

PARTIAL FRACTIONS

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1. ALGEBRAIC SIMPLIFICATION

Simplify $\frac{2}{x-3} - \frac{1}{x+2}$.

Common denominator is $(x-3)(x+2)$

$$\begin{aligned}\text{So } \frac{2}{x-3} - \frac{1}{x+2} &= \frac{2}{x-3} \frac{x+2}{x+2} - \frac{1}{x+2} \frac{x-3}{x-3} \\ &= \frac{2x+4}{(x-3)(x+2)} - \frac{x-3}{(x-3)(x+2)} \\ &= \frac{2x+4-(x-3)}{(x-3)(x+2)} \\ &= \frac{x+7}{(x-3)(x+2)}.\end{aligned}$$

2. IDEA APPLIED TO INTEGRALS

Integrate $\int \frac{x+7}{(x-3)(x+2)} dx$

The idea is to “unsimplify” this fraction by writing it as

$$\begin{aligned}&\int \frac{2}{x-3} - \frac{1}{x+2} dx \\ &= 2\ln(x-3) - \ln(x+2) + C\end{aligned}$$

The question is: How to unsimplify a fraction into a sum of fractions, which we can integrate?

3. EXAMPLE 1.

To unsimplify $\frac{x+7}{(x-3)(x+2)}$

write $\frac{x+7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$

To find the values of A and B , multiply by $(x-3)(x+2)$:

$$A(x+2) + B(x-3) = x+7$$

$$\text{Plug in } x=3: A(3+2) + B \cdot 0 = 3+7 \implies 5A = 10 \implies A = 2$$

$$\text{Plug in } x=-2: A \cdot 0 + B(-2-3) = -2+7 \implies -5B = 5 \implies B = -1$$

4. EXAMPLE 2

Integrate $\int \frac{5x^2 - 4x - 23}{(x-3)(x+2)}$

Since the numerator and denominator both have degree 2, we can actually divide:

$$5x^2 - 4x - 23 = 5(x^2 - x - 6) + x + 7$$

$$\text{So } \frac{5x^2 - 4x - 23}{(x-3)(x+2)} = 5 + \frac{x+7}{(x-3)(x+2)}$$

We did this last integral in Example 2.

Moral: if the degree of the numerator is $\geq$ the degree of the denominator, divide first.

5. EXAMPLE 3.

Integrate $\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$

Factor the denominator $x^3 + 2x^2 + x = x(x^2 + 2x + 1)$
 $= x(x + 1)^2$

Because of the square, this fraction unsimplifies as

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

Clear denominator by multiplying by $x(x + 1)^2$:

$$A(x + 1)^2 + Bx(x + 1) + Cx = 5x^2 + 20x + 6$$

Plug in $x = -1$: $A \cdot 0 + B \cdot 0 + -C = 5 - 20 + 6 \implies C = 9$

$$A(x + 1)^2 + Bx(x + 1) + 9x = 5x^2 + 20x + 6$$

Plug in $x = 0$: $A \cdot 1 + B \cdot 0 + 9 \cdot 0 = 6 \implies A = 6$

$$6(x + 1)^2 + Bx(x + 1) + 9x = 5x^2 + 20x + 6$$

Plug in $x = 1$: $6 \cdot 4 + B \cdot 1 \cdot 2 + 9 \cdot 1 = 5 + 20 + 6 \implies 24 + 2B + 9 = 31$
 $\implies 2B = 31 - 33 = -2 \implies B = -1$

6. EXAMPLE 3 CONTINUED

Integrate $\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

Clearing the denominator gave us

$$A(x + 1)^2 + Bx(x + 1) + Cx = 5x^2 + 20x + 6$$

Plugging in $x = -1$ and $x = 0$ gave us $C = 9$ and $A = 6$

$$6(x + 1)^2 + Bx(x + 1) + 9x = 5x^2 + 20x + 6$$

Plug in $x = 1$: Any value other than 0 or -1 will work.

$$6 \cdot 4 + B \cdot 1 \cdot 2 + 9 \cdot 1 = 5 + 20 + 6$$

$$\implies 24 + 2B + 9 = 31$$

$$\implies 2B = 31 - 33 = -2$$

$$\implies B = -1$$

7. EXAMPLE 3 FINALE

Integrate $\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$

Write

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$$

$$\begin{aligned} \text{So } \int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} &= \int \frac{6}{x} - \int \frac{1}{x+1} + \int \frac{9}{(x+1)^2} \\ &= 6 \ln(x) - \ln(x+1) + 9 \frac{(x+1)^{-1}}{-1} + C \end{aligned}$$

8. COMPLEX NUMBERS

A complex number is a number of the form

$$a + bi$$

where i has the property $i^2 = -1$.

We multiply complex numbers by using FOIL and the rule $i^2 = -1$.

$$\begin{aligned} &(5 + 2i)(3 - 6i) \\ &= \text{FIRST } 5 \cdot 3 + \text{OUTER } 5 \cdot (-6i) + \text{INNER } 3 \cdot (2i) + \text{LAST } 2i \cdot (-6i) \\ &= 15 - 30i + 6i - 12i^2 \\ &= 15 - 30i + 6i - 12(-1) \\ &= 27 - 24i \end{aligned}$$

9. FACTORING POLYNOMIALS

Fact: A polynomial of degree n factors into n linear pieces if we allow complex numbers.

