and only if A is closed and bounded.

THEOREM 42. Let $X = \mathbb{Q}$, the space of rational numbers and let B be the set of rational numbers x such that $0 \leq x \leq 1$. Then B is not compact.

6. Functions

If f is a function from X into Y , we write $f : X \rightarrow Y$. Here X is the *domain* of f , Y is the *codomain* of f , and we also say that f is from X to Y . Associated with a function f is the *graph* of f , a subset of $X \times Y$ such that no two distinct ordered pairs of the graph have the same first term. We sometimes use $\text{dom } f$ for the domain of f .

If $x \in \text{dom } f$, we denote the value of f at x by $f(x)$. The set $\{f(x) : x \in \text{dom } f\}$ is the *range* of f . f is *onto* its range.

If $A \subseteq X$, then $f(A) = \{f(x) : x \in A\}$; $f(A)$ is the *image* or *direct image*, under f of A . If $B \subseteq Y$, then $f^{-1}(B) = \{x : x \in \text{dom } f \text{ and } f(x) \in B\}$; $f^{-1}(B)$ is the *inverse image*, under f , of B .

THEOREM 43. Suppose $f : X \rightarrow Y$ is a function, $\mathcal{A}$ is a collection of subsets of Y , and B and C are subsets of Y . Then

$$(a) \quad f^{-1}(\cup\{A : A \in \mathcal{A}\}) = \cup\{f^{-1}(A) : A \in \mathcal{A}\}.$$

$$(b) \quad f^{-1}(\cap\{A : A \in \mathcal{A}\}) = \cap\{f^{-1}(A) : A \in \mathcal{A}\}.$$

- (c) $f^{-1}(B - C) = f^{-1}(B) - f^{-1}(C)$.
- (d) If $B \subseteq C$, then $f^{-1}(B) \subseteq f^{-1}(C)$.

THEOREM 44. Suppose $f : X \rightarrow Y$ is a function, $\mathcal{A}$ is a collection of subsets of X , and B and C are subsets of X . Then

- (a) $f(\cup\{A : A \in \mathcal{A}\}) = \cup\{f(A) : A \in \mathcal{A}\}$.
- (b) $f(\cap\{A : A \in \mathcal{A}\}) \subseteq \cap\{f(A) : A \in \mathcal{A}\}$.
- (c) $f(B) - f(C) \subseteq f(B - C)$.
- (d) If $B \subseteq C$, then $f(B) \subseteq f(C)$.

A function $f : X \rightarrow Y$ is *one-to-one* if and only if for each two distinct points x and y of X , $f(x) \neq f(y)$. If $f : X \rightarrow Y$ is one-to-one and onto Y , then f^{-1} , the *inverse* of f , is the function $f^{-1} : Y \rightarrow X$ obtained by reversing the ordered pairs in the graph of f . If $f : X \rightarrow Y$ is one-to-one and *not* onto Y , then we shall understand f^{-1} to mean the inverse of $f : X \rightarrow (\text{range } f)$. If $f : X \rightarrow Y$ is one-to-one, $x \in \text{dom } f$, and $y \in \text{range } f$, then $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

7. Continuity

DEFINITION 19. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \rightarrow Y$ is *continuous at a point* $x_o \in X$ if and only if for every $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$x \in B_\delta(x_o) \quad \text{implies} \quad f(x) \in B_\epsilon(f(x_o)).$$

DEFINITION 20. Let $A \subseteq X$. A function $f : X \rightarrow Y$ is *continuous on* A if and only if for each point $x_o \in A$, for every $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$x \in B_\delta(x_o) \cap A \quad \text{implies} \quad f(x) \in B_\epsilon(f(x_o)).$$

THEOREM 45. A function $f : X \rightarrow Y$ is continuous on X if and only if for every open set $V \subseteq Y$, $f^{-1}(V)$ is open.

THEOREM 46. Let f be continuous on a connected set X . Then $f(X)$ is connected.

