

1. A GEOMETRIC APPROACH TO TANGENT LINES

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2. TANGENT LINE OF A PARABOLA

Goal: Find a geometric definition of tangent line at a point P on a curve.

Let's work with the curve $y = x^2$ and try to find the tangent line at the point $P = (2, 4)$.

Where have we heard the phrase tangent line before?

In high school geometry.

3. TANGENT LINE FOR A CIRCLE

For a circle we know that the tangent has the following properties:

- perpendicular to the radius
- passes thru only one point on the circle

The first criterion does not generalize, because we have no notion of center or radius for a parabola.

The second criterion has a major flaw: There may be many lines thru the point P which do not intersect the curve at a second point. Only one of these is the tangent line.

We need another idea.

4. THE FLASHLIGHT IDEA

The tangent line points in the direction of the curve.

Think of a flashlight with a beam that can be focused. You can spread out the beam and illuminate a broad area or you can focus the light beam inside a very narrow acute angle. If your flashlight is pointing along the tangent line to a curve, then the light should illuminate the graph for a while, **no matter how narrow you focus the beam.**

5. GEOMETRIC DEFINITION OF TANGENT

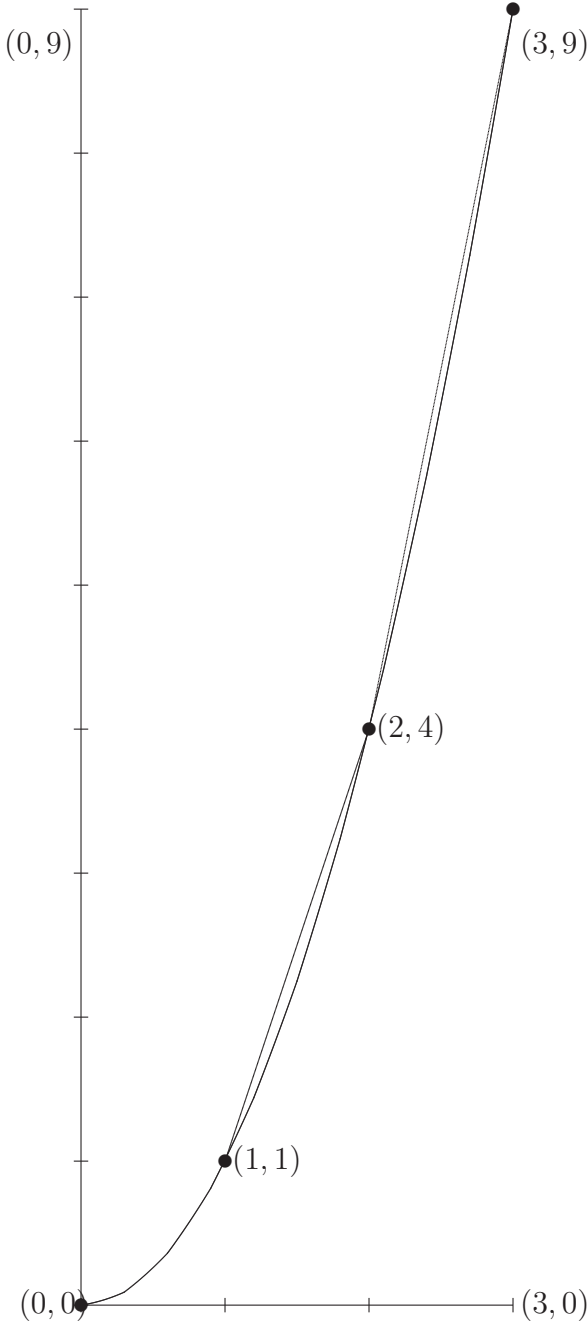
The tangent line $\mathbf{t}$ at point P has the property that for any acute angle with vertex P centered about either side of $\mathbf{t}$, the curve will lie inside this angle if you are near point P .