For example, $x^2 + 1 = (x + i)(x - i)$

Fact: If the coefficients of $f(x)$ are real, then complex roots come in pairs $a + bi$ and $a - bi$.

$$\begin{aligned}\text{For example, the polynomial } x^2 + 6x + 25 \quad x &= \frac{-6 \pm \sqrt{36 - 100}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= -3 \pm 4i\end{aligned}$$

We call the numbers $\alpha = a + bi$ and $\beta = a - bi$ **complex conjugates** of each other.

10. RECOVERING THE QUADRATIC POLYNOMIAL

Suppose we know that $\alpha = -3 + 4i$ and $\beta = -3 - 4i$ are the roots to a quadratic polynomial.

The we can recover this polynomial by using FOIL to multiply

$$\begin{aligned}&(x - (-3 + 4i))(x - (-3 - 4i)) \\ &= \text{FIRST } x^2 + \text{OUTER } -(-3 - 4i)x + \text{INNER } -(-3 + 4i)x + \text{LAST } (-3 + 4i)(-3 - 4i) \\ &= x^2 + (3 + 4i)x + (3 - 4i)x + (9 - 16i^2) \\ &= x^2 + 6x + 25\end{aligned}$$

The point is that when α and β are complex conjugates of each other, the polynomial $(x - \alpha)(x - \beta)$ is always a quadratic polynomial with real coefficients.

11. THEORY VERSUS PRACTICE

So in theory, every polynomial with real coefficients factors into a product of linear polynomials of the form $x - r$, where r is a real root, and quadratic polynomials of the form $x^2 + bx + c$ which do not further factor over the reals.

In practice, it is not easy to find these factors, and it may be impossible to find exact formulas for them.

12. EXAMPLE 4.

Evaluate

$$\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$$

We can factor $x^2 - x = x(x - 1)$.

We cannot factor $x^2 + 4$ without using complex numbers.

The trick is to unsimplify the integrand as

$$\frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}$$

Clear denominator by multiplying by $x(x - 1)(x^2 + 4)$:

$$A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)x(x - 1) = 2x^3 - 4x - 8$$

13. EXAMPLE 4 – INSTALLMENT 2.

$$A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)x(x - 1) = 2x^3 - 4x - 8$$

$$\text{Plug in } x = 1: A \cdot 0 + B \cdot 5 + 0 = 2 - 4 - 8 \implies 5B = -10 \implies B = -2$$

$$A(x - 1)(x^2 + 4) - 2x(x^2 + 4) + (Cx + D)x(x - 1) = 2x^3 - 4x - 8$$

$$\text{Plug in } x = 0: A \cdot (-4) + 0 + 0 = -8 \implies A = 2$$

$$2(x - 1)(x^2 + 4) - 2x(x^2 + 4) + (Cx + D)x(x - 1) = 2x^3 - 4x - 8$$

14. EXAMPLE 4 – INSTALLMENT 3

$$2(x - 1)(x^2 + 4) - 2x(x^2 + 4) + (Cx + D)x(x - 1) = 2x^3 - 4x - 8$$

To determine the values of C and D , plug in $x = -1$:

(the choice is again arbitrary)

$$2(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2) = 2(-1) - 4(-1) - 8$$

$$\implies -20 + 10 + 2(-C + D) = -2 + 4 - 8$$

$$\implies -10 + 2(-C + D) = -6$$

$$\implies 2(-C + D) = 4$$

$$\implies -C + D = 2$$

Plug in $x = 2$ (the choice is again arbitrary):

$$2(1)(8) - 2(2)(8) + (2C + D)(2)(1) = 2(8) - 4(2) - 8$$

$$16 - 32 + 2(2C + D) = 16 - 8 - 8$$

$$\implies -16 + 2(2C + D) = 0$$

$$\implies 2(2C + D) = 16 \implies 2C + D = 8$$

15. EXAMPLE 4 INSTALLMENT 4

To find C and D we must solve the system:

$$(1) -C + D = 2$$

$$(2) 2C + D = 8$$

Multiply (1) by 2 and add:

$$(1) -2C + 2D = 4$$

$$(2) 2C + D = 8$$

$$\text{Sum: } 3D = 12$$

$$D = 4$$

Plug $D = 4$ into equation (1) to get:

$$-C + 4 = 2 \implies C = 2$$

After considerable algebra we now know

$$\frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)} = \frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4}$$

16. EXAMPLE 4 FINALE

$$\int \frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)} = 2 \int \frac{1}{x} - 2 \int \frac{1}{x-1} + \int \frac{2x}{x^2+4} + 4 \int \frac{1}{x^2+4}$$

We know how to integrate each of these four integrals:

$$\int \frac{1}{x} = \ln x$$

$$\int \frac{1}{x-1} = \ln(x-1)$$

$$\int \frac{2x}{x^2+4} = -(x^2+4)^{-1}$$

To integrate $\int \frac{2x}{x^2+4}$, substitute $u = x^2 + 4$.

Then $du = 2x dx$ and

$$\int \frac{2x}{x^2+4} dx = \int \frac{du}{u}$$

$$= \ln u + C$$

$$= \ln(x^2+4) + C$$

17. EXAMPLE 4 NAIL IN COFFIN

To integrate $\int \frac{1}{x^2+4}$, substitute $x = 2 \tan \theta$.

$$\text{Then } \int \frac{1}{x^2+4} dx = \int \frac{2 \sec^2 \theta}{4 \sec^2 \theta} dx$$

$$= \frac{1}{2} \theta + C$$

$$= \frac{1}{2} \tan^{-1}(x/2) + C$$