THEOREM 47. Let f be continuous on a compact set X . Then $f(X)$ is compact.

8. Arcwise connected

DEFINITION 21. An *arc* in the plane is the set of images of points in the interval $[0, 1]$ under a continuous function $f : [0, 1] \rightarrow \mathbb{R}^2$. The points $f(0)$ and $f(1)$ are called the *endpoints* of the arc.

DEFINITION 22. A set A in the plane is *arcwise connected* if and only if for every pair of points a and b in A , there exists an arc $f : [0, 1] \rightarrow \mathbb{R}^2$ containing a and b as endpoints whose range $f([0, 1])$ lies in A .

THEOREM 48. If $A \subseteq \mathbb{R}^2$ is arcwise connected, then A is connected.

PROBLEM 49. Give an example of a set in $\mathbb{R}^2$ which is connected, but not arcwise connected.

THEOREM 50. If A is an open, connected subset of the plane, then A is arcwise connected.

9. Topological Equivalence

DEFINITION 23. Let (X, d_1) and (Y, d_2) be metric spaces. Sets $A \subseteq X$ and $B \subseteq Y$ are *homeomorphic* or *topologically equivalent* if and only if there exists a function $f : A \rightarrow B$ which is continuous, one-to-one, onto, and whose inverse f^{-1} is continuous. The function f is called a homeomorphism of A onto B .

THEOREM 51. If f is a homeomorphism from A onto B , the f^{-1} is a homeomorphism from B onto A .

THEOREM 52. Let A and B be homeomorphic.

- (a) If A is connected, then B is connected.
- (b) If A is compact, then B is compact.

THEOREM 53. Let $A \subseteq \mathbb{R}$. Then A is homeomorphic to the interval $[0, 1]$ if and only if A is a closed interval $[a, b]$.

THEOREM 54. The disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is homeomorphic to the square $\{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < y < 1\}$.

PROBLEM 55. Show that the interval $[0, 1] \subseteq \mathbb{R}$ is *not* homeomorphic to the square $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$.

CHAPTER 4

General Topology

1. Introduction

We began our work with metric spaces by abstracting the concept of distance, i.e., by deciding what are the essential properties of the ordinary concept of distance in the plane and then taking these as the defining properties of a distance function. We then found that many theorems true in the plane can be proved for general metric spaces. The real line, the plane, and 3-space became special cases of metric spaces.

Many of the theorems we have proved involve open sets or closed sets in a metric space. These concepts were defined in terms of ϵ -balls $B_\epsilon(x)$ which made use of the concept of metric. An open set could be defined as a union of such ϵ -balls. We could go one step further in abstraction if we could decide what are the essential properties of open sets. A closed set will be defined as the complement of an open set. Metric spaces will then be special cases of these more general spaces.

It turns out that it is not the properties of individual open sets which are important, but rather their relationship to each other.

2. Definitions and Examples

By a *space* we mean a set, and by a *point* we mean an element of a space (or set).

DEFINITION 24. Suppose X is a set. T is a *topology on X* if and only if T is a collection of subsets of X , called *open sets*, such that (1) $\emptyset$ and X are open, (2) the union of any number of open sets is open, and (3) the intersection of finitely many open sets is open.

DEFINITION 25. A *topological space* is a set X together with a topology on X . Usually we speak of a “topological space X ,” suppressing explicit mention of the associated topology.

DEFINITION 26. A set M in a topological space is *closed* if and only if M^c is open.

EXAMPLE 1. If X is a set, then $\{\emptyset, X\}$ is a topology on X , the *trivial* topology on X , and X with this topology is a *trivial space*.

EXAMPLE 2. The set of all subsets of X is a topology, the *discrete* topology on X , and X with this topology is a *discrete space*.

EXAMPLE 3. For any set X , let T be the set consisting of $\emptyset$, X , and every subset of X whose complement is finite. T is a topology on X , the *finite complement* topology.

