

Pails of Water

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1. THE PAILS OF WATER PROBLEM

You have a 5 and a 7 quart pail.

How can you measure exactly 1 quart of water, by pouring water back and forth between the two pails?

You are allowed to fill and empty each pail as many times as needed.

Split into groups and work on this problem.

The first group to finish should write their solution on the whiteboard.

2. ONE SOLUTION:

5-qt	7-qt	Explanation
5	0	fill 5-quart pail
0	5	transfer 5 quarts into 7
5	5	fill 5-quart pail a second time
3	7	pour 2 quarts from 5 into 7
3	0	empty 7-quart pail
0	3	transfer 3 quarts into 7
5	3	fill 5-quart pail a third time
1	7	transfer 4 quarts from 5 into 7
1	0	empty 7-quart pail a second time

3. AN ARITHMETIC EXPLANATION

What happened?

Notice that we filled the 5-quart pail **three** times

and emptied the 7-quart pail **twice**.

A little thought convinces us that the method given above can be written more succinctly as:

$$3 \times 5 - 2 \times 7 = 1.$$

5. POW PROBLEM #2

Solve the POW problem if $a = 10$ and $b = 19$. That is, find numbers m and n which satisfy the equation:

$$10m - 19n = 1$$

Hint: This is very easy.

7. POW PROBLEM #4

Solve the POW problem if $a = 11$ and $b = 8$. That is, find numbers m and n which satisfy the equation:

$$11m - 8n = 1$$

4. THE PAILS OF WATER PROBLEM

Given two positive integers a and b , the “Pails of Water Problem” (abbreviated as POW) means: find two numbers m and n so that

$$ma - nb = 1$$

6. POW PROBLEM #3

Solve the POW problem if $a = 7$ and $b = 9$. That is, find numbers m and n which satisfy the equation:

$$7m - 9n = 1$$

8. POW PROBLEM #5

Solve the POW problem if $a = 6$ and $b = 10$. That is, find numbers m and n which satisfy the equation:

$$6m - 10n = 1$$

9. TROUBLE BREWING

This last example $a = 6$ and $b = 10$ shows that not all POW problems can be solved, that is, you cannot always get a 1 in the equation

$$ma - nb = 1$$

What has to be true about a and b so that the POW problem has a solution?

11. SWITCHING a AND b

In general, in seeking a solution to the POW problem, we will be content with either

$$m \times a - n \times b = 1$$

or

$$m \times b - n \times a = 1$$

Either equation solves POW.

13. WHEN THE NUMBERS GET BIG

Now suppose you wish to obtain one quart with a 7-quart and a 19-quart pail.

That is, you want to solve the equation

$$19m - 7n = 1$$

Try it!

15. ARITHMETIC MAGIC

Here's how it works:

Equation	Explanation
$1 = 3 \times 5 - 2 \times 7$	solution for 5 and 7
$= 3 \times (19 - 2 \times 7) - 2 \times 7$	subst $5 = 19 - 2 \times 7$
$= 3 \times 19 - 6 \times 7 - 2 \times 7$	expand the parentheses
$= 3 \times 19 - 8 \times 7$	gather the 7's together.

The solution is to fill the 19-quart pair three times, a total of 57 quarts, and empty into the 7-quart pail eight times, a total loss of 56 quarts. The difference is precisely 1 quart.

10. SWITCHING a AND b

When $a = 5$ and $b = 7$ a solution to the POW problem is

$$3 \times 5 - 2 \times 7 = 1$$

What if we switch the 5 and 7 and ask for an equation

$$m \times 7 - n \times 5 = 1$$

Essentially, this means that we are filling the 7-quart pail and emptying into the 5-quart pail.

Can you solve this equation?

12. CONNECTING THE TWO SOLUTIONS

Do you see any way to connect the two solutions:

Solution 1. $3 \times 5 - 2 \times 7 = 1$ and

Solution 2. $3 \times 7 - 4 \times 5 = 1$.

Hint: look at the equation

$$3 \times 5 - 2 \times 7 = 3 \times 7 - 4 \times 5 = 1$$

14. WHEN THE NUMBERS GET BIG

Here is a secret to the POW Problem (19, 7). Fill the 19-quart pail two times from the 7-quart pail, thereby effectively converting the 19-quart pail into a 5-quart pail. The associated equation is

$$5 = 19 - 2 \times 7$$

But we know how to solve the POW problem with the numbers of 7 and 5. So we can use our previous solution for 5 and 7 to get a solution for 7 and 19.

16. ARE YOU READY?

Use this Super Secret Way to solve the POW Problem with the numbers 100 and 77.

17. POW(100,77)

18. TODAY'S POP QUIZ

Equation	Explanation
$1 = 3 \times 8 - 23$	solution for 8 and 23
$= 3 \times (77 - 3 \times 23) - 23$	subst $8 = 77 - 3 \times 23$
$= 3 \times 77 - 9 \times 23 - 23$	expand the parentheses
$= 3 \times 77 - 10 \times 23$	gather the 23's together
$= 3 \times 77 - 10 \times (100 - 77)$	subst $23 = 100 - 77$
$= 3 \times 77 - 10 \times 100 + 10 \times 77$	expand the parentheses
$= 13 \times 77 - 10 \times 100$	gather the 77's together.

Do these numbers check out? What's 13×77 and 10×100 .

Solve the POW Problem with the numbers 31 and 9.

19. QUOTIENT AND REMAINDER METHOD

In general, suppose you want to solve the POW problem with two numbers, a and b . Let's say that a is smaller than b . If you can't think of the solution right away because the numbers are too large, reduce the problem to two smaller numbers, by dividing the smaller number a into b , getting a quotient q and a remainder r .

Write this as

$$r = b - q \times a.$$

Record this equation as you will need it later.

20. WORKING BACKWARDS

The original POW problem a and b *reduces* to solving POW with the smaller numbers r and a . You can write this reduction as

$$(a, b) \rightarrow (r, a).$$

If the numbers r and a are still too big, reduce again, by dividing r into a .

Eventually you will reach two numbers that you can do in your head. Starting with this "easy" solution, carefully work backwards, substituting the "remainder" equations you previously recorded, until you reach a solution using the original numbers a and b .

21. GIVING CREDIT TO EUCLID

This method for solving the pails of water problem is called the *Euclidean algorithm*. An algorithm is a set of steps for completing some task. (No, the word is not named after Bill Clinton's vice president.) Who was that? A recipe from a cookbook is a good example of an algorithm.

This particular procedure dates back over two thousand years ago to the African mathematician Euclid, who also founded plane geometry.

22. WHO WAS EUCLID?

Ten minute activity: Google search "Euclid" and write down 3 interesting facts about him.