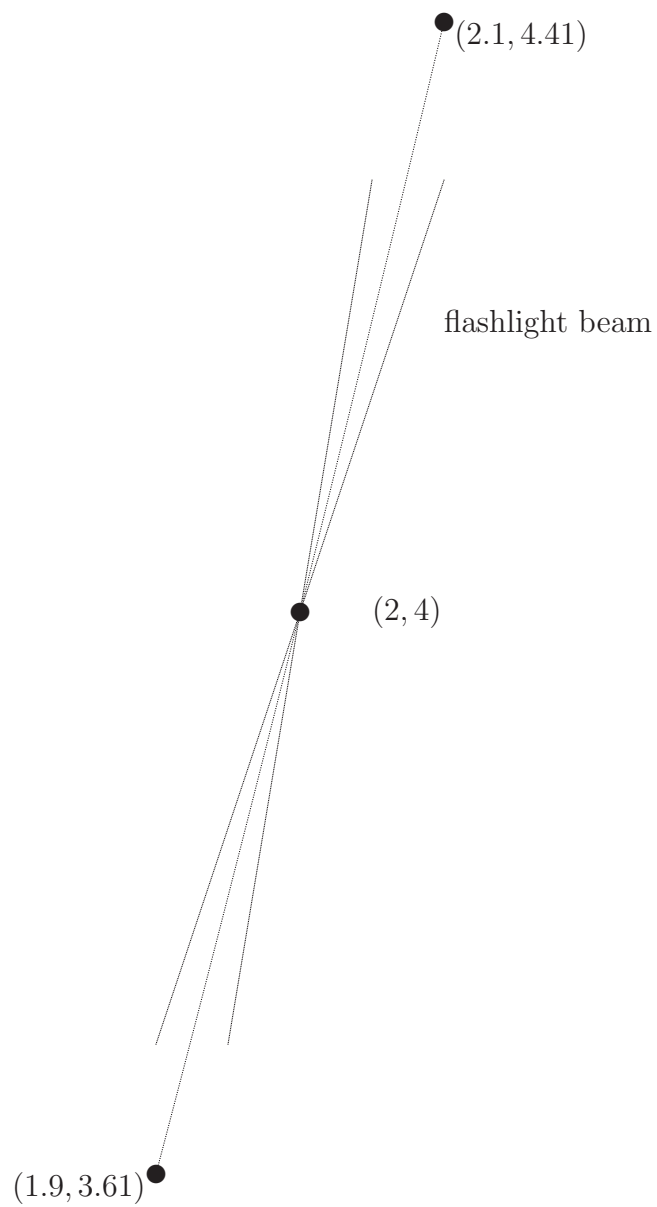
How do we find this tangent line $\mathbf{t}$?

Since point P lies on $\mathbf{t}$, it suffices to find the slope of this line.

But how do we do that?

6. THE CURVE $y = x^2$



7. MAGNIFICATION $\times 10$ 

8. OBSERATIONS ABOUT THE GRAPH

The graph shows the line segment connecting $(2, 4)$ and $(3, 9)$. It has slope $\frac{9-4}{3-2} = 5$.

If you look closely at the graph, or use a magnifying glass, you will see that the line segment connecting $(2, 4)$ and $(3, 9)$ lies **above** the parabola $y = x^2$ for value of x between 2 and 3.

So the slope of the tangent line at $(2, 4)$ must be less than 5.

9. TRY AGAIN.

Next consider the line connecting $(2, 4)$ to $(2.5, 6.25)$.

This line has slope

$$m = \frac{6.25 - 4}{2.5 - 2} = \frac{2.25}{.5} = 4.5.$$

Once again, this line lies above the parabola $y = x^2$ for values of x between 2 and 2.5.

Thus we have ruled out all lines thru $(2, 4)$ with slope 4.5 or greater.

10. TRY A THIRD TIME

The line connecting $(2, 4)$ to $(2.1, 4.41)$ has slope

$$m = \frac{4.41-4}{2.1-2} = \frac{.41}{.1} = 4.1.$$

Since this third line cannot be the tangent to the curve at $(2, 4)$, which rules out all lines thru $(2, 4)$ with slope 4.1 or greater.

Okay, this is getting boring.

11. GENERALIZE

Suppose that h is a little tiny positive number and that we connect a line ℓ from $(2, 4)$ to $(2 + h, (2 + h)^2)$.

The slope of ℓ is

$$m = \frac{(2+h)^2-4}{(2+h)-2} = \frac{4+4h+h^2-4}{h} = \frac{4h+h^2}{h} = \frac{h(4+h)}{h} = 4 + h$$

Our previous argument, which we have used three times so far, shows that the tangent line cannot have slope $4 + h$, no matter how small we choose the positive number h .

In other words, the tangent line cannot have slope greater than 4.

12. THE OTHER SIDE

Now work on the left side of $(2, 4)$.

What slopes do you get if you connect $(2, 4)$ to

- (a) $(1, 1)$
- (b) $(1.5, 2.25)$
- (c) $(1.9, 3.61)$
- (d) $(2 - h, (2 - h)^2)$

By the same reasoning as we used before, we can rule out all slopes smaller than 4.

13. WISDOM FROM SHERLOCK HOLMES

If we can eliminate **all statements but one** as being false, then the remaining statement, no matter how improbable, **must be true**.

We have eliminated all slopes bigger than 4 and all slopes less than 4.

Conclusion (via S. Holmes): The tangent line has slope $m = 4$.

Question 1: What is the equation of the tangent line?

14. USING LIMITS

The slope of the tangent line of $f(x)$ at $x = a$ is the limit

$$[h \rightarrow 0 \text{ Method}]: m_{tan} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

This can also be written

$$[x \rightarrow a \text{ Method}]: m_{tan} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Compute both of these limits when $f(x) = x^2$ and $a = 2$.

15. FUTURE HOPE

Does it always take this much work?

Happily, the answer is “no.”

The purpose of calculus is to give you an easy way to find the slope of the tangent line, even if the curve is gnarly and nasty.

Perhaps you have already seen some shortcuts from a previous high school course.

Bear in mind that knowing such methods does not exempt you from understanding the underlying process we used to define the concept of the tangent to a curve.

Just like owning a calculator does not mean you can forget how to add by hand, or owning a car does not mean you never have to use your legs again.