EXAMPLE 4. If X is a metric space, the family of all open sets (with respect to the metric on X) is a topology on X , the *metric topology* on X (or the *d-metric topology* on X).

3. Base for a topology

If T is a topology, then B is a *base* for T if and only if B is a subcollection of T such that for each open set U and each point p of U , there is a set V of B such that $p \in V$ and $V \subseteq U$. An equivalent condition is as follows: B is a base for T if and only if $B \subseteq T$ and each set of T is a union of members of B .

EXAMPLE 5. The set of all singleton subsets of a set X is a base for the discrete topology on X .

THEOREM 56. If X is a set and B is a collection of subsets of X , then B is a base for some topology on X if and only if (1) B covers X and (2) for each two sets U and V of B , and each point p of $U \cap V$, there is a set W of B such that $p \in W$ and $W \subseteq U \cap V$. (Note that (2) holds if it is true that for each two sets U and V of B , $U \cap V \in B$.)

EXAMPLE 6. If X is a metric space, the family of all ϵ -balls is a base for the metric topology on X .

EXAMPLE 7. The set of all open intervals (in $\mathbb{R}$) is a base for a topology for $\mathbb{R}$. The resulting topology is identical with the metric topology obtained using absolute value to compute distance. $\mathbb{R}$, with this topology, is the *real line*.

EXAMPLE 8. Let I denote $[0, 1]$, and let T be the set of all open intervals of the form $I \cap (a, b)$ where (a, b) is an open interval in $\mathbb{R}$. Then T is a base for a topology on I .

EXAMPLE 9. Let B be the family of all half-open intervals in $\mathbb{R}$ of the form $[a, b)$. B is a base for a topology for $\mathbb{R}$; we shall call the resulting space a *half-open interval space*.

4. Cartesian products and product spaces

If A and B are sets, their *cartesian product* (or *product*) $A \times B$ is defined as follows:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

For any finite number of sets, $A_1, A_2, \dots, A_n$, we may define a cartesian product as follows:

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \dots, x_n) : x_1 \in A_1, x_2 \in A_2, \dots, \text{ and } x_n \in A_n\}.$$

Other notations are

$$\prod_{i=1}^n A_i \quad \text{and} \quad \prod \{A_i : 1 \leq i \leq n\}.$$

Suppose X and Y are topological spaces. We then have *open sets in X* and *open sets in Y* . We may consider the following collection of subsets of $X \times Y$:

$$W = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

THEOREM 57. The set W is a base of a topology on $X \times Y$.

PROBLEM 58. Give an example to show that W is not necessarily a topology on $X \times Y$.

In the notation above, the *product topology* on $X \times Y$ is that having W as a base.

THEOREM 59. If X and Y are topological spaces with bases A and B respectively for their topologies, then $\{U \times V : U \in A \text{ and } V \in B\}$ is a base for the product topology on $X \times Y$.

The above results extend to finite cartesian products. Thus if $X_1, X_2, \dots$, and X_n are topological spaces, then

$$\{U_1 \times U_2 \times \cdots \times U_n : \text{if } 1 \leq i \leq n, U_i \text{ is open in } X_i\}$$

is a base for a topology on $X_1 \times X_2 \times \cdots \times X_n$. The resulting topology is the *product topology* for $\prod_{i=1}^n X_i$.

EXAMPLE 10. Suppose n is a positive integer. Then $\mathbb{R}^n$ denotes $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times). The set $\mathbb{R}^n$, with the product topology obtained from the real line topology on each factor, is *euclidean n -dimensional space* (or *euclidean n -space*).

5. Subspaces

THEOREM 60. If X is a topological space and $A \subseteq X$, then $\{U \cap A : U \text{ is open in } X\}$ is a topology on A .

The topology of Theorem 60 is the *relative topology on A* . A subset A of a topological space X is a *subspace* of X if and only if A has the relative topology.

