

1. MATH 353

Parallels

Professor Richard Blecksmith

richard@math.niu.edu

Dept. of Mathematical Sciences

Northern Illinois University

2. RAILROAD TRACKS

Given one rail of a railroad track,

is there always a second rail whose (perpendicular) distance from the first rail is exactly the width across the tires of a train,

so that the two rails never intersect?

Of course we believe this is so in Euclidean space.

3. PARALLELS ON THE SPHERE

We saw earlier that any two lines (great circles) on the sphere intersect.

Suppose we start our lines (railroad tracks) perpendicular to the equator, spaced 10 units apart.

These two lines eventually meet at the North (and South) Pole.

4. WHAT HAPPENS IF $\omega = \infty$?

We might expect:

Theorem. [Parallel lines exist]

Given a line ℓ and a point P not on ℓ .

Then there exists a line through P parallel to ℓ .

Okay, how do we go about proving this Theorem?

5. PERPENDICULARS TO THE RESCUE

One idea is to use perpendiculars.

Construct a line t through P and perpendicular to ℓ .

Now construct a line m through P and perpendicular to t .

Surely, lines m and ℓ must be parallel.

Suppose you are driving on 8th Avenue in New York City. You turn right on 34th Street, go two blocks, and then turn right on 6th Avenue.

Sixth Avenue must be parallel to 8th Avenue—and it is.

But can we prove it?

6. START OF THE PROOF

Suppose, to play Devil’s Advocate, that m and ℓ are **not** parallel.

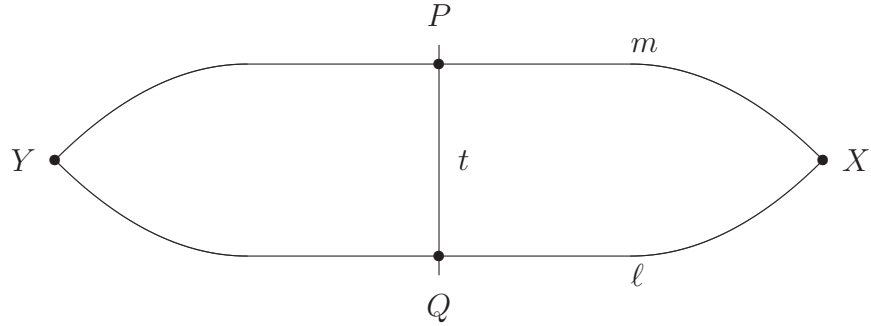
Then they must meet at a point which we will call X (the “**mysterious**” point).

Let Q be the point of intesection of line t and ℓ .

7. EARLY INNINGS OF THE PROOF

Note that $\angle QPX$ and $\angle PQX$ are both right angles.

Now let Y be a point on the opposite ray of $\overrightarrow{QX}$ so that $QY = QX$.



8. MIDDLE INNINGS OF THE PROOF

Consider the triangles $\triangle P Q Y$ and $\triangle P Q X$.

Since the angles $\angle P Q Y$ and $\angle P Q X$ are supplementary and $m\angle P Q X = 90$, it follows that $m\angle P Q Y = 90$.

By construction $Q Y = Q X$ and finally the two triangles share the side $\overline{P Q}$.

By Side–Angle–Side, $\triangle P Q Y \cong \triangle P Q X$.

9. SEVENTH INNING STRETCH

$\triangle P Q Y \cong \triangle P Q X$ implies

$$m\angle Q P Y = m\angle Q P X = 90.$$

By angle addition,

$$m\angle Y P X = m\angle Y P Q + m\angle Q P X = 90 + 90 = 180.$$

In other words, $\angle Y P X$ is a straight angle, implying that Y , P , and X are collinear.

10. NINTH INNING FINISH

We now have two different lines m and ℓ passing through the points X and Y .

That cannot happen unless $XY = \omega$.

If $\omega = \infty$ we violate axiom I4, which states that at most one line passes through two distinct points.

We were led to this contradiction by assuming that m and ℓ intersect;
so, in fact, m and ℓ must be parallel.

11. THE BIG QUESTION

Now here comes the question that mystefied mathematicians for over two thousand years:

*Is m the **only** parallel line through P ?*