EXAMPLE 11. Let I be $[0,1]$. I is a subspace of the real line if and only if I has as its topology $\{I \cap U : U \text{ is open in } \mathbb{R}\}$.

THEOREM 61. If X is a topological space, $A \subseteq X$, and B is a base for the topology of X , then $\{A \cap V : V \in B\}$ is a base for the relative topology on A .

EXAMPLE 12. Let I be $[0,1]$, and suppose that n is a positive integer. Let I^n denote $I \times I \times \cdots \times I$ (n times). Then $I^n \subseteq \mathbb{R}^n$, and let I^n have the relative topology. I^n is an *n -dimensional cube*.

EXAMPLE 13. For each positive integer n , let

$$S^n = \{x : x \in \mathbb{R}^{n+1} \text{ and } d(x, 0) = 1\}.$$

S^n is the unit *n -sphere* in $\mathbb{R}^{n+1}$ centered at 0. We also call S^n the *standard n -sphere*. We regard S^n as a topological space by giving it the relative topology for $\mathbb{R}^{n+1}$.

PROBLEM 62. Suppose X and Y are topological spaces, $A \subseteq X$, and $B \subseteq Y$. There are two “natural” ways of getting a topology for $A \times B$. Describe two, and determine their relationship.

6. Neighborhood, Interior, Exterior, and Boundary

If X is a topological space and $p \in X$, then W is a *neighborhood of p* if and only if $W \subseteq X$ and W contains an open set containing p . An open neighborhood of p is just an open set containing p .

DEFINITION 27. Let A be a set in a topological space X . A point p in X is an *interior point* of A if and only if there exists an open set $U \subseteq A$ such that $p \in U$. Equivalently, p is an interior point of A if and only if A is a neighborhood of p .

DEFINITION 28. Let A be a set in a topological space X . A point p in X is an *exterior point* of A if and only if there exists an open set $U \subseteq A^c$ such that $p \in U$.

DEFINITION 29. Let A be a set in a topological space X . A point p in X is a *boundary point* of A if and only for every open set U such that $p \in U$, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$. Equivalently, p is a boundary point of A if and only if each neighborhood of p intersects both A and A^c .

As before, the *interior* of A , $\text{Int } A$, is the set of all interior points of A ; the *boundary* of A , $\text{Bd } A$, is the set of all boundary points of A ; and the *exterior* of A , $\text{Ext } A$, is the set of all exterior points of A .

THEOREM 63. A set A is open if and only if every point in A is an interior point.

THEOREM 64. A set A is closed if and only if $\text{Bd } A \subseteq A$.

THEOREM 65. The interior of a subset A of a topological space is the largest open set contained in A .

The theorems about Int , Ext , and Bd in Chapter 3, Section 2.4 hold for general topological spaces.

DEFINITION 30. The closure of A , written $\overline{A}$, is $A \cup \text{Bd } A$.

The theorems about closure in Chapter 3, Section 2.6 (except Theorem 22.4) hold for general topological spaces.

7. Accumulation points

DEFINITION 31. If M is a set in a topological space X and $p \in X$, then p is an *accumulation point* of M if and only if each open set containing p contains at least one point of M distinct from p . An accumulation point is sometimes called a *cluster point* or *limit point*.

THEOREM 66. A set M in a topological space is closed if and only if each accumulation point of M belongs to M .

THEOREM 67. If M is a set in a topological space and M' is the set of all accumulation points of M , then $\overline{M} = M \cup M'$.

THEOREM 68. If A and B are subsets of a topological space, then each accumulation point of $A \cup B$ is an accumulation point either of A or of B .

THEOREM 69. If M is a set in a topological space and $K \subseteq M$, each accumulation point of K is also an accumulation point of M .

8. Separation axioms and Hausdorff spaces

Throughout this section, let X be a topological space.

DEFINITION 32. T_0 : X is a T_0 -space if and only if for any two distinct points of X , there is an open set containing one but not the other.

Any trivial space with more than one point fails to be T_0 .

EXAMPLE 14. Let $\mathcal{R}$ be the family consisting of $\emptyset$, $\mathbb{R}$, and all sets (a, ∞) where $(a, \infty) = \{x : x \in \mathbb{R} \text{ and } x > a\}$. $\mathcal{R}$ is a topology on $\mathbb{R}$. This space is T_0 . We shall call it the *open ray space*.

DEFINITION 33. T_1 : X is T_1 , or a T_1 -space, if and only if for each two distinct points x and y of X , there is an open set containing x but not y .

The open ray space above is not T_1 .

THEOREM 70. (a) A space is T_1 if and only if each singleton is closed.
 (b) In a T_1 -space, no finite set has an accumulation point.
 (c) Each finite complement space is T_1 .

DEFINITION 34. *Hausdorff* or T_2 : X is *Hausdorff*, or T_2 , or a T_2 -space, if and only if distinct points of X belong to disjoint open sets.

THEOREM 71. Every metric space is Hausdorff.

9. Sequences and Convergence

In these notes, $\mathbb{N}$ shall denote the set of all positive integers. A *sequence* is a function with domain $\mathbb{N}$. If $s : \mathbb{N} \rightarrow X$ is a sequence, we say it is *in* X or *into* X and the values $s(1), s(2), s(3), \dots$ are denoted by $s_1, s_2, \dots$, and called *terms* of the sequence. (Note that “sequence” is always used in the sense of “infinite sequence.”)

Suppose X is a topological space, x is a sequence in X , and $p \in X$. Then x *converges to* p , $x \rightarrow p$, if and only if for each open set U containing p , there is a positive integer k such that if $i \in \mathbb{N}$ and $i > k$, then $x_i \in U$. We may formulate this as follows: Each neighborhood of p contains all but finitely many terms of x .

EXAMPLE 15. Let $\mathbb{Q}$ be the set of all rational numbers and give $\mathbb{Q}$ the finite complement topology. Let f be a one-to-one sequence onto $\mathbb{Q}$. Then if $r \in \mathbb{Q}$, f converges to r .

THEOREM 72. In a Hausdorff space, convergent sequences have unique limits.

10. Continuity and Homeomorphism

DEFINITION 35. Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* if and only if for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X .

THEOREM 73. A function $f : X \rightarrow Y$ is continuous if and only if the inverse image of each closed set is closed.

Since the definition of homeomorphism depended only on the definition of continuous function, the same definition can be taken for general topological spaces.

DEFINITION 36. A function $h : X \rightarrow Y$ is a *homeomorphism from X to Y* if and only if h is one-to-one, onto, continuous, and has a continuous inverse h^{-1} . Note that homeomorphisms are *onto*. If $h : X \rightarrow Y$ is a homeomorphism from X to Y , then $h^{-1} : Y \rightarrow X$ is a homeomorphism from Y to X ; in this case, X and Y are *homeomorphic*, or *topologically equivalent*.

EXAMPLE 16. Consider the set of real numbers together with the following six topologies: (1) the trivial topology, (2) the discrete topology, (3) the finite complement topology, (4) the standard topology (Example 7), (5) the half-open interval space (Example 9), and (6) the open ray space (Example 14). No two of these six spaces are homeomorphic.

A property of topological spaces is a *topological invariant* if and only if when it is possessed by a space X , it is also possessed by each space homeomorphic to X .

THEOREM 74. Let (X, T) and (Y, S) be homeomorphic topological spaces. If (X, T) is Hausdorff, then so is (Y, S) .

DEFINITION 37. A topological space X with topology T is *metrizable* if and only if there is a metric d for X such that the d -metric topology on X is precisely T .

For example, each discrete space is metrizable (with the discrete metric).

THEOREM 75. Let (X, T) and (Y, S) be homeomorphic topological spaces. If (X, T) is metrizable, then so is (Y, S) .

11. Compactness

The previous definition of compactness for metric spaces depended only on the concept of open set, not on the metric, so we can keep this definition as previously stated.

DEFINITION 38. A set A in a topological space X is *compact* if and only if every open covering of A has a finite subcovering.

THEOREM 76. Let X be a compact space and let E be an infinite subset of X . Then E has an accumulation point.

THEOREM 77. Each compact subset of a Hausdorff space is closed.

THEOREM 78. Each closed subset of a compact space is compact.

THEOREM 79. If X and Y are spaces, X is compact, and $f : X \rightarrow Y$ is continuous, then $f(X)$ is compact.

THEOREM 80. If f is continuous and one-to-one function from a compact space onto a Hausdorff space, then f^{-1} is continuous, and hence f is a homeomorphism.

DEFINITION 39. A sequence $\{A_n\}$ is called a *nested sequence* if and only if for every positive integer n , $A_{n+1} \subseteq A_n$.

DEFINITION 40. A set A in a topological space X is *countably compact* if and only if every countable open cover $\{U_n\}_{n=1}^{\infty}$ has a finite subcover.

THEOREM 81. A space X is countably compact if and only if every nested sequence of non-empty closed sets has a non-empty intersection.

EXAMPLE 17. For any set X , let T be the set consisting of $\emptyset$, X , and every subset of X whose complement is countable. T is a topology on X , the *countable complement topology*.

It is obvious that every compact space X is countably compact.

PROBLEM 82. Find an example of a countably compact space, which is not compact.

WARNING: Theorem 39, which states that a compact metric space is bounded, has no analogue for general topological spaces, since the concept of boundedness requires that of a metric.

12. Connectedness

As with the definition of compactness, the previous definition of connectedness depended only on that of open set, not on the metric, so we can keep this definition as it was.

DEFINITION 41. A set A in a topological space X is *disconnected* if and only if there exist open sets U and V such that $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$, $A \subseteq U \cup V$, and $U \cap V = \emptyset$. A is *connected* if and only if it is not disconnected.

The theorems about connectedness in Chapter 3, Section 3 hold for general topological spaces.

THEOREM 83. Let A and B be connected and suppose $(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \emptyset$. Then $A \cup B$ is connected.

Examples of connected spaces

1. Each of the spaces $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \dots$ is connected.
2. A set M in $\mathbb{R}^n$ is *convex* if and only if for every two points x, y in M , the line segment between x and y lies entirely in M . In $\mathbb{R}^n$, each convex set is connected.
3. Each sphere $S^1, S^2, S^3, \dots$ is connected.

Exotic examples of connected spaces

1. The *Warsaw interval* is the space W which is the closure in $\mathbb{R}^2$ of $\{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}$. The Warsaw interval is connected.
2. For each positive integer n , let A_n be a horizontal ray in $\mathbb{R}^2$ extending from $(-n, 1 - \frac{1}{n})$ to the right, let B_n be a horizontal ray in $\mathbb{R}^2$ extending from $(-n, \frac{1}{n} - 1)$ to the right, let I_n be the interval from $(-n, 1 - \frac{1}{n})$ to $(-n, \frac{1}{n} - 1)$, and let $X_n = A_n \cup B_n \cup I_n$. Let X_0, Y , and Z be horizontal lines in $\mathbb{R}^2$ through $(0, 0)$, $(0, 1)$, and $(0, -1)$, respectively. Let $X = X_0 \cup Y \cup Z \cup (\cup_{n=1}^{\infty} X_n)$. X is connected.
3. Let Σ_0 be the “spiral” $\{(1 - \frac{1}{\theta}, \theta) : \theta \in \mathbb{R}, \theta > 0\}$ (given in polar coordinates for $\mathbb{R}^2$). Let $\Sigma = \Sigma_0 \cup S^1$; S^1 is the unit circle centered at 0. Σ is connected.
4. In $\mathbb{R}^3$, let X_0 be the xy plane and for each positive integer n , let X_n be the horizontal plane through $(0, 0, \frac{1}{n})$. For each such n , let Y_n be the interval from $(n, n, \frac{1}{n})$ to $(n, n, \frac{1}{n+1})$. Let M be $(\cup_{n=1}^{\infty} (X_n \cup Y_n)) \cup X_0$. M is connected.
5. In $\mathbb{R}^2$ let G be the set of all points with at least one irrational coordinate. G is connected.

13. Continuum

THEOREM 84. If $M_1, M_2, M_3, \dots$ is a decreasing nested sequence of compact connected sets in a Hausdorff space, then $\bigcap_{i=1}^{\infty} M_i$ is connected.

DEFINITION 42. A *continuum* is a compact connected set in a Hausdorff space.

EXAMPLE 18. If abc is a triangle in $\mathbb{R}^2$, it bounds a *triangular disc* Δ . Let x, y , and z be the midpoints of the sides of abc . The *interior* of triangle xyz will be called the *middle fourth* of abc . Note that Δ minus its middle fourth is the union of three congruent triangular discs.

Let M_1 be a triangular disc in $\mathbb{R}^2$. Let M_2 be M_1 minus its middle fourth. M_2 is the union of three congruent triangular discs. Let M_3 be the set obtained from M_2 by removing the middle fourths of the three triangular discs whose union is M_2 . Continue this process.

Let $M = \bigcap_{i=1}^{\infty} M_i$. M is a continuum.

EXAMPLE 19. Let I^3 be $[0, 1] \times [0, 1] \times [0, 1]$ in $\mathbb{R}^3$. Divide each face into nine square discs. These discs correspond to $[0, \frac{1}{3}] \times [0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}] \times [0, \frac{1}{3}]$, $[\frac{2}{3}, 1] \times [0, \frac{1}{3}]$, etc. The *middle ninth* is $[\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}]$. “Punch out” each middle ninth; this means remove the points of the sets $(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3}) \times [0, 1]$, $(\frac{1}{3}, \frac{2}{3}) \times [0, 1] \times (\frac{1}{3}, \frac{2}{3})$, and $[0, 1] \times (\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$.

The remaining set, M_1 , is the union of 20 cubes. Repeat the procedure above on each of these, obtaining M_2 . Continue. Let $M = \bigcap_{i=1}^{\infty} M_i$. M is a continuum.

EXAMPLE 20. Let D be a circular disc in $\mathbb{R}^2$. Let $D_1, D_2, D_3, \dots$ be a sequence of mutually disjoint circular discs in the *interior* of D . Let $S = D - \bigcup_{i=1}^{\infty} D_i$. S is a continuum.

14. Number Theory and Topology

Let $X = \mathbb{Z}$ be the set of integers. If $m > 0$ and a are integers, let $[a]_m$ denote the equivalence class $\{n : n \equiv a \pmod{m}\}$. For example, $[1]_2$ is the set of odd integers and $[3]_5$ is the set of all integers which are congruent to 3 mod 5. Let $\mathcal{B} = \{[a]_m : m \in \mathbb{Z}^+, a \in \mathbb{Z}\}$.

PROPOSITION 85. Suppose $[a]_m \cap [b]_n \neq \emptyset$ and let t lies in the intersection $[a]_m \cap [b]_n$. Then

$$[a]_m \cap [b]_n = [t]_{\text{lcm}(m,n)},$$

where $\text{lcm}(m, n)$ denotes the least common multiple of m and n .

PROPOSITION 86. $\mathcal{B}$ is a base for a topology on $\mathbb{Z}$.

PROPOSITION 87. Each set $[a]_m$ is infinite.

PROPOSITION 88. Each nonempty open set is infinite.

PROPOSITION 89. Each set $[a]_m$ is both open and closed.

PROPOSITION 90. Let $M = \bigcup_{p \text{ prime}} [0]_p$, where the union is taken over all prime numbers p . Then $M^c = \{1, -1\}$.

THEOREM 91. There exist infinitely many primes.